Thursday 5th March 2020

Zeta function approach to V.C.

	Colored Jones	Twisted Alex Poly
• Rationality	✓	✓
• Functional eq.	Zagier's modularity Conjecture	Symmetric
R.H.	?	? ← R.H. for Ihara Zeta function

Recall Ihara zeta function

G finite connected unoriented graph

$G^+ =$ doubled G $\quad \bullet\!\!-\!\!\bullet \longrightarrow \rightleftarrows$

Zeta function: $\zeta_G(u) = \prod_{\substack{\text{primitive} \\ \text{reduced} \\ \text{cycles } P}} (1-u^{|P|}) \overset{\text{Bass}}{=\!=} \det(I-uT)$

$|P|$ = length P

G regular d-graph: V with d-edges, like $d=4$ for knot diagrams

R.H.: $\zeta_G(u)$ has a R.H. (ie every zero with $0 < Re(s) < 1$, then $Re(s) = \frac{1}{2}$) if G is Ramanujan.

Ramanujan: Connected d-regular for which $-d \leq$ spectrum $\leq d$ with $\lambda_1 = d$. Further $\max_{i=2,\dots,n}(|\lambda_i|) \leq 2\sqrt{d-1}$

(Correspondence is $u = q^{-s}$, $q = d-1$)

Zeta function approach to Melvin-Morton Conjecture (X.S. Liu and Z.M)

Side remark. $A : V \to V$, $\dim V < \infty$. Then ① $\det(e^A) = e^{\operatorname{tr} A}$

② $\det(I - A) = \sum_{i \geq 0} (-1)^i \operatorname{Tr}(\Lambda^i A)$

extend to exterior product $\Lambda^i A : \Lambda^i V \to \Lambda^i V$

Claim $\frac{1}{\det(I-A)}$ should be regarded as a zeta function (we will define A later)

define zeta function of A. $\zeta_A(z) = \exp\left(\sum_{n\geq 1}^{\infty} \frac{z^n}{n} \mathrm{tr}(A^n)\right)$

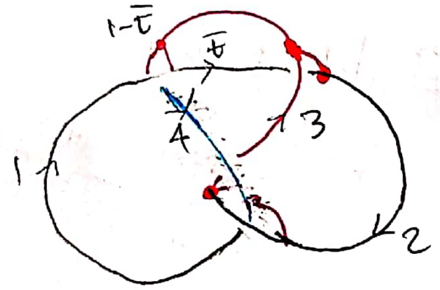
Note that

why? $-\log(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \rightsquigarrow \frac{1}{1-x} = \exp \sum \frac{x^n}{n}$

$\Rightarrow$ So for diagonal matrices it is true that ~~det~~ $\frac{1}{\det(1-zA)} = \exp\left(\sum \frac{z^n}{n} \mathrm{Tr}(A^n)\right)$

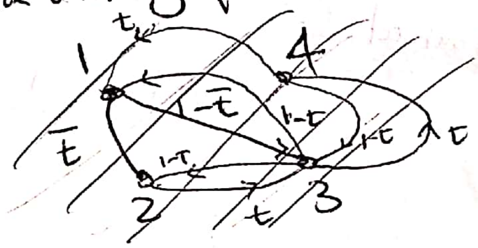
and by Jordan decomposition the rest follows.

Recall Wirtinger presentation for $\pi_1(S^3 \setminus K)$



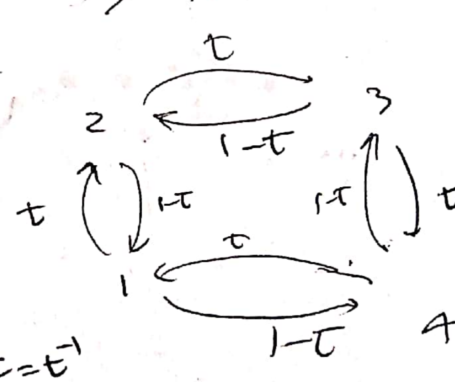
define a walk graph, vertices = arcs colored

W_D



$\tilde{B} =$

	1	2	3	4
1		$\bar{t}$		$1-\bar{t}$
2	$1-t$		t	
3		$1-\bar{t}$		$\bar{t}$
4	t		$1-t$	



?

where $\bar{t} = t^{-1}$

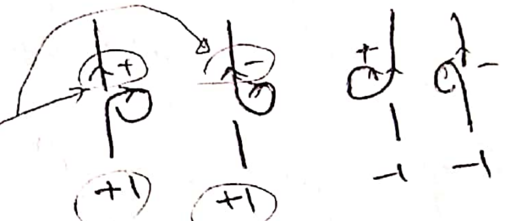
We claim determinant of $I - B_{(ij)}$ = Alex poly where $B_{(ij)}$ is delete i-th row, j-th column of B. For example $I - B_{11} = \begin{pmatrix} 1 & -t & 0 \\ \bar{t}-1 & 1 & -\bar{t} \\ 0 & t-1 & 1 \end{pmatrix}$

gives determinant $= 3 - t - \bar{t} = 1 - z^2$ for $z = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$

What does it count? They should count "closed orbits" count closed geodesics.

delete one row and one column
which corresponds to This →

Knot ⟷ 1-string link



Side remark: For Types of kinks in the plane
we associate +, - based on
the crossing. We also associate
a rotation number

So we have labeling ++, -+, +-, --

We choose the 1 string using ++ kink To connect
and we do some random walk that do not touch the
red part

arXiv 98.12039
Foata, Zeilberger: Combinatorial proof of Bass's evaluation

Thm. Given a 1 string T as above

① $\dfrac{J(K_T)}{q+q^{-1}} = t^{wr(T)-rot(T)}\left(1+\sum_{k=1}^{\infty}\sum_{C=(C_1,\dots,C_k)\in Q_s^k} t^{rot(C)-\beta_1(C)}\, W(C_1)\cdots W(C_k)\right)$

$t=q^{2}$

② $\lim_{d\to\infty}\dfrac{J_{d+1}(K, e^{h/d})}{[d+1]} = t^{\frac{rot(T)}{2}}\left(1+\sum_{k=1}^{\infty}\sum_{(C_1,\dots,C_k)\in Q^k} W(C_1)\cdots W(C_k)\right)$

$= t^{-\frac{rot(T)}{2}} \Big/ \det(I-B)$ ← Alex. poly

- The nontrivial part in the proof is why $\frac{1}{\det(I-B)} = 1 + \sum_{k} \sum_{(c_1 \dots c_k) \in Q^k} w(c_1) \dots w(c_k)$

Need to define Lyndon words: Let A be finite alphabet set totally ordered $A = \{0, 1, 2, \dots, n\}$. Let $A^* =$ all words including $\emptyset$.

Case we are interested in is $A = \{0, 1\}$.

<u>Def.</u> a Lyndon word is a non-empty word which is ① not a power ② <u>minimal</u> in its cyclic class
↳ in lexicographic order

e.g. 0, 1, ~~00~~, 01, ~~10~~, ~~11~~, ~~000~~, 001, ~~010~~, ~~100~~, 011, ~~101~~, ~~110~~, ~~111~~, ...

Theorem: $\beta(\Lambda) = \det(1 - B)$ for any finite matrix B, where $B = (b_{ij})_{i,j=0}^{n-1}$ and b_{ij} as a set of commuting variable. Let $A = \{0, 1, \dots, n-1\}$ with $L_A =$ all Lyndon words in $A^* = \bigsqcup_{n=0}^{\infty} A^n$ and $\Lambda = \prod_{\ell \in L_A} (1 - [\ell])$ is a formal commuting variable.

$[\ell]$ ← variable labeled by $[\ell]$

Finally $\beta(\underbrace{a_1 \dots a_n}_{\text{a Lyndon word}}) = b_{a_1 a_2} b_{a_2 a_3} \dots b_{a_n a_1}$

So $\beta : \mathbb{Z}[\{[\ell]\}_{\ell \in L_A}] \longrightarrow \mathbb{Z}[\{b_{ij}\}]$

↑ infinite generated algebra

e.g. $B = \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix}$ $\det(I - B) = (1-b_{00})(1-b_{11}) - b_{01}b_{10}$ ⑤

$\beta(\Lambda) = (1-b_{00})(1-b_{11})(1 - b_{01}b_{10})(\ldots$

$\beta(0) = b_{00}$ Turns out everything else cancels!

$\beta(1) = b_{11}$

$\beta(01) = b_{01}b_{10}$

• Going back to the Theorem T:

② For Alex. poly
Look for finite simple cycles $= Q$

then $(c_1, \ldots, c_k) = Q^k$

and Q^k's $\in Q^k$ are the ones that do not share any edges. Then apply $\beta(\Lambda) = \det(I - B)$.

① For Jones poly: R-matrix $q^{*}\begin{pmatrix} q & & & \\ & q-\bar{q} & 1 & \\ & 1 & 0 & \\ & & & q \end{pmatrix} = q^{*}q \begin{pmatrix} 1 & & & \\ & 1-q^{-2} & qq^{-2} & \\ & qq^{-2} & 0 & \\ & & & 1 \end{pmatrix}$

$1-t \quad t$