Tuesday 10th March 2020

- No lecture on Thursday

1. 4-dim Topology (3+1)-D TQFT
2. VC: Configuration approach

Top interpretation of Jones poly

Braid groups : fundamental group of n points in D^2

n distinct pts, unordered

- define Config. space

$$C^n(X) = (D^2)^n \setminus \Delta$$

where $\Delta = \{ (x_i) \mid x_i = x_j, \exists\, i \neq j \}$

it is well-known $\pi_1(C^n(D^2), *) \cong B_n$

- $\pi_n M$ of a manifold M for $n = 1,2,3,4$ is a homotopy invariant of M so not that useful.
- Therefore we replace $\pi_n M$ by $C^n(M)$ which is not a topological invariant. In general $X_1 \overset{hom}{\simeq} X_2 \not\Rightarrow C^n(X_1) \overset{hom}{\simeq} C^n(X_2)$

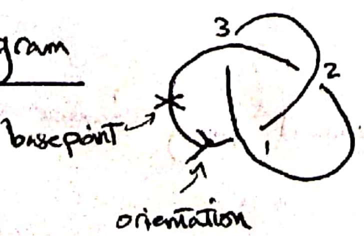
$pt \simeq D^2$ and $C^n(\underset{\simeq\, pt}{pt}) \not\simeq C^n(D^2)$

Volume Conjecture: For a hyperbolic knot $K \subset S^3$, Colored Jones Poly $J_d'(K;q)$ (normalized J_d)

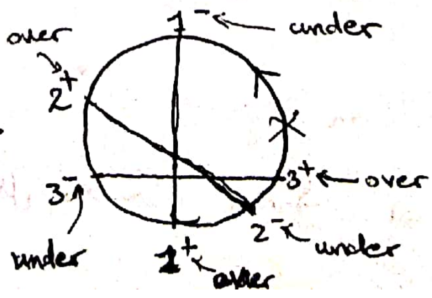
then
$$\lim_{d\to\infty} \frac{\log|J_d'(K; e^{2\pi i/d})|}{d} = \frac{1}{2\pi}\mathrm{Vol}(S^3\setminus K)$$

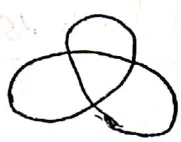
Some kind of zeta functions of graph

Knot diagram



$\xleftrightarrow[\text{diagram}]{\text{Gauss}}$

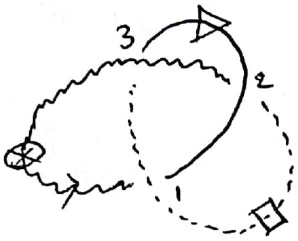


Two graphs: 1) U_D :  The Universe graph

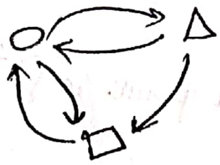
2) Walk graphs $W_D^{\pm}$: + chose all + pts in Gauss diagram

− " " − " " ,

than turn each over arc into a vertex



From each vertex v two edges.

$\xrightarrow{W_D^+}$ 

The reason to introduce these: statistical mechanics on knot diagrams/

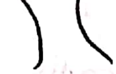
State-sum construction.

① Kauffman ⟶ a state on a knot diagram is an assignment of A and B to each crossing ⟶ $2^{\#}$ every states

② Turner

For every state s, define $\langle S\rangle = \sum A^{\# \text{ of A's in S}} B^{\# \text{ of B in the states}}$ $(d^{\# \text{edges}})$

Given a crossing resolve all crossing

 A $\xrightarrow[\text{rotate counter clockwise sweep out 2 of 4 regions}]{\text{start with over crossing}}$ 

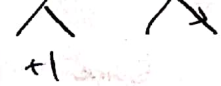
Jones poly

define $J(K_D, A) = (-A)^{-3w(K_D)} \sum_s \langle s\rangle$

writhe index

replacing

Similarly resolve (clockwise)  B ⟶ 



+1

$B = A^{-1}$

$d = -A^2 - A^{-2}$

$q = A^{-4}$ (or A^{-2})

② Turaev State sum

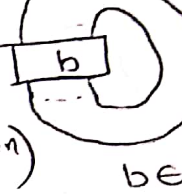
to get Quant invariant find (R, α, β, μ) ⟵ enhanced Yang-Baxter operator"

	00	01	10	11
00	$-q$			
01		$\bar{q}-q$		
10		1	1	
11				$-q$

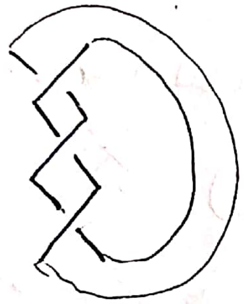
$R =$ (table above) $: (\mathbb{C}^2)^{\otimes 2} \circlearrowleft$

basis $|i\rangle|j\rangle$ $i,j \in \{0,1\}$

$\bar{q} = q^{-1}$, $\alpha = -q^2$, $\beta = 1$, $\mu = \begin{pmatrix} \bar{q} & 0 \\ 0 & q \end{pmatrix}$

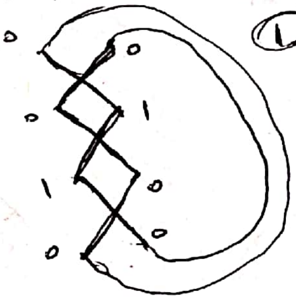
__Theorem__: $J(K,q) = \alpha^{-w(D)} \beta^{-n} \, \mathrm{Tr}(\varphi_R(b) \cdot \mu^{\otimes n})$ $K =$ 

$= (-q^2)^{-w(D)} \, \mathrm{Tr}(\varphi_R(b) \mu^{\otimes n})$ $b \in B_n$

Trefoil:  $b \in B_2$ $= \sigma_1^3$

$\Longrightarrow$ __Always leads to a state sum__

Take any knot diagram, a state on D is an assignment of $\underline{0}$ or $\underline{1}$ to each edge of U_D



① Weight of a state $\pi(S) = \prod_{\text{all vertices}} (R^{\pm})^{c,d}_{a,b}$

$v =$ crossing with a, b below and c, d above, $\pm$ crossing

② a state is admissible if $\pi(S) \neq 0$

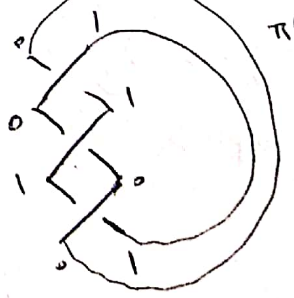
__Thm__: $J(K,q) = (-q^2)^{-w(K_D)} \sum_{\substack{\text{all admissible} \\ \text{states } S}} \underline{q^{\mathrm{rot}(\bar{S}) - \mathrm{rot}(S)}} \, \pi(S)$ defined later

looking at nonzero R entries to see admissability, __Examples:__

These two must be the same

$\pi(S) = (\bar{q}-q)^3$

$\pi(S) = (\bar{q}-q)^2 \cdot 1$



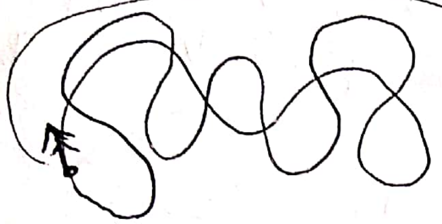
$\pi(S) = (-q)^3$

Note There are $2^6 = 64$ possibilities but admissability dramatically drops This number.

Rotation number rot(S) invented by whitney

draw the immersed curve which is in general position (not having any $\ast$)

The Tangent vector

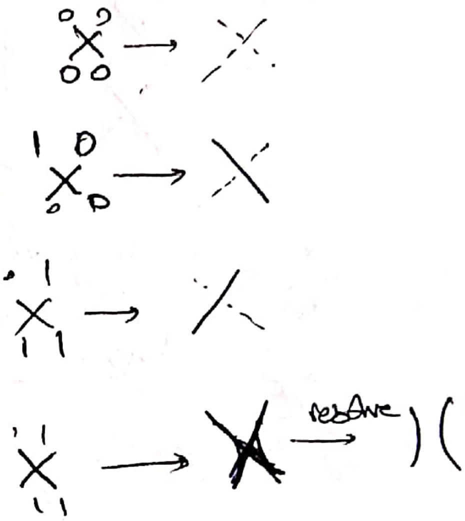


map it to a circle $\longrightarrow S^1$ and look how many times you loop around circle. Called the rotation number

$q^{rot(\bar{S}) - rot(S)}$

Complement of S: $0 \leftrightarrow 1$, $1 \leftrightarrow 0$

Now take S and replace 0 labeled arcs by ⋰ and 1 labeled ones by \ , e.g.



Then you follow only \ (after resolving) arcs along the knot. If

- it's clockwise -1
- anti " $+1$

$\Rightarrow$ Jones poly counts disjoint cycles of random walks which essentially implies the MM conjecture.