Thursday 6th February 2020

- We want to study knots $K \subset S^3$ and their fundamental group

$$\pi_1(\underbrace{S^3 \setminus K}_{\overset{\shortparallel}{E_K}}, *) = \Pi_K$$ using combinatorial group theory.

$*$: any point in E_K

- In general, a group with finite presentation has presentation $G = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$

where r_i are relators e.g. $G_n = \langle x, y \mid \underbrace{xyx = yxy}, \underbrace{x^{n+1} = y^n} \rangle$

is the Trivial group.

These are relations

relators: $xyx(yxy)^{-1}$, $x^{n+1}y^{-n}$

- The way we construct G is by taking the free group $F(x_1, \dots, x_n)$

and quotient it by the normal subgroup generated by $r_1, \dots, r_m$: $\dfrac{F(x_1, \dots, x_n)}{N(r_1, \dots, r_m)}$

- Proof for G_n = Trivial:

  - $xyx = yxy \underset{\omega = xy}{\Longrightarrow} \omega x \omega^{-1} = y$ and $x = \omega^{-1} y \omega \Rightarrow x^n = \omega^{-1} y^n \omega$ $(\star)$

but y^n commutes with ω: $y^n \omega = y^n xy = x^{n+1} xy = x x^{n+1} y =$

$$x y^n y = x y y^n = \omega y^n$$

Therefore $(\star) \Rightarrow y^n = x^n$ but $y^n = x^{n+1} \Rightarrow x^n = x^{n+1} \Rightarrow x = 1$

and so $x = y = 1 \Rightarrow G_n$ is Trivial.

• In general, There are no algorithms To decide G is Trivial or not.

Now given $G = \langle x_1 \ldots x_n \mid r_1 \ldots r_m \rangle$, construct the group ring $\mathbb{Z}[G]$

defined as : $\{x \mid x = \sum n_g g, \, n_g \in \mathbb{Z}, \, g \in G\}$ the finite

finite sum

formal Sums of elements in G, with addition $x+y = \sum (n_g + m_g) g$

and multiplication $xy = \sum_t \left(\sum_{gh=t} n_g m_h\right) t.$

There is a map (a ring map) called augmentation

$$\alpha : \mathbb{Z}G \longrightarrow \mathbb{Z} \quad : \alpha\left(\sum n_g g\right) = \sum n_g$$

There is also the map $F(x_1, \ldots, x_n) \xrightarrow{\gamma} G = \langle x_1, \ldots x_n \mid r_1, \ldots, r_m \rangle$

$\gamma(x_i) = x_i$

which defines a map also called $\gamma : \mathbb{Z}F(x_1, \ldots x_n) \rightarrow \mathbb{Z}G$

• We want To define The Alexander polynomial as analog of the order of a finite abelian group A :

(Recall discussion in previous section)

$$\mathbb{Z}^n \xrightarrow{A_m} \mathbb{Z}^n \longrightarrow\!\!\!\!\rightarrow A$$

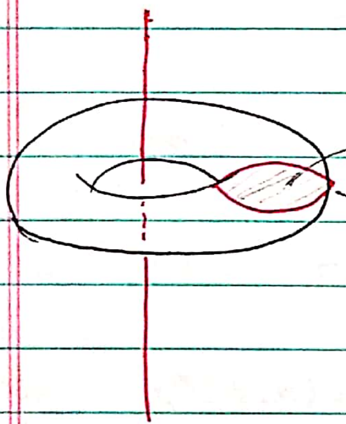
$$A \cong \mathbb{Z}^n / A_m(\mathbb{Z}^n) \cong \bigoplus_{i=1}^{s} \mathbb{Z}/d_i\mathbb{Z} \quad , \; |\det A_m| = \text{order of } A = d_1 \cdots d_s$$

e.g. $\mathbb{Z}^2 \xrightarrow{\begin{pmatrix}2&1\\1&2\end{pmatrix}} \mathbb{Z}^2 \to \mathbb{Z}_3$ (A_m is symmetric & even $a_{ii} \overset{2}{\equiv} 0$).

and $|\det A_3| = |\mathbb{Z}_3| = 3$

$H_1(\tilde{E}_K^{ab}, \mathbb{Z})$ as a Λ-module

universal abelian cover of E_K



To visualize $\tilde{E}_K^{ab}$ for K = unknot.

Seifert surface

cut at here you get:

and you glue them. Deck Transformation is shifting by a unit.

In general take a knot on the boundary torus of the tubular neighborhood, then take its Seifert surface and do the same thing we did for unknot.

- There is a theorem that for any finitely presented Λ-module, you can find

a presentation $\Lambda^n \xrightarrow{A_K} \Lambda^n \twoheadrightarrow H_1(\tilde{E}_K^{ab}, \mathbb{Z})$

then $\Delta_K(t) = \det A_K$ up to $t^{\pm m}$.

- A_K is given by Fox Calculus,

R.H. Fox

- Define Fox derivative $D: \mathbb{Z}F_n \to \mathbb{Z}F_n$ be a ring morphism

such that $D(uv) = Du\,\alpha(v) + u\,D(v)$

1) Trivial example $D = 0$.

2) Let us take $g, h \in F_n$:

$$D(gh) = Dg\,\alpha(h) + g\,D(h)$$

$$\Rightarrow D(gh) = Dg + g\,D(h)$$

• This comes from group Cohomology. Let $H^1(G, M)$, for M a G-module, and 1-cochain $\varphi \in C^1(G, M)$ and its coboundary $\delta\varphi \in C^2(G, M)$ is defined by $\delta\varphi(g, h) = g\varphi(h) - \varphi(gh) + \varphi(g)$

Assume $\delta\varphi = 0 \Rightarrow \varphi(gh) = g\varphi(h) + \varphi(g)$

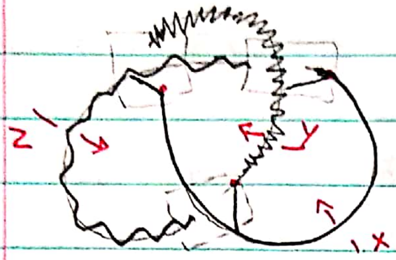
So a Fox derivative is the extension of a 1-cocycle to the group ring $\mathbb{Z}F_n$.

3) $Dg = g - 1$ is a Fox derivative as

$$D(gh) = Dg + g\,D(h) \Leftrightarrow gh - 1 = g - 1 + g(h-1) \checkmark$$

• Going back to Alex. poly. we use Wirtinger presentation for $\pi_1(S^3 \setminus K, *) = \langle x_1, \dots, x_n \mid r_1, \dots, r_n \rangle$ (notice number $\prod r_i = 1$ of relators = number of generators)

Recall how we obtained π_1..



for each overstrand, get a generator, and for each crossing a relation.

And for all loops we get trivial element that is $\prod r_i = 1$.

In case of Trefoil,

$$\langle x, y \mid xyx = yxy \rangle$$

$xyxy^{-1}x^{-1}y^{-1}$ ← relator

Now for Trefoil, $F_2 \longrightarrow \Pi_{\text{Trefoil}} \xrightarrow[\text{abelianization}]{ab} \mathbb{Z}$

The map ab computes the linking $\quad G \longrightarrow G/[G,G]$

number of the representative in Π_{trefoil} with the trefoil knot.

Abelianization for trefoil implies $\begin{cases} xy = yx & \leftarrow \text{abelianize} \\ xyx = yxy \end{cases} \Rightarrow \underline{x = y}$

So that is why we get $\mathbb{Z}$.

Extend the maps to group rings

$$\mathbb{Z}[F_2] \xrightarrow{\gamma} \mathbb{Z}\Pi_{\text{Trefoil}} \xrightarrow{\alpha} \mathbb{Z}[\mathbb{Z}]$$

Now define the Alexander matrix $A_K = (a_{ij})$

$$a_{ij} = (\alpha \circ \gamma)\frac{\partial r_i}{\partial x_j} \in \mathbb{Z}[t^{\pm}] = \Lambda$$

where $\frac{\partial}{\partial x_i}(x_j) := \delta_{ij}$ and $\frac{\partial}{\partial x_i} : \mathbb{Z}F_n \to \mathbb{Z}F_n$ is a Fox derivative.

The determinant of A_K will be zero because we deleted one relator (so it is NOT a square matrix). Dropping one column would make it square and we get the Alexander poly.

- For example for Trefoil: $A_K = \left(\frac{\partial r}{\partial x} \quad \frac{\partial r}{\partial y}\right)$, $\Delta_K = t-1+t^{-1}$

where $r = xyxy^{-1}x^{-1}y^{-1}$. We will compute $\frac{\partial r}{\partial x}$.

- Properties of Fox Calculus:

① $D(r_1 r_2^{-1}) = Dr_1 - r_1 r_2^{-1} D r_2$ for $r_1, r_2 \in F_n$.

proof: $\underbrace{Dr_1 + r_1 D r_2^{-1}}$ and $D(r_2 r_2^{-1}) = 0 \Rightarrow Dr_2 + r_2 D r_2^{-1} = 0$

$\Rightarrow Dr_2^{-1} = -r_2^{-1} D r_2$

which implies $\overset{\shortparallel}{\rightarrow}$ $Dr_1 - r_1 r_2^{-1} D r_2$

But if we take a relator defined as $r = r_1 r_2^{-1}$ then

since $r=1$ in the group, $Dr = Dr_1 - Dr_2$.

② $\frac{\partial}{\partial x} x^2 = \frac{\partial}{\partial x} x \cdot 1 + x \cdot 1 = 1 + x$

③ $\frac{\partial}{\partial x}(w_1(x) \cdot w_2(y)) = \frac{\partial}{\partial x} w_1(x) \cdot 1 + 0$

$\frac{\partial}{\partial x}(w_1(x) \cdot w_2(y)) = 0 + w_2(y) \frac{\partial}{\partial x} w_1(x)$

where $w_1(x), w_2(y)$ are words in x, y.

Now let us take: $\frac{\partial}{\partial x}(r_1 r_2^{-1}) = \frac{\partial}{\partial x} r_1 - \frac{\partial}{\partial x} r_2$

$r_1 = xyx, r_2 = yxy$

$$= \frac{\partial}{\partial x}(xy)\,1 + xy\,1 - \frac{\partial}{\partial x}(yx).\,1 - 0$$

$$= 1 + 0 + xy - y$$

which after abelianization $\xrightarrow[\substack{x \to t \\ y \to t}]{a} 1 + t^2 - t = t\underbrace{(t - 1 + t^{-1})}_{\text{Alex. poly up to } t^{\pm1}}$

And This is det A_K when we drop second column.

$$A_K = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \end{pmatrix}$$

You can check that $\frac{\partial r}{\partial y}$ also gives the same thing up to

$$\frac{\partial r_1 r_2^{-1}}{\partial y} = -(1 + yx - x) = -(1 + t^2 - t)$$

a sign (which depends on which column we take out in general).

- $$\begin{tikzcd} \Pi_K \arrow[r, "a"] \arrow[dr, "\delta"'] & \mathbb{Z} \\ & GL(n, \mathbb{C}) \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathbb{Z}\Pi_K \arrow[r, "a"] \arrow[dr, "\delta"'] & \mathbb{Z}[t^{\pm1}] \\ & M_n(\mathbb{C}) \end{tikzcd}$$

$$\Rightarrow \mathbb{Z}\Pi_K \xrightarrow{\rho \otimes a} M_n(\mathbb{C}[t^{\pm1}])$$

This is what you do for twisted, where the same construction (taking determinant) applies but every $t^{\pm1}$ is blown up to a matrix.