Thursday February 20th 2020

1) Where does the V.C. come from? (2+1)-gravity with negative cosmological constant

Witten's papers 1. Exactly...

(2 Revisited)

3. Analytic continuation of CS-theory

· X^3 manifold spacetime, eg X^3 closed oriented

Einstein Hilbert action (EH):

$$I(X^3, g) = \int_{X^3} d^3\mathrm{vol}(R - 2\Lambda)$$

scalar curvature; cosmological constant $\Lambda = -\frac{1}{\ell^2}$

Einstein equation $\rightsquigarrow$ $R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \Lambda g_{\alpha\beta} = 0$ $(T_{\alpha\beta})$

There is a spin connecti ω and dreiben (Veilben)

Fact: If you have a 3 manifold closed oriented then $TX^3 \cong X^3 \times \mathbb{R}^3$ but cannot be trivialized canonically, so there is a framing (dreiben) a choice of the trivialization $e: TX^3 \xrightarrow{\cong} X^3 \times \mathbb{R}^3$

So instead of g can use (ω, e) Then $I(X^3, g)$ becomes $\rightsquigarrow k_R\, CS(A_+) - k_L\, CS(A_-)$

$SO(2,2)$ gauge field

(two copies of CS theory)

But $SO(2,2) \overset{\text{locally}}{\cong} SO(2,1) \times SO(2,1)$ This was all in Lorentzian signature, we should go

$SO(4) \cong SU(2) \times SU(2)$

into the Euclidean signature by doing a wick rotation.

This makes $SU(2,1) \to SU(2)$ and we end up in classical CS theory.

Witten said if all these make sense we should get a version of V.C. out of this (as from the integration I we get the volume.)

• What is the V.C. for closed manifolds?

$SU(2)_k$ CS   TQFTs ← Reshetikhin–Turaev (doubled) Turaev–Viro

$$\frac{\log |Z_k(X^3)|}{N} \xrightarrow{N\to\infty} \mathrm{Vol}(X^3)$$

for $N = k+2$ where k is the level ← Naive generalization to 3 mnfld

If Z_k comes from $SU(2)_k$–CS TQFTs then it is known that

$$|Z_k(X^3)| \leq \alpha \cdot D^{2\,\text{genus}(X^3)}$$

where $D^2 = \sum d_i^2$ the total quant. dim. of the TQFT.

α is a constant

(Ref: look at book of Turaev "Quant. Invariants of 3 mnflds")

Sketch of proof for the bound:

$$X^3 = H_g \cup_{\Sigma_g} H_g \qquad Z(X^3) = \langle Z(H_g), Z(H_g) \rangle_{V(\Sigma_g)}$$

in unitary, by Verlinde's formula

This is bounded by D^{2g}

But then $\frac{|\log Z_k(X^3)|}{N} \leq \frac{\log(\alpha D^{2g})}{N}$ but $D^2 \sim N^{3/2}$ and so $N\to\infty$ this should go to zero!

So there needs to be a new version of V.C. (by Q Chen and T Yang)

For "unitary theories". And the idea (with strong numerical evidence) is to square the q corresponding to $SU(2)_k$ so making the theory non-unitary. Essentially since the Hamiltonian is NOT hermitian therefore when taking $\mathrm{Tr}\, e^{-itH}$ we can have a rate of growth that is proportional to N.

The second possibility is:

(2) We should not rotate the fibers but just the base so we get CS theory for non compact (unlike $SU(2)$) Lie groups.

(It is in the paper Analytic continuation).

Going back to V.C for hyperbolic knots

$$\lim_{N\to\infty} \frac{\log |V_N(K, e^{2\pi i/N})|}{N} = \frac{1}{2\pi} \mathrm{Vol}(S^3 \setminus K)$$

$$V_N = \frac{J_N}{[N+1]} \quad \text{so that } V_N(\text{unknot}) = 1$$

Now for $SU(2)$ TQFT at $q = e^{2\pi i/N}$ colors are $1, 2, \dots, K+1 = N-1$ so there is no V_N so V.C. above does not have a $SU(2)$ TQFT interpretation

- But it is likely to have a TQFT interpretation on $SL(2,\mathbb{C})$ or $SL(2,\mathbb{R})$. So we need to do TQFT on non compact theories.

- For $V_N(K, q)$ as long as $q \in \mathbb{C}^{*} = \mathbb{C} \setminus \{0\}$ V_N is well-defined. For $N=2$ this gives Jones poly. Is there a generalization of Jones poly to 3-mfld?

Witten found generalization only for some roots of unity, $q = e^{\pm\frac{2\pi i}{r}}$

After many years, we can do almost any root of unity. We claim that the obstruction to the generalization, is the Volume, of Jones poly to an analytic fcn for 3mflds

• Why are we talking about this?

To relate the Jones colored poly to twisted Alex. poly we want to look at analytic expansion (Taylor series) of J_N and $\Delta_K(t)$ and the series better be similar to each other if N.C is true.

• Melvin-Morton Conj.

Three proofs:

1) Bar-Natan, Garoufalidis.

2) Lin-W (Zeta function, Lyndon words)

3) "L. Rozansky Expansion" boson fermion correspondence

It's a "physicist" proof

J. Milnor "infinite cyclic covering" discovered that $\frac{1}{\text{Alex.}}$ = zeta function which is counting some periodic orbits

fixed points of some diffeomorphism

We will discuss (3).

$X \times I$ 

$(x,0) \sim (f(x),1)$

• In colored Jones q should be interpreted as e^h. The reason is $V_N(K,q) \underset{CS}{=} \int_{A} \mathcal{D}A\, e^{ikCS(A)}$ (some operator)

$\overset{h=\frac{1}{K}}{=} \sum a_n h^n$ Then a_n are called Vassiliev invariants. In other words expanding $V_N(K; e^h) = \sum_{n=0}^{\infty} a_n h^n$

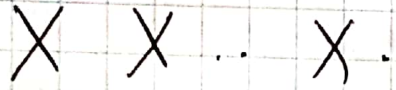
gives invariants a_n.

<u>Thm</u> (XS Lin): a_n are <u>finite type</u> invariants.

ie $\forall n$, $\exists m$ s.t. for any singular link L with m singularities

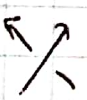
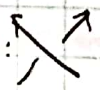
$a_n(L) = 0$ as defined below:

A link L with m crossings that are transversal intersections

like this 

Now define $a_n(L) = \sum_{\epsilon_1, \dots, \epsilon_m} \epsilon_1 \dots \epsilon_m \, a_n(L_{\epsilon_1 \dots \epsilon_m})$

where $\epsilon_i \in \{\pm 1\}$ determine how we resolve the crossing

above. $+1$:  or -1:  In other words

 So this is like derivative and

being of finite type of order $\leq m$ means "taking derivative

m times" gives zero.

• <u>Conjecture</u>: Given a finite type invariant V of order m, then $\exists$ a

Universal constant C s.t. $|V(K)| \leq C\,(\#\text{crossings of } K)^m$

Proved by Bar-Norton "poly invariants of poly".

So implication is that a_n should grow polynomially.

This means V_N is almost like a modular form. At the same time it is a Zeta function. So maybe it's a Mellin Transformation

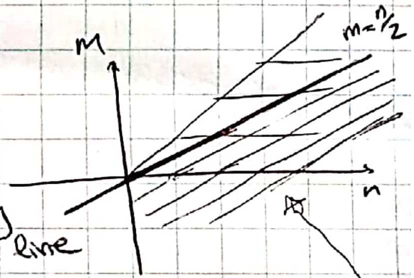
- Let us change notation $N \to d$. Notation depends on d.

Consider approx. of $e^h \sim 1+h$. Calculate

$$V_d(k, 1+h) = \sum_{n=0}^{\infty} \underbrace{v_{n,d}(k)}_{\text{finite type invariants}} h^n$$

- Melvin Morton notices that $v_{n,d}(k)$ is a polynomial function of d which can be written as

$$v_{n,d}(k) = \sum_{0 \le m \le n} D_{m,n}\, d^{2m}$$



They also notices $D_{m,n} = 0$ if $m > n/2$ summing over this line

Now take $\sum_{m=0}^{\infty} D_{m,2m}\, a^{2m}$ formal series with a formal var.

This is equal to $\dfrac{1}{\Delta(k, e^{\pi i a} - e^{-\pi i a})}$ (This is also a zeta function)

But what if we sum other lines $m = 2n+s$ for any $s \le 0$?

The first line it gives $\dfrac{P_2(z^2)}{\Delta^3}$ $\quad z = e^{\pi i a} - e^{-\pi i a}$

The second line gives $\dfrac{P_3(z^2)}{\Delta^5}$ where P_i are polynomial invariants.

Physical interpretation: boson–fermion are 'inverse' of each other.

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boson ~ colored Jones and Alex poly ~ free fermions (by Saleur, Kauffman)

So that is why Alex poly appears in denominator.