Tuesday 28th Jan 2020 ①

We want to prove the Volume Conjecture for the figure 8 Knot.

Recall Volume Conjecture:

$$\lim_{N\to\infty} \frac{\log\left|\frac{J_N(K;e^{q})}{[N]_q}\right|}{N} = \frac{Vol(S^3\setminus K)}{2\pi}$$

where $q = e^{\frac{2\pi i}{N}}$ and $[n]_q = \dfrac{q^{n/2}-q^{-n/2}}{q^{1/2}-q^{-1/2}}$

★ In This section, we will use normalized colored Jones poly.

Therefore Volume Conjecture is: $\lim_{N\to\infty} \dfrac{\log|J_N|}{N} = \dfrac{Vol(S^3\setminus K)}{2\pi}$

• We will do The last step of the proof first. We will assume (later prove) that Jones poly for figure 8 Knot K is:

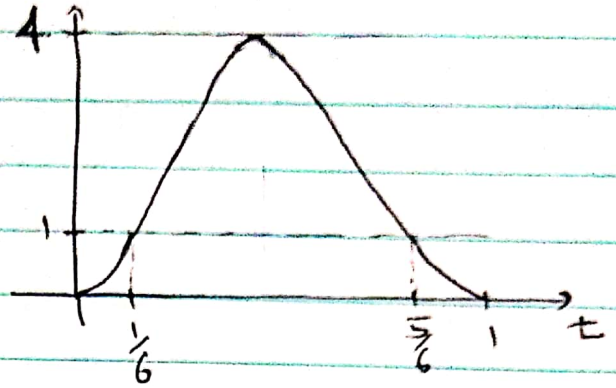
$$J_N(K;q) = \frac{1}{\{N\}} \sum_{j=0}^{N-1} \frac{\{N+j\}!}{\{N-1-j\}!}$$

where $\{n\} = [n]_q\,(q^{\frac{1}{2}}-q^{-\frac{1}{2}}) = q^{n/2}-q^{-n/2}$, Thus:

$$J_N(K;q) = \sum_{j=0}^{N-1} \prod_{k=1}^{j} \left(q^{\frac{N-k}{2}}-q^{-\frac{(N-k)}{2}}\right)\left(q^{\frac{N+k}{2}}-q^{-\frac{(N+k)}{2}}\right)$$

$$\overset{q=e^{\frac{2\pi i}{N}}}{\Longrightarrow} \quad = \sum_{j=0}^{N-1} g_N(j) \quad \text{for } g_N(j) = \begin{cases} \prod_{k=1}^{j} 4\sin^2\left(\frac{k\pi}{N}\right), & j\neq 0 \\ 1, & j=0 \end{cases}$$

We plot $4\sin^2(t\pi)$:



Therefore for

$$\begin{cases} 4\sin^2\left(\frac{k\pi}{N}\right)<1, & 0<k<\frac{N}{6} \\ 4\sin^2\left(\frac{k\pi}{N}\right)>1, & \frac{N}{6}<k<\frac{5N}{6} \\ 4\sin^2\left(\frac{k\pi}{N}\right)<1, & k>\frac{5N}{6} \end{cases}$$

As $g_N(j)$ are product of $4\sin^2\left(\frac{k\pi}{N}\right)$, we deduce that $g_N(j)$ is:

① decreasing for $0<j<\frac{N}{6}$

② increasing for $\frac{N}{6}<j<\frac{5N}{6}$

③ decreasing for $\frac{5N}{6}<j<N$

Hence $\max g_N(j)$ occurs at $j=\left\lfloor\frac{5N}{6}\right\rfloor$. So

$$g_N\left(\left\lfloor\frac{5N}{6}\right\rfloor\right) < \sum_{j=0}^{N-1} g_N(j) < N\, g_N\left(\left\lfloor\frac{5N}{6}\right\rfloor\right)$$

$$\Rightarrow \frac{\log g_N\left(\left\lfloor\frac{5N}{6}\right\rfloor\right)}{N} < \frac{\log \sum_0^{N-1} g_N(j)}{N} < \frac{\log N}{N} + \frac{\log g_N\left(\left\lfloor\frac{5N}{6}\right\rfloor\right)}{N}$$

$$\Rightarrow \lim_{N\to\infty} \frac{\log |J_N|}{N} = \lim_{N\to\infty} \frac{\log g_N\left(\left\lfloor\frac{5N}{6}\right\rfloor\right)}{N}$$

But
$$\lim_{N\to\infty} \frac{\log g_N\left(\left\lfloor\frac{5N}{6}\right\rfloor\right)}{N} = \lim_{N\to\infty} \sum_{k=1}^{\left\lfloor\frac{5N}{6}\right\rfloor} \frac{2\log\left(2\sin\frac{k\pi}{N}\right)}{N}$$

$$\underset{(\star)}{=} \frac{2}{\pi}\int_0^{\frac{5\pi}{6}} \log(2\sin x)\,dx = -\frac{2}{\pi}\Lambda\left(\frac{5\pi}{6}\right)$$

$(\star)$ is due to Riemann sum approx, and $\Lambda(\theta) = -\int_0^\theta \log|2\sin x|\,dx$

$\Lambda(\theta)$ is the Lobachevsky function which satisfies:

- Periodic $\Lambda(\theta+\pi)=\Lambda(\theta)$
- odd $\Lambda(-\theta)=-\Lambda(\theta)$
- $\Lambda(2\theta)=2\Lambda(\theta)+2\Lambda(\theta+\frac{\pi}{2})$

All properties can be proved using elementary facts about $\sin(x)$

- $\sin(\theta+\pi)=-\sin(\theta)$
- $\sin(-\theta)=-\sin(\theta)$
- $\sin 2\theta = 2\sin\theta\cos\theta = 2\sin\theta\sin(\frac{\pi}{2}-\theta)$

$$\Rightarrow \log|2\sin(2x)| = \log|2\sin x| + \log|2\sin(x+\tfrac{\pi}{2})|$$

Using the above: $\Lambda(\frac{5\pi}{6})=\Lambda(\pi-\frac{\pi}{6})=-\Lambda(\frac{\pi}{6})=\frac{-\Lambda(\frac{\pi}{3})}{2}+\Lambda(\frac{2\pi}{3})$

$$=\frac{-3}{2}\Lambda(\tfrac{\pi}{3})$$

$$\Rightarrow \lim \frac{\log|J_N|}{N} = \frac{3}{\pi}\Lambda(\tfrac{\pi}{3}) = \frac{\mathrm{Vol}(S^3\setminus K)}{2\pi}$$

as $\mathrm{Vol}(S^3\setminus K)=6\Lambda(\frac{\pi}{3})$

(Recall that hyperbolic volume of tetrahedron with angles α,β,γ is $\Lambda(\alpha)+\Lambda(\beta)+\Lambda(\gamma)$; Also $S^3\setminus K$ is <u>two</u> ideal tetrahedrons with <u>all angles</u> $\frac{\pi}{3}$).

- Next, we prove the formula for colored Jones poly:

$$J_N(K;q)=\frac{1}{\{N\}}\sum_{j=0}^{N-1}\frac{\{N+j\}!}{\{N-1-j\}!}$$

Recall (R, μ, α, β) enhanced R-matrix definition.

Recall for

$$(1)\ R^{ij}_{kl} = \sum_{m=0}^{\min(N-1-i,j)} \delta_{l,i+m}\,\delta_{k,j-m} \frac{\{l\}!\,\{N-1-k\}!}{\{i\}!\,\{m\}!\,\{N-1-j\}!} (q^{-1})^{\frac{N-i-1}{2}\cdot\frac{N-j-1}{2} + \frac{m(i-j)}{2} + \frac{m(m+1)}{4}}$$

$$(2)\ (R^{-1})^{ij}_{kl} = \sum_{m=0}^{\min(N-1-l,j)} \delta_{l,i-m}\,\delta_{k,j+m} \frac{\{k\}!\,\{N-1-l\}!}{\{j\}!\,\{m\}!\,\{N-1-i\}!} (q^{-1})^{\frac{i-(N-1)}{2}\cdot\frac{j-(N-1)}{2} + \frac{m(i-j)}{2} - \frac{m(m+1)}{4}}$$

$$(3)\ \mu^i_j = \delta_{i,j}\, q^{\frac{(2i-N+1)}{2}} \qquad (4)\ \alpha = q^{\frac{N^2-1}{4}},\ \beta = 1$$

(R, μ, α, β) is an enhanced R-matrix and

$\forall$ knots K: $\quad J_N(K;q) = \frac{\{1\}}{\{N\}} T_{(R,\mu,\alpha,\beta)}(K)$

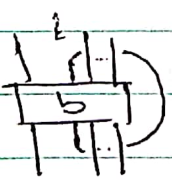
where $T_{(R,\mu,\alpha,\beta)}(K) = \alpha^{-w(K)} \beta^{-n} \operatorname{Tr}(\rho_R(\hat{b})\, \mu^{\otimes n})$

where $K = \hat{b}$ is braid closure of n-strand braid b

and $\rho_R(\hat{b})$ is obtained by placing R or R^{-1} instead of braidings

in b (σ_i or σ_i^{-1}: ╳ or ╳). $w(K)$ is writhe index of K.

- Note $J_N(\text{unknot}) = 1$ as $\operatorname{Tr}(\mu) = \sum_{i=0}^{N-1} q^{\frac{2i-N+1}{2}} = \frac{\{N\}}{\{1\}}$
- Note $\operatorname{Tr} = \operatorname{Tr}_1 \operatorname{Tr}_2 \cdots \operatorname{Tr}_n$ where Tr_i is obtained by taking

closure on i-th strand only: "  " $= \operatorname{Tr}_i(b)$".

- Every Tr_i gets the average (trace) of the endomorphism from $V^{\otimes n} \to V^{\otimes n}$ on the i-th tensor factor.
- For $K =$ figure eight knot, $b = \sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1}$, $w(b) = 0$, b acting on 3 strands, Thus $n = 3 \Rightarrow J_N(K; q) = \frac{\{1\}}{\{N\}} Tr_1 Tr_2 Tr_3 (\rho_R(\hat{b}) \mu^{\otimes 3})$
- Consider instead $Tr_2 Tr_3 (\rho_R(\hat{b}) \mu^{\otimes 2})$ where we have gotten rid of two tensor factors. Thus we have an endomorphism of V.
- FACT: The endomorphisms given by R and μ using braids are intertwiners of representations of some quantum group (called $U_q(\mathfrak{sl}_{\mathbb{C}}(2))$). In particular, V is an irreducible representation of that quantum group.

A map $f: V \to W$ is an intertwiner if it is compatible with the representations on V and W, ie

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\phi_V"'] \arrow[dr, phantom, "\circlearrowright"] & W \arrow[d, "\phi_W"] \\ V \arrow[r, "f"'] & W \end{tikzcd}$$

where $\phi_{\_}$ is the representation action.

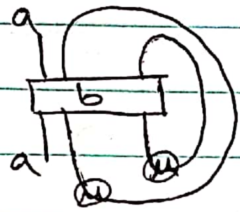
- Hence $Tr_2 Tr_3 (\rho_R(\hat{b}) \mu^{\otimes 2}) \in End(V)$ is an intertwiner of an irreducible representation. Schur's lemma from representation theory applies:

This means $Tr_2 Tr_3(\varphi_{R}(\hat{b})\mu^{\otimes 2}) = S \times Id_V$ for some $S \in \mathbb{C}$.

Therefore $J_N(K,q) = \frac{\{1\}}{\{N\}} Tr_1(S \times \mu) = \frac{\{1\}}{\{N\}} S \, Tr(\mu)$

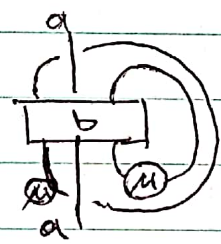
$= S$

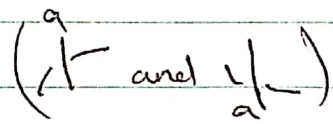
- So we only need to compute S which is any diagonal element of the matrix $Tr_2 Tr_3(\varphi_R(\hat{b})\mu^{\otimes 2})$ like the following for any index $0 \leq a \leq N-1$:



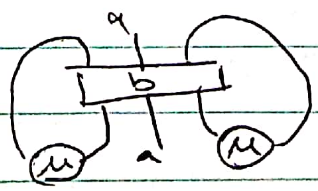
- It turns out that computing the above is still not easy. Instead, we would like to take Trace on <u>first</u> & <u>third</u> factor instead of <u>second</u> & <u>third</u>.

This means the following picture,

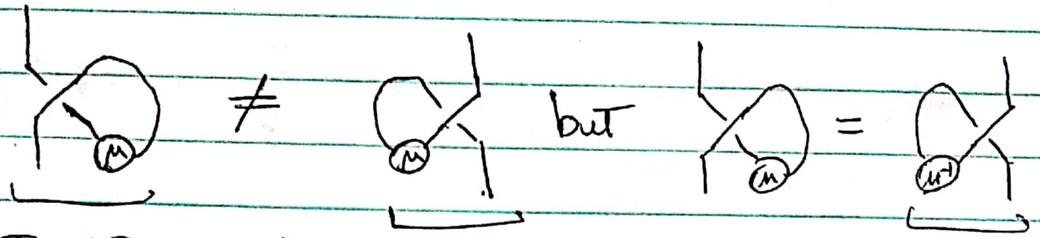


- But due to braidings present at top and bottom () this becomes still hard to compute.

- We would like to close the first strand from the left to right,

This means: 

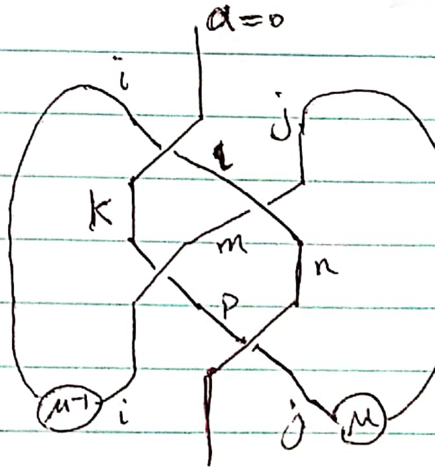
It turns out that this is <u>NOT</u> equal to the previous picture. The reason is that to take closure from left to right, one needs to put $\underline{\underline{\mu^{-1}}}$ instead of $\underline{\underline{\mu}}$. The reason behind this is outside scope of this section but as an exercise you can check that:



$$\mathrm{Tr}_2(R(\mathrm{Id}\otimes\mu)) \qquad \mathrm{Tr}_1(R(\mu\otimes\mathrm{Id})) \qquad \mathrm{Tr}_1(R(\mu^{-1}\otimes\mathrm{Id}))$$

$$\sum_i R^{ji}_{ji}\mu^i_i \qquad \sum_j R^{ji}_{ji}\mu^j_j \qquad \sum_j R^{ji}_{ji}(\mu^{-1})^j_j$$

Therefore, we need to compute the following, where to make computations easier we choose index $a=0$.

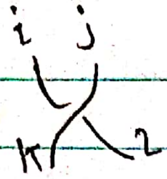


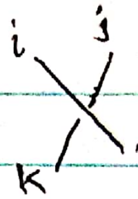
$$\sum_{i,j,k,l,m,n,p} R^{io}_{kl}\,(R^{-1})^{lj}_{mn}\,R^{km}_{ip}\,(R^{-1})^{pn}_{oj}\,(\mu^{-1})^i_i\,\mu^j_j$$

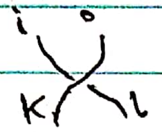
Recall the equations (1) & (2) for R & R^{-1}:

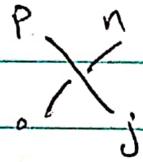
$$\underset{\sim\sim}{R^{ij}_{kl}} = \sum \underset{\sim\sim\sim\sim\sim}{\delta_{l,i+m}\,\delta_{k,j-m}}\cdots \quad \& \quad \underline{\underline{(R^{-1})^{ij}_{kl}}} = \sum \underline{\delta_{l,i-m}\,\delta_{k,j+m}}\cdots$$

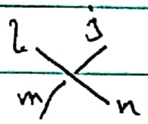
For the $\underline{\text{terms above}}$ to be nonzero, we have the following $\underset{R}{(+)}$, $\underset{R^{-1}}{(-)}$ rules

R $(+)$  $i+j=k+l$, $l \geqslant i$, $k \leqslant j$

R^{-1} $(-)$  $i+j=k+l$, $l \leqslant i$, $k \geqslant j$

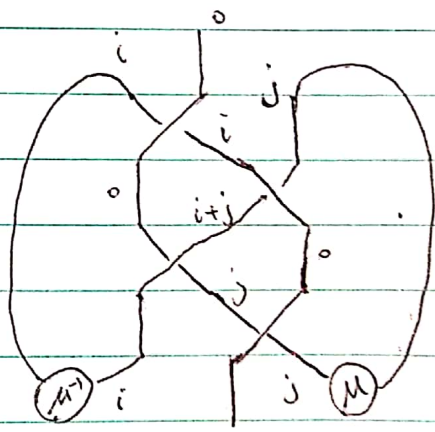
Now apply $(+)$ on  $i+0=k+l$, $l \geqslant i$, $k \leqslant 0$ $\Rightarrow$ $k=0$, $l=i$

apply $(-)$ on  $p+n=0+j$, $n \leqslant 0$, $p \geqslant j$ $\Rightarrow$ $n=0$, $p=j$

apply $(-)$ on  $l+j=m+n$, $n=0$, $l=i$ $\longrightarrow$ $m=i+j$

So we get

$$\sum_{\substack{0\leqslant i\leqslant N-1\\ 0\leqslant j\leqslant N-1\\ 0\leqslant i+j\leqslant N-1}} R^{i0}_{0i}\,(R^{-1})^{ij}_{i+j\,0}\,R^{0\,i+j}_{ij}\,(R^{-1})^{j0}_{0j}\,(\mu^{-1})^{i}_{i}\,\mu^{j}_{j}$$



$$= \sum_{\substack{0\leqslant i,j\leqslant N-1\\ 0\leqslant i+j\leqslant N-1}} (-1)^i \frac{\{i+j\}!\,\{N-1\}!}{\{i\}!\,\{j\}!\,\{N-1-j-i\}!} \times q^{-\frac{(N-1)i}{2}+\frac{(N-1)j}{2}-\frac{i^2}{4}+\frac{j^2}{4}-\frac{3i}{4}+\frac{3j}{4}}$$

Put $k=i+j$ $\Rightarrow$ $\displaystyle\sum_{k=0}^{N-1} \frac{\{N-1\}!}{\{N-1-k\}!}\, q^{\frac{k^2}{4}+\frac{Nk}{2}+\frac{k}{4}} \times$

$$\underbrace{\left(\sum_{i=0}^{k} (-1)^i \frac{\{k\}!}{\{i\}!\,\{k-i\}!}\, q^{\frac{(-2N-k-1)i}{2}}\right)}$$

For the inner sum, we shall use the following fact.

• Define $T(k,l) = \sum_{i=0}^{k} (-1)^i q^{\frac{li}{2}} \begin{bmatrix} k \\ i \end{bmatrix}_q$ where $\begin{bmatrix} k \\ i \end{bmatrix}_q = \frac{\{k\}!}{\{i\}!\,\{k-i\}!}$

Then $T(k,l) = \prod_{j=1}^{k} \left(1 - q^{\left(\frac{l+k+1}{2}\right) - j}\right)$

*Proof: exercise. Use Pascal's identity $\begin{bmatrix} k \\ i \end{bmatrix}_q = q^{-\frac{k-i}{2}} \begin{bmatrix} k-1 \\ i-1 \end{bmatrix}_q + q^{i/2} \begin{bmatrix} k-1 \\ i \end{bmatrix}_q$

to get recersive relation $T(k,l) = \left(1 - q^{\frac{l+k+1}{2} - k}\right) T(k-1, l+1)$. ∎

Plugging $l = -2N-k-1$ in above we get by direct calculations

$$T(k,l) = \frac{\{N+k\}!}{\{N\}!} q^{-\frac{k^2}{4} - \frac{Nk}{2} - \frac{k}{4}}$$

This implies

$$J_N(K;q) = \sum_{k=0}^{N-1} \frac{\{N-1\}!}{\{N-1-k\}!} q^{\frac{k^2}{4} + \frac{Nk}{2} + \frac{k}{4}} T(k, -2N-k-1)$$

$$= \sum_{k=0}^{N-1} \frac{\{N-1\}!}{\{N-1-k\}!} \frac{\{N+k\}!}{\{N\}!}$$

$$= \frac{1}{\{N\}} \sum_{k=0}^{N-1} \frac{\{N+k\}!}{\{N-1-k\}!} \qquad ∎$$