Thursday 9th January, 2020

The class will have no homework or exam but we will leave some exercise; Office hours only occasionally Friday 3-4pm or by appointment (only if there is a geometry topology seminar SH 6713).

The topics discussed in this class will be centered around the famous *volume conjecture*.

References that will be used along the way:

- $\frac{1}{3}$ of class : Murakami and Yokota : Volume conjecture for knots. I will not follow the second half of the book because I believe there is a more promising approach.
- $\frac{1}{3}$ of class from two books: The first is by Thurston: geometry and topology of 3 manifolds, which you can find here. I will talk about just one or two chapters. The second book is by Jessica Purcell: hyperbolic knot theory which you can find here. This explains one chapter of the previous book and is much more readable.
- The last third of the class, I will figure it out! We will likely discuss proofs of some special cases for the volume conjecture.

To introduce volume conjecture, it is useful to have a general idea of what worlds this conjecture is trying to connect. There are two worlds of low dimensional topology (dim ≤ 4). There is a quantum world and a classical world. It is of great interest to see how the two worlds are related. Normally people on each side do not talk to each other much. Classical is more or less geometry and topology (homotopy theory, homology, etc.) while quantum is more or less algebraic (Quantum Field Theory, and the stuff you hear about recently).

A very special case of the relation is in the study of knots. We have quantum invariants discovered more recently, and classical topological invariants which are historically older. The volume conj is the most pronounced relationship between these invariants. It relates the quantum invariant which is called the Jones polynomial and the classical invariant which is called the hyperbolic volume of the complement of a hyperbolic knot in S^3. One can generalize this invariant to the Gromov norm, which also works for non-hyperbolic knots, i.e. knots which complement in S^3 is not a hyperbolic manifold.

The history of the volume conjecture starts at roughly 1987, by E. Witten; He wrote a paper on exactly solvable 3d gravity and on the last paragraph he mentioned that if his thinking is correct, then there should be some relation between the Gromov norm and quantum invariants on the knots.

The next important work is that of R. Kashaev, where he defined something called Kashaev invariant of knots. He then formulated a precise volume conjecture which he verified for the figure 8 knot.

This was followed by two works by Murakami. In one the Kashaev invariant depending on N was formulated, which turns out to be the colored Jones polynomial evaluated at some root of unity $q = e^{2\pi i/N}$.

There are few knots for which this conjecture can be verified. Some have been done by numerical computation so you can check it.

If you explore the literature, you will see some superficial connection between the two subjects which might make you think it is an easy conjecture, but this is really not the case!

One difficulty of this conjecture is that while the Gromov norm is easy to calculate (a package called *snap* can be used to calculate it), the colored Jones polynomial can only be computed efficiently on a quantum computer. So perhaps the advent of quantum computers and the numerical simulations that will follow, could help us gain more insight.

Side discussion: It is unknown whether Jones polynomial is a complete invariant or not. In fact we do not know if it can detect even the unknot! There are invariants like Khovanov homology that can detect the unknot (proven in the previous decade).

We want to understand classically what colored Jones polynomial means; more precisely, what classical information can be obtained from the sequence of $N-$colored Jones polynomial of a knot. Any such classical information is a good theorem!

My plan is to explain the colored Jones polynomial in two different ways. I will give today the useless but most elementary definition. Then I will define it using Yang Baxter equation.

Definition 1 *A knot K is a smooth embedding of the circle S^1 into S^3 or $\mathbb{R}^3$ up to isotopy.*

We will always assume a knot is oriented. There are four flavors of orientations. as a knot is in S^3 which itself has the usual $\pm$ orientation, and the knot itself which has also two possible arrows on it. The orientation on S^3 determines the overcrossing or under crossing and the knot arrow helps to compute the sign of the over/undercrossing (used in computing the linking number for example).

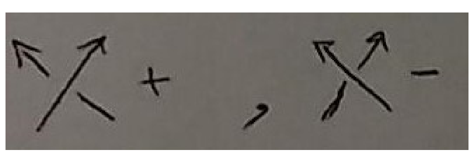

Figure 1: Orientation

The most powerful invariant is of course the complement $S^3 \backslash K$. This is actually a deep theorem that this is a complete invariant and determines the knot uniquely.

Like mentioned previously, classical invariants are the ones coming from homology/homotopy of the knot complement. While quantum invariants usually come from quantum physics and partition functions (this is all a rough classification so do not take it too seriously).

We define next the colored Jones polynomial of oriented links L (put an arrow on each component).

Each component of L is associated with a positive integer N. This is the color of the component. N also references the dimension of irreducible representation of $\mathfrak{su}(2)$.

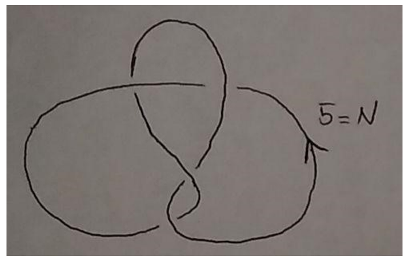

Figure 2: Colored figure eight knot

Normally I would write $L = \cup_i L_i$ with integer c_i attached to L_i. I may not be consistent with my notation throughout the quarter.

Side-discussion: There are speculations on the version of volume conjecture where the colors correspond to the irreps of $\mathfrak{su}(n)$. It is also conjectured instead that by taking can HOMFLY polynomial (a generalization of Jones polynomial) one will get more than the volume on the classical side.

The colored Jones polynomial of colored link (L, c) is a Laurent polynomial $J(L, c; q) \in \mathbb{Z}[q^{\pm 1/2}]$ with variable $q^{\pm 1/2}$ and $q \in \mathbb{C}^* = \mathbb{C} - \{0\}$. We will be interested the most in (K, N), giving the polynomial $J_N(K, q)$. Though we will repeatedly call it a polynomial, note this is not a polynomial.

To make the calculation of Jones polynomial easier, we need to introduce quantum *integers* for $q \in \mathbb{C}^*$:

$$[n]_q := \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}$$

There are different conventions and once has to be careful. Sometimes the 1/2 is forgotten.

If we do l'Hospital's rule and take $\to 1$, this gives us n. This corresponds physically to taking the famous Planck constant $\hbar$ to zero as $q = e^{\alpha\hbar}$ where α is some coefficient. So $q \to 1$ corresponds to going from quantum to classical. Mathematically, once can view $[n]_q$ as a $q-$*deformation* of integers.

Show the following as an exercise:

$$[n+1] + [n-1] = [2][n]$$

Note

$$[2] = q^{1/2} + q^{-1/2}$$

Every expression of quantum numbers ultimately becomes a polynomial of $[2]$ and $[1] = 1$, if one uses the above simple identities recursively.

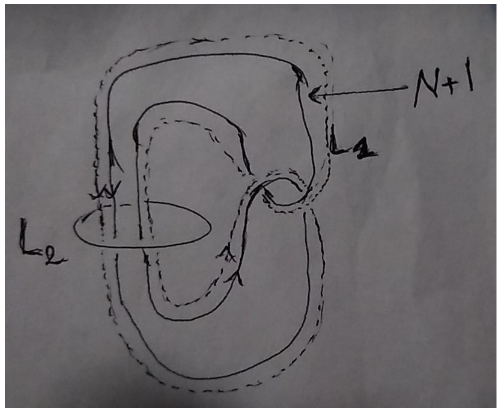

Figure 3: Whitehead link

We would also like to use this relation to define $J(L, c; q)$ recursively.

To have a complete definition we need to first define $J(L, c; q)$ for the base case which corresponds to the coloring by [2] for all components. Of course if any component has color [1], we can safely ignore it:

$$J(L, c_1 \cup \ldots \cup c_n; q) = J(L', c'_1 \cup \ldots \cup c'_n; q)$$

where L' is obtained by dropping all components colored by 1. Physically this corresponds to the vacuum sector which amplitude is always one.

For the nontrivial base case, we define:

$$J(L, 2 \cup \ldots \cup 2; q) := J(L; q)$$

where $J(L; q)$ is the Jones polynomial of L, which will be defined later. Using the exercise above, we can define:

$$J(L = L_1 \cup \ldots, (N+1) \cup \ldots; q) = J() - J(L_1 \cup ..., (N-1) \cup \ldots; q)$$

where $J()$ is corresponding to $[2][N]$:

$$J(L_1^{(2)} \cup \ldots, N \cup 2 \cup \ldots; q)$$

where $L_1^{(2)}$ has two components with color $N, 2$.

The term color goes back to doubling or tripling the knot. So recursively, this says the *colored* Jones polynomial is some linear combination of Jones polynomials of links where we have multiplied the knot N times as shown below for the whitehead link:

Essentially, one considers N parallel running copy of the knots. The way these parallel copies are drawn is by using *0-framing* **push-off** of the knots. The way you produce the **push-off** is by walking along the knot diagram, holding out your right hand, and drawing a parallel knot. The linking number between the knot and this push-off is concentrated at the crossings of the original knot. A framing is a trivialization of the normal vector bundle, up to isotopy. Equivalently, it is a choice of normal vector field along the knot, up to isotopy. The framing is completely characterized by a single integer, the linking number between the knot and a push-off along the chosen normal vector field. There is a special framing (the *0-framing*) given by a Seifert surface of the knot: the neighborhood of the boundary of the surface gives a normal vector field, and the linking number of the push-off with the knot is zero.

Now let me define the Jones polynomial to complete the definition. Using the skein relation:

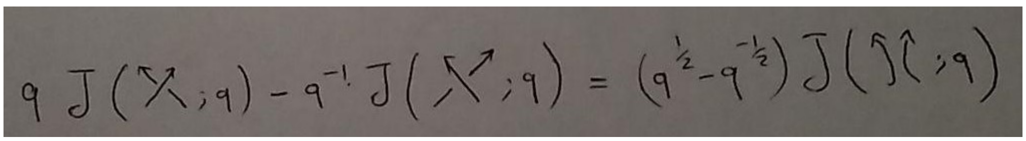

Figure 4: Skein relation

Hence, Jones polynomial of oriented links $J(L;q) \in \mathbb{Z}[q^{\pm 1/2}]$ is defined by

- $J(unknot; q) = [2]$. Sometimes you may have seen the convention that this is one.
- Use skein relation to recursively resolve crossings and get to unknot.

Let us calculate the Jones polynomial of figure eight:

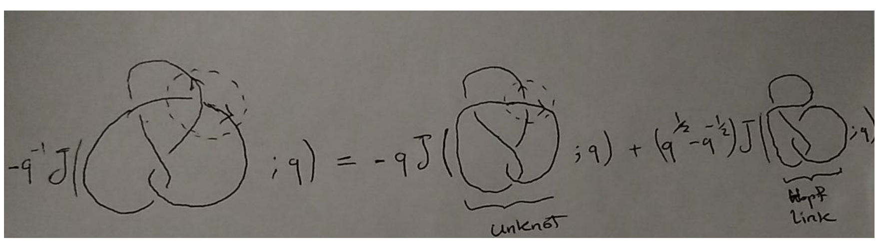

Figure 5: Figure eight Jones polynomial

As an exercise, try to finish the above calculation by computing the Jones polynomial of the Hopf link. You can also find the Jones polynomial of the figure eight knot on its wikipedia page.

Thus taking any crossing of L, which its alternated and resolved version can make the link simpler (there is *always* such a crossing as long as the link is not a collection of unknots), you always get to a place where you have to calculate the Jones polynomial of a simpler knot. But how do we know it is consistent and we get the same answer no matter which crossings we choose to apply the skein relation to? This is actually a (not easy) theorem.

Many significant classes of knots have their closed formula for Jones polynomial found. Now let us discuss the other side of the Volume conjecture which has to do with the Hyperbolic volume. First

Theorem 1 *(Reiley but rediscovered by Thurston) There exists a Riemannian metric on* $S^3 \backslash K$ *where* $K =$ *the figure eight, with sectional curvature* $= -1$*.*

Thurston's idea was to see the noncompact three manifold $S^3 \backslash K$ as a gluing of two tetrahedrons. For a full reference on polyhedral decomposition of any knot, starting with figure eight, we refer to chapter 2 of Jessica Purcell's book in References. More details are also provided in future sections. If one knows that there is a polyhedral decomposition of the complement, it is not hard to see why figure eight gives tetrahedron decomposition, as it divides the plane into 6 regions, number of tetrahedron faces.

The volume conjecture is:

Conjecture 1 *If* K *is hyperbolic, then*

$$\lim_{N \to \infty} \frac{\log |\frac{J_N(K;e^q)}{[N]_q}|}{N} = \frac{Vol(S^3 - K)}{2\pi} \tag{1}$$

where $q = e^{\frac{2\pi i}{N}}$ *and* $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$.