Tuesday 3rd March 2020

Zeta function approach to V.C.

	Riemann & Weil	Colored Jones	Twisted Alex
• Rationality	✓	✓	✓
• Functional eq.	✓	Modularity conj	?
• Riemann Hypothesis	✓	?	?

Grothendieck Philosophy, essence of Zeta function?

⇓

counting with weights

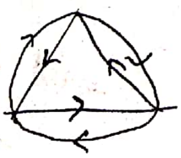
We will start with Ihara-Selberg zeta function of graphs $\xrightarrow{\text{then}}$ knot diagrams
(J.P. Serre)

Given a finite unoriented graph G $\qquad$ $G \quad\Rightarrow\quad G^+$

Let G^+ = be the doubled oriented graph

Ihara Zeta function $\quad \zeta_G(u) = \prod_{p \in PG^+} (1-u^{|p|})$ $\quad$ u some variable

- $p \in PG^+$ = The set of all primitive, reduced cycles on G^+
  - primitive: $\nexists$ path x s.t. $p = x^n$
  - reduced: $e_i e_{i+1}$, $e_{i+1} \neq \bar{e}_i$
- $|p|$ = length = # of edges in p



$\zeta(u) = (1-u^3)(1-u^3)$

here you have $|PG^+| = \infty$

So $\zeta_G(u)$ is an infinite product

Bass evaluation: $\zeta_Q(u) = \det(1-uT)$ where T is a finite matrix! (2)

So the infinite product ultimately gives something finite.

Def. T: Let $V(E) = \mathbb{C}$-span of all edges in G^+

Let $J: V(E) \to V(E)$

$e_{ij} \to \bar{e}_{ij} = e_{ji}$

Succession map $Succ: V(E) \to V(E)$

$$e_{ij} \to \sum_{e_{jl} \neq e_{ji}} e_{jl}$$

And then define

$$\boxed{T = Succ - J}$$

Example:

1D Ising model: Given lattice S, write $\zeta_S(z,\beta) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} Z_n(S)\right)$

$Z_n(S)$: partition of model of size n

Fact: This is rational and equal to $\frac{1}{\det(sth)}$ → determinant of something related to Transfer matrix T.

X. L. Liu and Z. W.

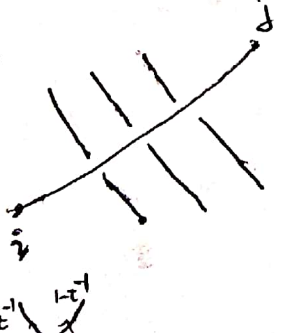
(Random walks on knot diagrams

⟹ a new proof of the Melvin-Morton Conjecture

Remark by V. Jones: The Burau representation has an interpretation as bowling on braids.

Burau rep $\varphi: B_n \to GL_n(\mathbb{Z}[t^{\pm 1}])$

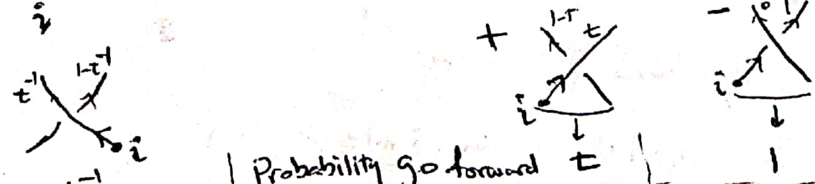
Given $b \in B_n$



$$\varphi(b)_{ij} = \sum_{\substack{\text{all paths } p \\ i \to j}} w(p), \quad 1 \le i,j \le n$$

p goes upwards

$w(p) = \prod$ weights at each crossing



		+	−
Probability go forward	t^{-1}	t	1
" go down	$1-t^{-1}$	$1-t$	0

Note $1-t^{-1}$ or $1-t < 0$ but here we are doing formal calculus.

R-matrix gives rise to Jones Poly

$$R = q^{\frac14}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-q^{-1} & q^{-\frac12} & 0 \\ 0 & q^{-\frac12} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = q^{-\frac14}\begin{pmatrix} q^{\frac12} & & & \\ & q^{\frac12}-q^{-\frac12} & 1 & \\ & 1 & 0 & \\ & & & q^{\frac12} \end{pmatrix}$$

$$\xrightarrow{q \to q^2} q^{-\frac12}\begin{pmatrix} q & & & \\ & q-q^{-1} & 1 & \\ & 1 & 0 & \\ & & & q \end{pmatrix} \leftarrow \text{Call this } R_J$$

we shall replace q by q^2.

define also $R_A = \begin{pmatrix} q & & & \\ & q-q^{-1} & 1 & \\ & 1 & 0 & \\ & & & -q^{-1} \end{pmatrix}$

R_J gives Jones, R_A gives Alex. Note how close they are.

$R_A : \mathbb{C}^2 \otimes \mathbb{C}^2 \circlearrowleft$ write $\mathbb{C}^2 = \mathbb{C}|i\rangle \otimes |j\rangle$ then labels on matrix

Rows & columns are $00 \ldots 11$ $(i, j \in \{0,1\})$

$$\begin{array}{c|c} & 00 \quad \cdots \quad 11 \\ \hline \begin{matrix} 00 \\ \vdots \\ 11 \end{matrix} & \begin{pmatrix} & & \\ & & \\ & & -q^{-1} \end{pmatrix} \end{array}$$

This says interchanging two fermions gives $-q^{-1}$ instead of q. Also note the "phase factor" $q^{-\frac12}$ for R_J.

Another fact: $R_{J,A} - R_{J,A}^{-1} = (q-q^{-1})\cdot Id$

Get link invariants from enhanced R-matrix:

Both R-matrix give rise to rep of B_n

$\varphi_J : \sigma_i \in B_n \longrightarrow id \otimes \cdots \otimes R_J \otimes \cdots \otimes id$ (i-th place)

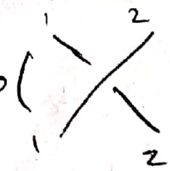
$\varphi_A :$ similar to above

For Jones $J(\hat{b}; q) = \; ? \operatorname{Tr}(\mu^{\otimes n} \varphi_J(b))$ (? = normalization)

For Alex $\Delta(\hat{b}; q) = \; ? \operatorname{Str}(\mu\mu^{+} \varphi_A(b))$

Str = supertrace $\longrightarrow$ a basis of $(\mathbb{C}^2)^{\otimes n}$ is a length n-bit string $|I\rangle = |i_1 \cdots i_n\rangle$. Sum $\sum i_j$

the odd ones pick up a negative sign in the trace (fermion)

• For example: $\varphi($  $) = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 1-t & t \\ 2 & 1 & 0 \end{array}$ (4)

• Claim: Burau rep is always reducible. Since $\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}$ is an eigenvector.
due to preservation of probability $\varphi(b)\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} = \begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$.

So define the reduced Burau by restricting to
the space spanned by $\{ v_i = \begin{pmatrix} 0 \\ \vdots \\ -t \\ 0 \\ \vdots \end{pmatrix} \leftarrow$ i-th place $\}$. This is invariant subspace.

$\tilde{\varphi}(b)$ = reduced Burau, Then define $\Delta(\hat{b}) = \dfrac{\det(1-\tilde{\varphi}(b))}{1+t+\dots+t^{n-1}}$ is the Alex. poly.

if we were to take $\varphi(b)$ then det above would be zero.

• Extend the Burau reps to String links

by using the exact same definition.

n disjoint arcs in $\mathbb{R}^2 \times [0,1]$ relative to $\mathbb{R}^2 \times 0$, $\mathbb{R}^2 \times 1$

it is a semi-group.

examples

FACTS:

1) $\varphi(L) = 1$
for any 1-string link
because no matter how the bowling goes, it will end up in the same place as there is only one string, with one end and beginning.

2) Each entry in $\varphi(L)$ is a rational function of t. $\varphi(\,) = \dfrac{1}{2-t}\begin{pmatrix} 1 & 1-t \\ t^{-1}-1 & 3-t-t^{-1}\end{pmatrix}$

We want to generalize this idea to Jones poly.
random walk / state-sum models of quant invariants

Amitsur-Levitzki identity: Take any $2n$ matrices $A_1, \dots, A_{2n}$ of size $n \times n$

$$\sum_{\sigma \in S_{2n}} \underbrace{\pi(\sigma)}_{\text{parity of permutation}} A_{\sigma(1)} \cdots A_{\sigma(2n)} = 0$$

For $n=1 \to$ it says complex numbers commute

Zeta function approach to MM Conjecture:

Given a knot K

$$J_d(K, q) \cong J(K^{\otimes d}, q)$$

$$/ \longrightarrow \underbrace{// \cdots /}_{d}$$

$$R = q \begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline 00 & q & & & \\ 01 & & 1-q^{-2} & q & \\ 10 & & q & 0 & \\ 11 & & & & q \end{array}$$

notice how this is like the bowling situation.