Thursday 23rd Jan 2020

- In previous section, we discussed how to obtain a morphism

$$\varphi : \pi_1(S^3 \setminus K) \longrightarrow PSL(2,\mathbb{C})$$

Recall $PSL(2,\mathbb{C})$ acts on $\mathbb{H}^3 = \{(x,y,t) \mid t>0\}$ by using the quaternion representation. $q \in \mathbb{H}^3 \Rightarrow q = x + iy + jt$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{C}) \Rightarrow \gamma(q) = (aq+b)(cq+d)^{-1}$$

Denote $\Gamma = \mathrm{Im}(\varphi)$. Then ρ is discrete if Γ orbits of any $p \in \mathbb{H}^3$ has a discrete Topology, ie $|\Gamma p \cap C| < \infty$ for all compact set $C \subseteq \mathbb{H}^3$.

- For figure eight knot, φ is discrete & faithful ($\ker \varphi = \{id\}$).
- To prove the above, you need an algorithm to get the fundamental domain of φ and then it will be easy to show discreteness. See the reference by Thurston: Three dim manifolds, Kleinian Groups and hyperbolic geometry. (Bulletin AMS)

To see what Thurston did, we will need to decompose the complement to two ideal Tetrahedra.

• Side-discussion on what will come later: Twisted Alexander Poly

• The Alexander poly is purely classical: Take the fundamental group of the knot and abelianize it, **ab** : $\pi_1(S^3\setminus K) \to \dfrac{\pi_1(S^3\setminus K)}{[\pi_1(S^3\setminus K), \pi_1(S^3\setminus K)]}$

• The abelianization is first homology which is $\mathbb{Z}$.

• Take the kernel of **ab**, which is a normal subgroup. By standard Topology you can find a space $\tilde{X}$ which is covering space of $S^3\setminus K$ and has $\pi_1(\tilde{X}) = \ker(ab)$. It is called the universal abelian cover.

• Another standard Topology fact states that $\mathbb{Z}$ in the range acts on $\tilde{X}$. Therefore $H_1(\tilde{X};\mathbb{Z})$ is a $\mathbb{Z}[\mathbb{Z}] \simeq \mathbb{Z}[t^{\pm 1}]$ module where the first $\mathbb{Z}$ are integers and the second is the abelianization. One then observes that $H_1(\tilde{X};\mathbb{Z}) \simeq \dfrac{\mathbb{Z}[\mathbb{Z}]}{\langle A(t)\rangle}$ where $A(t)$ is the Alexander Polynomial.

• Let K be hyperbolic. We have two morphisms

$$\begin{tikzcd} \pi_1(S^3\setminus K) \arrow[r, "P_{ab}"] \arrow[dr, "\rho"'] & \dfrac{\pi_1(S^3\setminus K)}{[\pi_1(S^3\setminus K), \pi_1(S^3\setminus K)]} \\ & PSL(2,\mathbb{C}) \end{tikzcd}$$

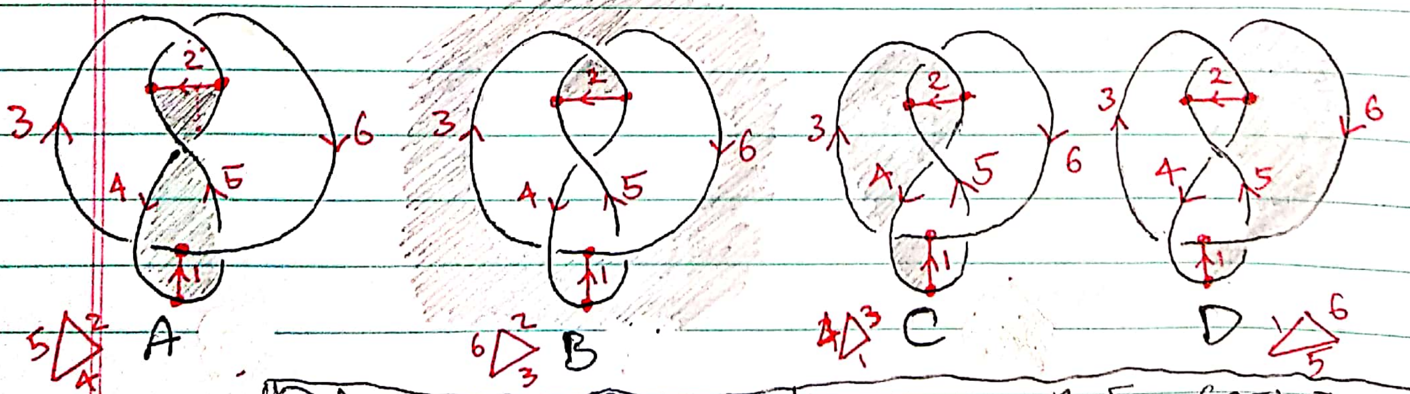
Assume we can lift ρ to $SL(2,\mathbb{C})$.

• Now there is a theorem that $(\text{twisted Alex poly})_{N} \xrightarrow[N \to \infty]{} \text{hyperbolic volume}$.

We need to find the relationship between J_N & $(\text{twisted Alex poly})_N$.

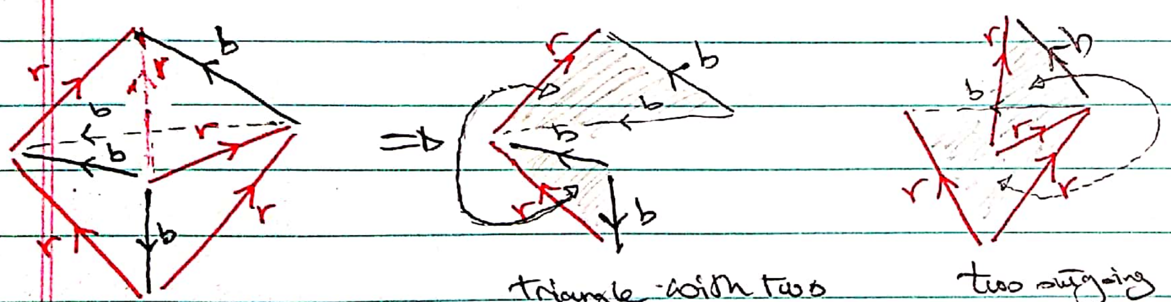
Perhaps finding some quantum version / R matrix from $(\text{twisted Alex poly})_N$ is the way to go.

• We will show how to decompose $S^3 \setminus K = \text{figure } 8$ to two tetrahedra. We shall consider K as being on $\mathbb{R}^2$ (except at the crossings) and the two tetrahedra faces meeting each other on the plane and around the crossings. The final picture for how the faces of the top tetrahedra fit is the following.



Ref.: Alexander Gutierrez (Hyp. geom. on the Figure Eight knot compl.)

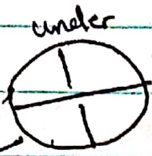
Here, A, B, C, D are the faces of the tetrahedron and the edges are shown by red arrows and numbered from one to six. Notice how two edges are NOT on the strands and instead connect two strands of bigons. This is a general pattern in the decomposition of knot complement to polyhedra, where the edges are the overstrands and the bridge between bigons. The vertices lie on the knot of course. The picture from the bottom for the bottom tetrahedron is very similar. One can track that the identification of edges and faces is similar to the picture below, where one must match faces with the same pattern of edges like the examples shown.

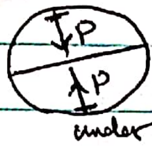


triangle with two outgoing black

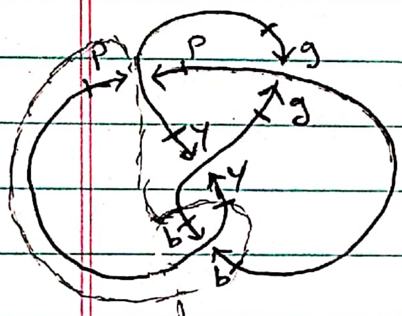
two outgoing red

Now we shall go through the general argument which also works for decomposition to polyhedra for any knot complement.

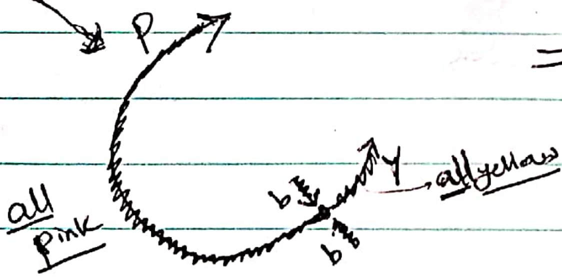
- Take each crossing 

a neighborhood
and color it with two opposite arrows 

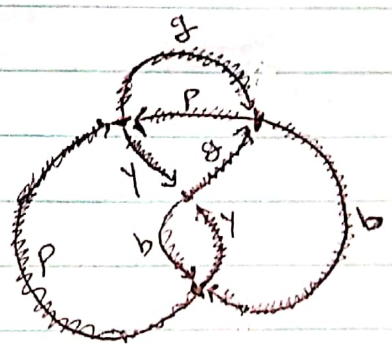
where p: pink. We will use colors y: yellow, g: green, b: blue. We get this picture:



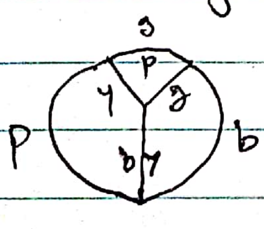
Next you identify the overstrands to a point up to the crossings. This gives:



$\Rightarrow$



The next step is "let bigons be bygones". We shrink the bigons (same as using a bridge in previous pictures). We get the following:



The bottom tetrahedron is similarly built and identified with the top using the colors.