Tuesday February 23th 2020

Paper recommendation: B. Mazur: Knots, primes, and P_0

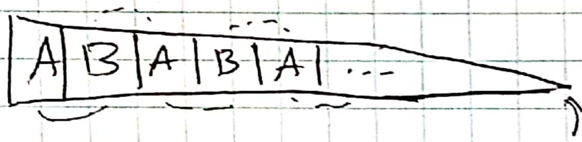
his "propaganda": hyp knots $K \subset S^3$ $\longleftrightarrow$ primes # $p\mathbb{Z} \subset \mathbb{Z}$

$$\mathrm{Vol}(S^3 \setminus K) \longleftrightarrow \log p$$

① Mazur manifold: compact $M^4 \underset{\text{contractible to}}{\simeq} *$ but it is not D^4.

② He also proved nontrivial knots have no inverse by 'Swindle' argument

$$A \# B = \text{unknot} \implies A \text{ unknot}$$



= unknot

you have a "singularity" here which you can resolve.

③ Twisted Alex polynomial, Colored Jones, Riemann Zeta, Weil Zeta, zeta function

- Weil $\Rightarrow$ Artin-Mazur zeta function

diffeo $f: M^n \to M^n$ then define $\zeta_f(z) = \exp\left(\sum_{n=0}^{\infty} \frac{z^n}{n}(\#\text{fixed pts of } f^n)\right)$

① ζ_f is a rational function

② Functional equation something like "$\zeta(s) = \Gamma(s)\zeta(1-s)$" is a Poincare duality, proven by Grothendieck

Gamma function

③ Product equation $\sum \frac{1}{n^s} = \prod_{\text{primes}} (1-p^{-s})$ proven by Deligne

Milnor: proved why $\zeta_f = \frac{1}{\Delta(K,t)}$.

Melvin-Morton Conj: "colored Jones is zeta function"

D. Zagier: (Quant Modular Forms

Melvin-Morton-Rozansky expansion:

Recall given $K \subset S^3$ $J_d(K;q)$: $J_d(\text{unknot}) = [d]$

$$V_d = \frac{J_d}{[d]}, \quad V_d(\text{unknot}) = 1$$

$$V_d(K, q=1+h) = \sum_{n\geq 0}\Big(\sum_{0\leq m\leq 2n} D_{m,n}\, d^m\Big) h^n$$

(everything in here is in formal calculus).

$$= \sum_{n\geq 0}\sum_{0\leq 2m'\leq 2n} D_{m',n}\, d^{2m'} h^n$$

$$= \sum_{n\geq 0}\sum_{0\leq m'\leq n} D_{m',n}\, d^{2m'} h^n$$

$\star$ Thm of MM. $D_{m',n} = 0 \ \forall m' > \frac{n}{2}$ $\star$

$$= \sum_{n=0}^{\infty}\sum_{0\leq m'\leq \frac{n}{2}} D_{m',n}\, d^{2m'} h^n \quad (n' = n - 2m')$$

$$= \sum_{n'=0}^{\infty}\Big(\underbrace{\sum_{m'\geq 0} D_{m',2m'+n'}\,(dh)^{2m'}}\Big) h^{n'}$$

MMR expansion

Theorem by Rozansky

$$V_n(K,z) = \frac{P_n(z^2)}{\Delta^{2n+1}(z)} \qquad z = q^{\frac{d}{2}} - q^{-\frac{d}{2}}, \quad z = t^{\frac12} - t^{-\frac12}$$

Examples:

① trefoil knot : $V_d(q) = 1 + q^{d-1}\sum_{m=1}^{d-1} q^{md}(1-q^{d-1})\cdots(1-q^{d-m})$

$$\Delta = 1+z^2, \quad V_1 = 2z^2 + z^4, \quad V_2 = 1 - 3z^2 + z^4, \ \ldots$$

② figure 8 : $V_d = 1 + \sum_{m=1}^{d-1}\prod_{j=1}^{m}(q^d + q^{-d} - q^j - q^{-j})$

$$\Delta = 1 - z^2, \quad P_2 = -1 - z^2, \quad P_4 = 4 + 20z^2 + 14z^4 + 2z^6$$

We will show outline of the proof in the next few lectures.

$V_d(K, 1+h)$? Defined using R-matrix.

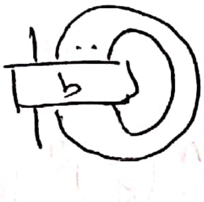
$U_q\mathfrak{sl}(2,\mathbb{C})$: $\exists$ an irrep of each dim V_d $\{f_0, \ldots, f_{d-1}\}$

- Lie algebra of $Sl(2,\mathbb{C})$ generated by X, Y, H. $X = \begin{pmatrix} \cdot & 1 \\ \cdot & \cdot \end{pmatrix}, Y = \begin{pmatrix} \cdot & \cdot \\ 1 & \cdot \end{pmatrix}$, $H = \begin{pmatrix} 1 & \cdot \\ \cdot & -1 \end{pmatrix}$
- quantum groups : Hopf algebras HA: $U_q sl(2,\mathbb{C})$ (univ enveloping algebra)
- $\exists$ universal R-matrix $R \in HA \otimes HA$ (q-deformed)

$[H,X] = 2X, \quad [H,Y] = 2Y,$
$[X,Y] = H$

gives $\rightarrow$ (R, μ) enhanced $\longrightarrow$ link invariant

$J_d(K, q)$ = link invariant from R-matrix of $U_q sl(2,\mathbb{C})$' =  $b \in B_N$

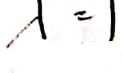
What about V_d?  leaving first strand open by Schur's lemma gives a scalar $= V_d(K, q)$

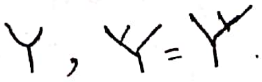
- The boson-fermion interpretation: We have a rep on $\mathbb{C}[f_0, \ldots, f_{d-1}] = V_d$

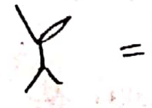
$$X f_m = [m] f_{m-1} \qquad [n] = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$$
$$Y f_m = [d-1-m] f_{m+1}$$
$$H f_m = (d-1-2m) f_m$$

a basis for $V_d^{\otimes N}$ is $\{ f_{m_1} \otimes \cdots \otimes f_{m_N} \}$ which can be mapped to monomials $\mathbb{C}[z_1^{m_1} \cdots z_N^{m_N}]$. Now $\Delta^{N-1}\left(\frac{1}{2}[N(d-1) - H]\right)$ acts on $V_d^{\otimes N}$ where Δ is the comultiplication for the Hopf algebra $U_q sl(2,\mathbb{C})$.

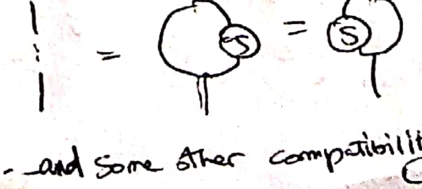
- A quick review on HA: ① μ :  multiplication satisfying  associativity

$\exists$ unit  s.t. 

② Δ: 

③ 

S: antipode
like taking 'inverse'

④ 

example: $\mathbb{C}[G]$ the group algebra is a HA.

... and some other compatibility equations.

Eigenspace decomp of $V_d^{\otimes N}$ using $\frac{1}{2}[N(d-1)+H] = C$: (4)

$$C(f_{m_1}\otimes\cdots\otimes f_{m_N}) = \underbrace{\left(\sum_{i=1}^{N} m_i\right)}_{n}(f_{m_1}\otimes\cdots\otimes f_{m_N})$$

$$V_d^{\otimes N} = \bigoplus_{n=0}^{\infty} V_N^{(n)} \quad , \quad V_N^{(1)} \cong \mathbb{C}[z_1,\dots,z_N] \text{ as } f_j \to z_j$$

since $\sum m_i = 1 \to \exists!\, j : m_j = 1$ and all others $= 0$.

Rozansky's deformation of R-matrix:

$$\check{R}[a;\varepsilon_1,\varepsilon_2,\varepsilon_{12}](f_{m_1}\otimes f_{m_2})$$
$$= \sum_{n\geq 0} \frac{\left(\prod_{m_1-n+1\leq l\leq m_1} l\right)}{n!} \left(e^{\varepsilon_{12}} a\right)^n \left(e^{\varepsilon_1}\right)^{m_1} \left(e^{\varepsilon_2} q^{-d}\right)^{m_2} f_{m_2+n} f_{m_1-n}$$

a perturbation of R-matrix using a (spectral parameter) and small numbers $\varepsilon_1,\varepsilon_2,\varepsilon_{12}$

Taking derivative at zero recovers the R-matrix.

Technical Lemma: $\exists$ polynomials $T_{j,k}^{[R,1]}$, $T_{j,k}^{[R,2]}$ s.t.

$$R = q^{\frac{d^2-1}{4}} q^{\frac{-d-1}{2}} \left(1+\sum_{j\geq 1} h^j \partial_a^j \sum_{k\geq 0} h^k T_{j,k}^{[R,1]}(\partial\varepsilon_1, \partial\varepsilon_2+\partial\varepsilon_{12})\right)$$
$$\left(1+\sum_{j\geq 1} h^j T_j^{[R,2]}(\partial\varepsilon_1,\partial\varepsilon_{12})\right)\left(1+\sum_{j\geq 1} h^j \frac{\prod_{0\leq l\leq j-1}(\partial\varepsilon_2 - l)}{j!}\right)$$

$$\check{R}[a,\varepsilon_1,\varepsilon_2,\varepsilon_{12}]\Big|_{\substack{a=1-q^{-d}\\ \varepsilon_1=\varepsilon_2=\varepsilon_{12}=0}}$$ gives the R matrix.

Define:

- $V_{d,\infty}^{\otimes N} \leftarrow f_0, f_1, \dots, f_{d-1}, f_d, \dots$ where you add the f_j, $j > d-1$
- $V_N^{(1)} \cong \mathbb{C}[z_1,\dots,z_N]$, $\forall i$ consider $\mathbb{C}^2 \cong \mathbb{C}[z_i, z_{i+1}]$ and take the restriction of $\check{R}$ to this $\mathbb{C}^2$, called $\tilde{\check{R}} = \begin{pmatrix} e^{\varepsilon_1+\varepsilon_2} a & e^{\varepsilon_2} q^{-d} \\ e^{\varepsilon_1} & 0 \end{pmatrix}$

If you set $q^{-d} = t$, $a = 1-q$ $\xrightarrow{\varepsilon_1 = \varepsilon_2 = 0}$ $\begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix}$ which is the Burau rep.

Using this linear algebra lemma you get the Alex. poly..

Lemma: $\mathcal{O}$ operator on the poly algebra $\mathbb{C}[z_1, \dots, z_N]$ coming from $\tilde{\mathcal{O}}$ on $\mathbb{C}[z_1] \oplus \mathbb{C}[z_2] \oplus \dots \oplus \mathbb{C}[z_N]$. If spectrum $\lambda \tilde{\mathcal{O}} \in \mathring{D}^2$ for some $\lambda \in \mathbb{C}$, then
$$\sum_{n \geq 0} \lambda^n \operatorname{Tr}_{V_N^{(n)}} \mathcal{O} = \frac{1}{\det_{V_N^{(1)}} (1 - \lambda \tilde{\mathcal{O}})}.$$