Thursday 27 February 2020

- $\lim_{d\to\infty} \dfrac{\log|V_d(K; e^{\frac{2\pi i}{d}})|}{d} = \lim_{d\to\infty} \dfrac{\log|A_d(K,t)|}{d^2}$ ← Zeta function

Two important Zeta function:

Riemann

$\zeta(s) = \sum_{1}^{\infty} \frac{1}{n^s}$

$Z = q^{-s}$

Weil

X nonsingular projective n-dim variety on $\mathbb{F}_q$

$\zeta_X(z) = \exp\left(\sum_{m=1}^{\infty} \frac{z^m}{m} N_m \right)$

\# pts of X defined over $\mathbb{F}_{q^m}$

★ Rieman Zeta Function ★

1. Functional Equation

$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s)$

2. Euler product: $\sum_{n=1}^{\infty} n^{-s} = \prod_{\text{primes}} (1-p^{-s})^{-1} = \frac{1}{1-p_1^{-s}} \cdot \frac{1}{1-p_2^{-s}} \cdots$

3. Rieman Hypo: Trivial zeros $s = -2, -4, \ldots$ (you have to do analytic extension to see this)

nontrivial zeros: all have $Re(s) = \frac{1}{2}$

★ Weil Zeta Function

1. Rationality (done by Dwork) $\zeta_X(z)$ is rational

$\zeta_X(s) = \prod_{i=0}^{2n} P_i(z)^{(-1)^{i+1}}$ and $P_i(z)$ is a poly

$P_0(z) = 1 - z \quad P_{2n}(z) = 1 - q^n z$

$P_i(z) = \prod_j (1 - \alpha_{ij} z)$ just factorization by the zeros

Deligne proved that α_{ij} satisfy a version of Rieman Hypo

where $|\alpha_{ij}| = q^{\frac{1}{2}}$

2. Functional Equation $\zeta_X(q^{-n}z^{-1}) = \pm q^{n\chi/2} z^{\chi} \zeta_X(z)$ ②

Euler characteristic

Grothendieck built motif cohomology to prove Riemann Hypo for X but Deligne proved it using p-adic cohomology without Grothendieck motif Cohomology.

• The question we are interested is what is the analog of all these for Colored Jones or the (twisted) Alex. poly. ? Most likely related to Weil instead of Riemann.

	$\frac{1}{\Delta_K(t)}$	$V_d(K;q)$
• rationality	✓	✓ → as both are rational already
• Functional Equation	?	Probably Zagier's modularity
• Riemann Hypo	?	?

"B. Mazur has some ideas" in "Knots, Primes & P."

• Zagier's modularity conjecture : K knot ⟷ prime p in $\mathbb{Z}$

Given a knot K, define $J_K : \mathbb{Q}/\mathbb{Z} \longrightarrow J_c(K; e^{2\pi i \frac{a}{c}})$

$\{a/c \mid (a,c)=1\}$

So every knot gives a function on rationals.

But $SL(2,\mathbb{Z})$ acts on $\mathbb{Q}/\mathbb{Z}$ by $x \in \mathbb{Q}/\mathbb{Z}$: $\gamma(x) = \frac{ax+b}{cx+d}$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Conjecture: Define $\hbar = \frac{2\pi i}{x + \frac{d}{c}}$, CS invariant of the hyp structure coming from $\varphi: \pi_1(S^3 \setminus K) \to PSL(2,\mathbb{C})$

$$J_K(\gamma(x)) = J_K(x) \cdot \left(\frac{2\pi}{\hbar}\right)^{3/2} \cdot e^{i(Vol + i\,CS)/\hbar} \cdot [\text{const} + \text{h.o.t}]$$

hyp knot

$x = \frac{a_x}{c_x}$ ‖ $J_{c_x}(K; e^{2\pi i x})$

higher order terms in $\hbar$.

Note This implies Volume Conjecture by choosing $x = N-1$, $\gamma = \begin{pmatrix}1&0\\1&1\end{pmatrix}$.

• For the rest we will try to argue what functional equ. & R-H. are for the $\frac{1}{\Delta_K(t)}$.

• Recall Milnor's Theorem $\frac{1}{\Delta_K(t)}$ is a Weil zeta function.

$K \subset S^3$ $\quad E_K = S^3 \setminus K$, $\tilde{E}_K^{Ab} \leftarrow \pi_1(\tilde{E}^{ab}) = [\pi,\pi]$

$$\begin{array}{ccccc} \mathrm{Ker}\,Ab & \longrightarrow & \pi_1(E_K) & \xrightarrow{Ab} & \mathbb{Z} \\ \cong & & \| & & \\ [\pi,\pi] & & \pi & & \end{array}$$

Then $\mathbb{Z}$ acts on $\tilde{E}_K^{Ab}$ as covering transformations

generator $1 \mapsto t : \tilde{E}_K^{Ab} \circlearrowleft$

$\frac{1}{\Delta_K(t)}$ is called the torsion of a knot.

$\tau(\lambda) = f_0(\lambda) f_1(\lambda)^{-1} f_2(\lambda) f_3(\lambda)^{-1}$ where $f_i(\lambda)$ $\left(= \det(\lambda I - t_*^{(i)})\right)$ is the characteristic poly of $t_* : H_i(\tilde{E}_K^{ab}; \mathbb{Q}) \circlearrowleft$

Then we can setup the 'Weil' zeta function:

$$\zeta_K(z) = \exp\left(\sum_{n=0}^{\infty} \frac{z^n}{n} L(t^m)\right) \quad \text{where } L(f) = \sum_{i=0}^{\infty} (-1)^i \mathrm{Tr} f_{*i}$$

where $f_{*i} : H_i(M;\mathbb{Q}) \circlearrowleft$ for diffeo $f : M \to M$

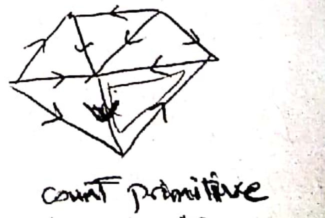
Milnor's thm: $\tau_K(z)\,\zeta_K(z^{-1}) = z^{\chi}$

and $\frac{1}{\Delta_K(\lambda)} = \tau_K(\lambda)$ so $\frac{\zeta_K(z^{-1})}{z^{\chi}} = \Delta_K(z)$

• A. Terras (H. Stark): Garden of zeta functions

Given any finite oriented graph Γ, you can define

$\zeta_\Gamma(z)$: count primitive loops with weight $e^{-s \cdot \text{length}}$



count primitive loops with weight $e^{-s\,\text{length}}$