Thursday February 13th 2020

1- No Class Next Tuesday 18th

- Recall the two families of invariants for knots $K \subset S^3$
  - ★ Colored Jones $\{J_N(K;q)\}_{N=2}^{\infty}$
  - ★ Twisted Alex. Poly. $\varphi : \pi_1(S^3 \setminus K, *) \to SL(2,\mathbb{C})$ giving $\{\Delta_{K,\varphi}(t)\}$

Volume Conjecture is likely coming from some relation between the two, perhaps for a particular representation φ.

★ If you want to study repr. φ, you can restrict to real/complex parts of $SL(2,\mathbb{C})$, ie $SL(2,\mathbb{R})$ / $SU(2)$. This makes things easier.

- If K is hyperbolic, then we know there is a canonical (holonomy) map $\varphi : \pi_1(S^3 \setminus K, *) \to PSL(2,\mathbb{C})$ which gets lifted to $SL(2,\mathbb{C})$. Composed with symmetrization map we get a map to $SL(N,\mathbb{C})$

$$\varphi_N : \pi_1(S^3 \setminus K, *) \longrightarrow SL(N,\mathbb{C})$$

how many lifting = how many spin structure

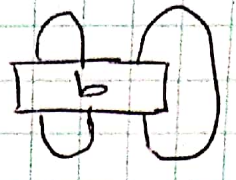
Volume conjecture is:

$$\lim_{N\to\infty} \frac{\log |J_N'(K, e^{\frac{2\pi i}{N}})|}{N} = 2 \lim_{N\to\infty} \frac{\log|\Delta_{K,\varphi_N}(\xi)|}{N^2}$$

$$\forall |\xi| = 1$$

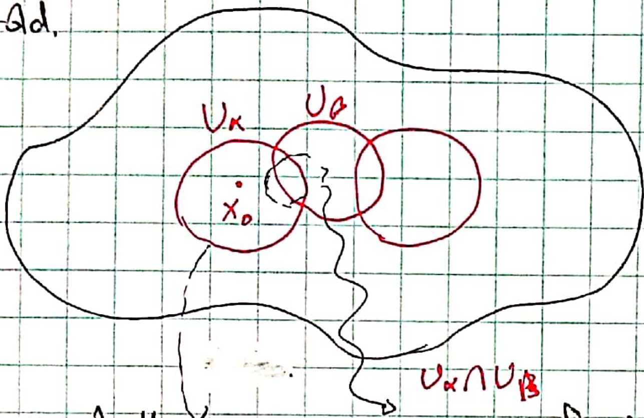
Midterm: (Try to) prove for figure 8.

Goal: Try to prove for all 2-bridge knots.

a 2-bridge knot is of the form  for $b \in B_3$.

It has the fundamental group $\langle x, y \mid wx = yw \rangle$ which is same as that of figure 8.

(1) Holonomy representation: Suppose M is a hyperbolic complete manifold.



open ball in $\mathbb{H}^3$

Transition function must be an isometry of $\mathbb{H}^3$ (it can be extended to $\mathbb{H}^3$)

So an element of $PSL(2, \mathbb{C})$

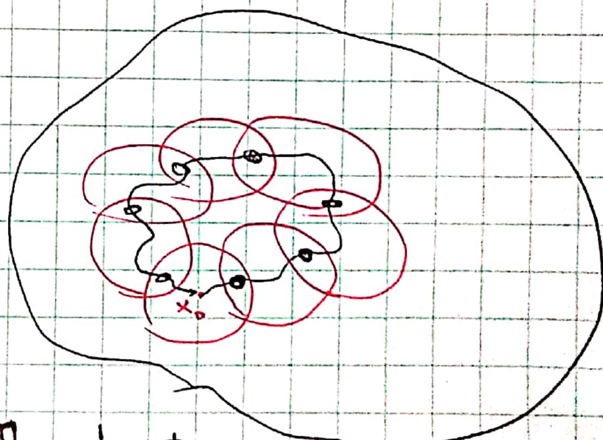
it is a fact that conformal maps in 3-dim extend uniquely. $\Uparrow$

holonomy means

Take a path and collect those elements

But when you come back to x_0

you may not get back to the same element. This means, as

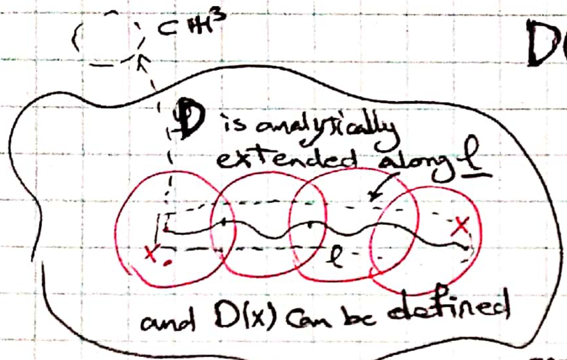


This element is dependent only on the homotopy class of the loop, you get a representation $\varphi : \pi_1 \to PSL(2, \mathbb{C})$

(2) Developing map: $D : \tilde{M} \longrightarrow \mathbb{H}^3$

Recall:

$\tilde{M} = \{ (x, [\ell]) \mid \ell$ a path from x_0 to $x \}$ — Therefore



$D(x,[\ell])$ depends on x_0 and chart we pick around it only up to a composition by an element of $PSL(2,\mathbb{C})$.

D(x) is only dependent on $[\ell]$ and on x_0 and the chart at x_0 up to an element of $PSL(2,\mathbb{C})$

Facts from hyperbolic geometry:

Have representation $\varphi : \pi_1 M^3 \longrightarrow PSL(2,\mathbb{C})$ and $M^3 \cong \mathbb{H}^3/\Gamma$

where $\Gamma = \varphi(\pi_1 M^3)$ is discrete & faithful.

φ is projective representation and the scalar $c(g,h)$ in

write associativity rule to see this

$$\varphi(gh) = c(g,h)\, \varphi(g)\, \varphi(h)$$

is a two-cycle. But $H^2(PSL(2,\mathbb{C})) = 0$ so φ can always be lifted to $SL(2,\mathbb{C})$. The different lifting correspond to the different spin structures.

- Recall for figure 8: $\langle x,y \mid wx = yw \rangle \quad w = [x, y^{-1}]$

holonomy $\varphi_2 : \; x \to \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \; y \to \begin{pmatrix} 1 & 0 \\ -w & 1 \end{pmatrix}$

$$\varphi_N : \Pi_K \to SL(2,\mathbb{C}) \xrightarrow{\ ?\ } SL(N,\mathbb{C})$$

- By repr. Theory, $SL(2,\mathbb{C})$ has a so-called fundamental (defining) representation : $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$ = matrix representation on $\mathbb{C}^2$.

Let $V := \mathbb{C}^2$ above. Then we can define representation on $V^{\otimes m}$ by $g \cdot (v_1 \otimes \dots \otimes v_m) \Longrightarrow gv_1 \otimes \dots \otimes gv_m$.

- Now This representation is <u>NOT</u> irreducible. For example:

$$V \otimes V \cong \underset{\downarrow}{\textcircled{1}} \oplus \mathbb{C}^3 \longrightarrow \text{"Triplet" representation}$$

"Singlet" representation

- We can prove by induction That $V^{\otimes m}$ decomposition has always some irreducible representation of dimension <u>$m+1$</u> called V_{m+1}.

- By using the fact That permutations commute with The representation we can show V_N has a basis $\langle x^{N-1}, x^{N-2}y, \dots, y^{N-1} \rangle$ for which the representation φ_N is:

As an example $N=3$, for figure 8: φ_2: $a \to \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
$b \to \begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix}$

Then to get φ_3 first compute $\varphi(a)^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$
$= \begin{pmatrix} x-y \\ y \end{pmatrix}$

Then plug in first coordinate into x and second into y : $\langle x^2, x^1y^1, y^2 \rangle \xrightarrow{\varphi_3} \langle (x-y)^2, (x-y)y, y^2 \rangle$

which means $\varphi_3(a) = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix}$.

Similarly for b we have: $\varphi^{-1}(b)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ \omega x + y \end{pmatrix}$

Thus $\langle x^2, xy, y^2 \rangle \xrightarrow{\varphi_3} \langle x^2, x(\omega x + y), (\omega x + y)^2 \rangle$

Therefore $\varphi_3(b) = \begin{pmatrix} 1 & \omega & \omega^2 \\ 0 & 1 & 2\omega \\ 0 & 0 & 1 \end{pmatrix}$

We can then compute (using previous section materials):

$$\Delta_{\infty, \varphi_3} = -t^{-3}(t-1)$$

Similarly $\Delta_{\infty, \varphi_4} = t^{-4}(t^2 - 4t + 1)^2$ and more interesting is:

$$\Delta_{\infty, \varphi_5} = -t^{-3}(t-1)(t^4 - 9t^3 + 44t^2 - 9t + 1)$$

in Volume conjecture if $t=1$ we get all zero which log is $\underline{-\infty}$! so need to do normalization for N odd.

For Figure 8 The sequence converges like This:

$$\frac{4\pi \log|\Delta_{\infty, \varphi_N}(1)|}{N^2}$$

N	4	12	24	32
Value	0.54...	1.86...	1.98...	2.006...

Data from H. Goda paper

The Volume is $\underline{2.0298832\ldots}$

- One of the difficulties in matching up The colored Jones sequence and The Twisted Alex. is the specific choice of value $q = e^{\frac{2\pi i}{N}}$

for the colored Jones and the generic choice of ξ, $|\xi|=1$ on the other side.

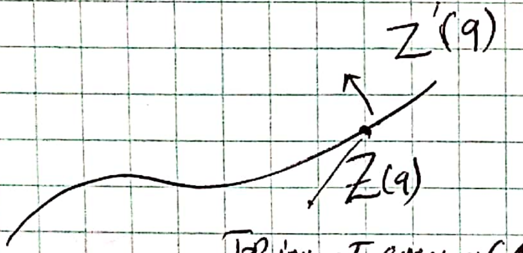
- We believe there is a version where colored Jones is evaluated at generic values. Some "philosophical" reasons:

J_N is a top. invariant. A Turaev-Viro inv where one triangulates the manifold and puts labels on edges, faces, vertices and take the "state-sum". This is a top. inv. when labels come from a "modular tensor category". also called a partition function $Z(q)$ (due to state sum formulation)

Now we ask if for $q \in \mathbb{C}$ we are getting a top. inv

"Then looking at the normal direction of this,

we believe we should get the volume"



Top. inv. at every $q \in \mathbb{C}$