Tuesday 11th February 2020 (D)

- Alexander Poly. of knots. Recall notations $K \subset S^3$, $E_K = S^3 \setminus K$ knot complement

$\Pi_K = \pi_1(S^3\setminus K, *)$, abelianization $\Pi_K \xrightarrow{a} \mathbb{Z}$

$x=y=\tau$ $\quad$ $a=\tau^2, b=\tau^3$ or $\tau = a^{-1}b$

e.g. $\Pi_{\text{Trefoil}} = \langle x\, y \mid xyx = yxy \rangle = \langle a, b \mid a^3 = b^2 \rangle$

(abelianization $x=y$) $\quad$ (abelianization $ab = ba$)

- Wada's version. Let G be a grp with presentation $G = \langle x_1, \dots, x_n \mid r_1, \dots, r_{n-1} \rangle$

This is called a deficiency $\underline{1}$ presentation due to $\underline{n-1}$ relators & $\underline{n}$ generators.

$$\downarrow a$$

$$\mathbb{Z} \cong G/[G,G]$$

- Fox Calculus F_n - free grp with $\underline{n}$ gen. $x_1, \dots, x_n$

Fox derivative: a $D : \mathbb{Z}F_n \longrightarrow \mathbb{Z}F_n$ (group ring) $\mathbb{Z}G = \{ \sum n_g g \mid g \in G, n_g \in \mathbb{Z} \}$

s.t. $\forall x, y \in F_n$ $\quad D(xy) = Dx + x \cdot Dy$

e.g. $\partial/\partial x_j$ is a Fox deriv where $\partial/\partial x_j (x_i) = \delta_{ij}$

- Let G be a grp with a presentation $G = \langle x_1, \dots, x_n \mid r_1, \dots, r_{n-1} \rangle$

$$\downarrow \alpha \text{ (the same old } a\text{)}$$

$G = F_n / $ Normal subgrp gen. by $r_1, \dots, r_{n-1}$ $\qquad \mathbb{Z} = G/[G,G]$

Then $\mathbb{Z}F_n \xrightarrow{\gamma} \mathbb{Z}G \xrightarrow{\alpha} \mathbb{Z}[\mathbb{Z}] = \Lambda = \mathbb{Z}[\tau^{\pm 1}]$

Def. of Alex. poly of a finitely presented grp G with abelianization $\mathbb{Z}$

$$\text{Alexander matrix} := \left(\alpha\cdot\gamma\left(\frac{\partial r_i}{\partial x_j}\right)\right) \text{ of size } (n-1)\times n$$

Delete a column (the ℓ-th column) to get $(n-1)\times(n-1)$ matrix $A_G^{(\ell)}$

Then $$\Delta_G(t) = \left[\frac{\det A_G^{(\ell)}}{\alpha(x_\ell)-1}\right](1-t)$$

• It does NOT depend on the presentation (highly nontrivial). Let us check this: $\pi_{\text{trefoil}} = \langle x,y \mid xyx\,y^{-1}x^{-1}y^{-1}\rangle = \langle a,b \mid a^3b^{-2}\rangle$

• Recall lemma: If you have relator $r = r_1r_2^{-1}$ Then $D(r_1r_2^{-1}) = Dr_1 - Dr_2$ for any fox derivative in $\mathbb{Z}G$. Proof: last section.

Application: • $\frac{\partial}{\partial x}(xyx - yxy) = 1 + x\frac{\partial}{\partial x}(yx) - y\frac{\partial}{\partial x}(xy)$

$= 1 + xy - y(1 + x\cdot 0) = 1 + xy - y \xrightarrow{\alpha} 1 - t + t^2 = -\frac{1-t+t^2}{t-1}(1-t)$

• $\frac{\partial}{\partial a}(a^3 - b^2) = 1 + a + a^2$ $\left(\begin{array}{l}\text{general rule:}\\ \frac{\partial}{\partial x}x^n = 1 + x + \cdots + x^{n-1}\end{array}\right)$

$\xrightarrow{\alpha} \frac{1+t^2+t^4}{t^3-1}(1-t)$ (which is equal to $-\frac{1-t+t^2}{t-1}(t-1)$ up to some power of t)

2 We want to show independence of Alex-poly with respect to presentation.

Tietze Thm: If $\langle x_1,\dots,x_n \mid r_1,\dots,r_m\rangle$ and $\langle y_1,\dots,y_s \mid R_1,\dots,R_t\rangle$

are presentation of the same group. Then they are related by the following moves. ("you can always rename")

1. Change a relator $r_i \to r_i^{-1}$
2. Change r_i to $r_i, r_i r_j$ for any $i \neq j$ (add $r_i r_j$ $\forall j \neq i$)
3. add $w r_i w^{-1}$ for any $w \in F_n$
4. add a new generator X and a new relator X.

- Andrew - Curtis Conjecture (Supposed to be wrong!)

★ Given any $\underbrace{\text{balanced}}_{\text{number of relators = number of generators}}$ presentation of the Trivial group

Then it can be reduced to the Trivial representation $\langle x_1, \dots, x_n \mid x_1, \dots, x_n \rangle$

using ① ③ ④ and ②' ★

it's ② modified where you delete r_i after adding $r_i r_j$ (to keep balance).

e.g. $\langle xy \mid xyx = yxy, x^{n+1} = y^n \rangle$

- For $n \geq 4$ gen. it is unknown, and believed to be wrong.
- There are examples where as a lower bound there needs to be at least $2^{2^{\cdot^{\cdot^{2}}}}\}\log_2 n$ many moves.

- Using Tietze's Thm, we can show the Alex. poly. is indep. of presentation as Δ_G is invariant under those moves.

- Twisted Alex. poly. Wada's version:

Let $G = \langle x_1, \dots, x_n \mid r_1, \dots, r_{n-1} \rangle \xrightarrow{\alpha} \mathbb{Z}$ and
$\varphi : G \longrightarrow GL(N, \mathbb{C})$

This time define $\mathbb{Z}F_n \xrightarrow{\gamma} \mathbb{Z}G \xrightarrow{\alpha} M_N(\mathbb{Z}[t^{\pm 1}]) = M_N(\Lambda)$

and take $A_{G,\varphi} = \left[(\alpha \otimes \varphi)\, \gamma \frac{\partial r_i}{\partial x_j} \right]$ as the twisted Alexander matrix

Example: $(\alpha \otimes \varphi)(1 + xy - y) \underset{\substack{x \to t \\ y \to t}}{=} 1 + t^2 \varphi(xy) - t\,\varphi(y)$

Then twisted Alex. poly. is obtained by deleting ℓ-th column and taking:

$$\Delta_{G,\varphi}(t) = \frac{\det(A^{(\ell)}_{G,\varphi})}{\det((\alpha \otimes \varphi)(x_\ell) - 1)} (1-t)$$

Example.

- Compute twisted Alex. poly. of figure 8: $\Pi_K = \langle x, y \mid wx = yw \rangle$
$w = xy^{-1}x^{-1}y = [x, y^{-1}]$

where we choose $\varphi :\ x \longrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (which is the rep. coming
$y \longrightarrow \begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix}$ from the hyperbolic structure)
where $\omega = e^{2\pi i/3}$.

Recall φ is faithful and discrete and $\Gamma = \varphi(\Pi_K)$ satisfies
$\mathbb{H}^3/\Gamma \cong S^3 \setminus \infty$.

Need to compute: $\frac{\partial}{\partial x}(wx - yw) = 1 - xy^{-1}x^{-1} + w - y + yxy^{-1}x^{-1}$

$$\xrightarrow[\substack{x \to T \\ y \to t}]{(\alpha \circ \varphi)\gamma} I - t^{-1}\varphi(xy^{-1}x^{-1}) + \varphi(w) - t\varphi(y) + \varphi(yxy^{-1}x^{-1})$$

Taking its determinant gives $\underline{t^{-2} - 6t^{-1} + 10 - 6t + t^2}$ and multiplied $(1-t)$

which should be normalized by $\det((\alpha \circ \varphi)(y) - 1) = (t-1)^2$

giving $\overset{?}{\pm} t^{-2}(t^2 - 4t + 1)$

- Given any hyperbolic knot $K \subset S^3$ with the corresponding

$$\underbrace{\varphi : \pi_1(S^3 \setminus K) \to PSL(2,\mathbb{C}) \leftarrow SL(2,\mathbb{C}) \xrightarrow{Sym^{N-1}} SL(N,\mathbb{C})}_{\varphi_N}$$

Then take the twisted Alex. poly. Δ_{K,φ_N}

- The Theorem is: $\forall \xi \in S^1$.

$$\lim_{N \to \infty} \frac{\log |\Delta_{K,\varphi_N}(\xi)|}{N^2} = \frac{1}{4\pi} \operatorname{Vol}(S^3 \setminus K)$$

How the $\begin{cases} \text{Jones colored } J_N(K;q) \\ \text{twisted Alex. } \Delta_{K,\varphi_N} \end{cases}$ are related?

① Both are zeta functions, an <u>Ihara</u> zeta function.

② They are both intersection pairing