Tuesday 4th February 2020

• We can define twisted Alex poly $\Delta_K^{\xi_N}(t) \in \mathbb{C}[t^{\pm 1}]$ s.t $\forall \xi \in \mathbb{C}$, $|\xi| = 1$, we have (arXiv 1912.12946)

Theorem:
$$\lim_{N\to\infty} \frac{\log|\Delta_N^{\xi_N}(\xi)|}{N^2} = \frac{1}{4\pi} \mathrm{Vol}(S^3 \setminus K)$$

• We will study the definition of twisted Alex. poly not the proof of the above.

• Goal is to show that Colored Jones Poly and twisted Alex. poly are related.

But note the latter is classical. So either we make the latter more quantum or the former less quantum.

• There is an interpretation by Stephen Bigelow of Jones poly (not colored) in terms of "semi-"classical constructions (using "intersection forms & configuration space, ...).

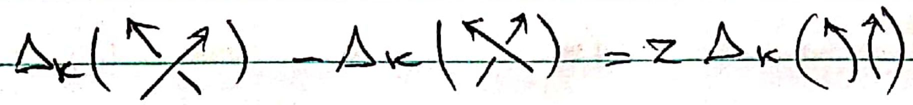
First, we start with just Alexander polynomial. There are many ways to define this polynomial.

• Let $K \subset S^3$, then the Alexander poly is $\Delta_K(t) \in \mathbb{Z}[t^{\pm 1}]$ up to any power t^n $(n \in \mathbb{Z})$. There is also the Alexander-Conway poly

$\Delta_K(z)$ which after change of variable $z = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$ gives $\Delta_K(t)$.

The easiest definition is using The Skein-relation:

- $\Delta_K(z) = 1$ if $K =$ unknot.
- $\Delta_K(\ ) - \Delta_K(\ ) = z\,\Delta_K(\ )$



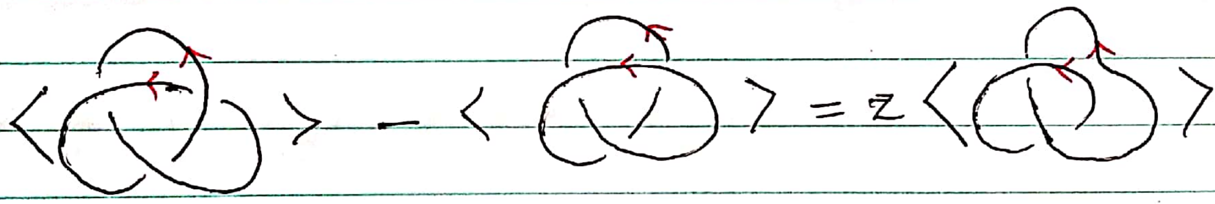
Example: ~~Trefoil~~ Trefoil knot. We claim $\Delta_K(z) = 1 + z^2 = 1 + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2$

$= t^{-1} - 1 + t$

"$=$" $1 - t + t^2$

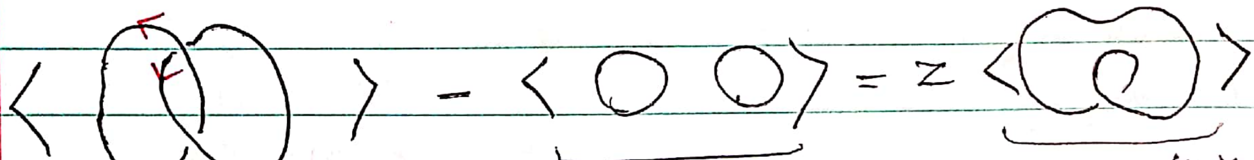
"up to any power t^n"

Proof:



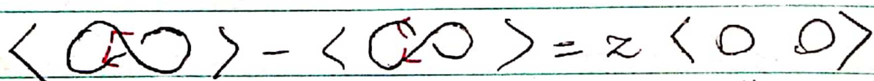
$\Rightarrow \Delta_{\text{trefoil}}(z) = 1 + z\,\Delta_{\text{Hopf link}}(z)$

And for the Hopf link:



1 (unknot)

$\Downarrow$



1 unknot $-$ 1 unknot $\Rightarrow$ 0

Therefore $\Delta_{\text{Hopf link}}(z) = z \Rightarrow \Delta_{\text{Trefoil}}(z) = 1 + z^2$

Exercise: Show $\Delta_{\text{figure eight}}(z) = 1 - z^2$.

* But what is the Topological meaning of Alexander poly?

Recall that for $E_K = S^3 \setminus K$ the knot complement (the knot exterior), $H_*(E_K, \mathbb{Z}) \cong H_*(S^1, \mathbb{Z})$ and $\pi_n(E_K, *) = 0$ if $n \neq 1$. This means E_K is a $K(\pi_1(E_K), 1)$ Eilenberg–McLane space.

- Take the abelianization map $\varphi : \pi_1(S^3 \setminus K) \twoheadrightarrow \dfrac{\pi_1}{[\pi_1, \pi_1]} \cong H_1$

$$\langle x_1, \dots, x_n \mid r_1, \dots, r_n \rangle \mapsto \langle x_1, \dots, x_n \mid x_i x_j = x_j x_i, r_1, \dots, r_n \rangle$$

but $H_1 \cong \mathbb{Z}$ with generator being the meridian of the knot.

Taking the following short exact sequence

$$1 \to \ker\varphi \to \pi_1(S^3 \setminus K) \to \mathbb{Z} \to 1$$

Then by standard topology, this corresponds to

a covering space $\tilde{E}_K^{ab}$ of E_K $\longrightarrow E_K$ such that $\mathbb{Z}$ is the deck transformations of $\tilde{E}_K^{ab}$.

with $\pi_1(\tilde{E}_K, *) = \ker\varphi$

Definitions and standard topology facts:

Let X be any finite CW complex and $\varphi : \pi_1(X) \twoheadrightarrow G$ onto

Then $1 \to \ker\varphi \to \pi_1(X) \twoheadrightarrow G \to 1$

and we can construct a space $\tilde{X}$ such that $\tilde{X}$ is covering space of X with $\pi_1(\tilde{X}) = \ker \varphi$ and all homeomorphisms that commute with the covering map p is G.

$$\begin{tikzcd} \tilde{X} \arrow[dr, "p"'] & \tilde{X} \arrow[l, "h"', "\text{homeo}"] \arrow[d, "p"] \\ & X \end{tikzcd}$$

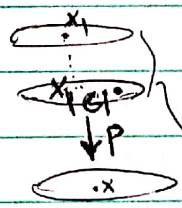
$p \longrightarrow$ a local homeo (e.g. $\mathbb{R} \xrightarrow{p} S^1 = \mathbb{R}/\mathbb{Z}$ for $\varphi : \mathbb{Z} \to \mathbb{Z}$ identity)

All such h's above give a group which coincides with G. These are called the deck transformations.

In fact $p^{-1}(x) \cong G$ as a set $\forall x \in X$. Thus we can say that:

pick a basepoint $x_0 \in X$, $\tilde{X} = \{ (y, [\alpha]) \mid y \in X$ and α is a path from x_0 to y and $\alpha \sim \beta$ if $\beta^{-1}\alpha \in \ker\varphi$, in other words α, β are in the same coset of $\pi_1(X)/\ker\varphi$ which is isomorphic to G as φ is surjective. $\}$

Locally, p^{-1} gives decks as p is a local homeo



and G elements permute these decks.

$|G|$ many decks

Apply the above theorem to $\varphi : \pi_1(S^3 \setminus K) \twoheadrightarrow \mathbb{Z}$, then we get ($S^3 \setminus K = E_K$)

$$\begin{array}{c} \tilde{E}_K^{ab} \\ \downarrow \\ E_K \end{array}$$

(universal abelian cover) with $\pi_1(\tilde{E}_K^{ab}) \cong \ker\varphi \cong [\pi_1, \pi_1]$

with deck transformations being $H_1(E_K, \mathbb{Z})$ isomorphic to $\pi_1(E_K)/\ker\varphi$

- Denote $J = \mathbb{Z}$ as a group, let $\Lambda = \mathbb{Z}[J]$ be a group ring which is isomorphic to $\mathbb{Z}[t^{\pm 1}]$.

- Take a deck transformation $T : \tilde{E}_K^{ab} \circlearrowleft$ then $T_* : H_1(\tilde{E}_K^{ab}; \mathbb{Z}) \circlearrowleft$

Therefore $H_1(\tilde{E}_K^{ab}, \mathbb{Z})$ is a $\mathbb{Z}[J]$-module where J are all deck tranformations of $\tilde{E}_K^{ab} \cong \mathbb{Z}$. We have the following Theorem.

- <u>Thm</u>: $H_1(\tilde{E}_K^{ab}, \mathbb{Z})$ as a Λ-module $\cong \Lambda / \langle \Delta_K(t) \rangle$ where $\langle \Delta_K(t) \rangle$ is the ideal generated by $\Delta_K(t)$.

- Notice the analogue ; $\mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^n$ $\quad A = n \times n$ matrix, $\det A \neq 0$

e.g $\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$

Then we get some abelian subgroup $G_A = \mathbb{Z}^n / \operatorname{Im} A$ which by standard group theory is isomorphic to $\bigoplus_{i=1}^{n} \mathbb{Z}/d_i\mathbb{Z}$ where order of $G_A = |\det A| = \prod_{i=1}^{n} d_i$.

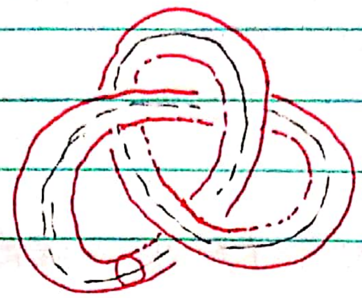
Of course, in our case, $\Lambda = \mathbb{Z}[J]$ is not a PID but a UFD (unlike $\mathbb{Z}$).

but for our particular case, the analogue does hold.

- Next section, we will discuss Fox calculus, which will allow us to compute $\Delta_K(t)$.

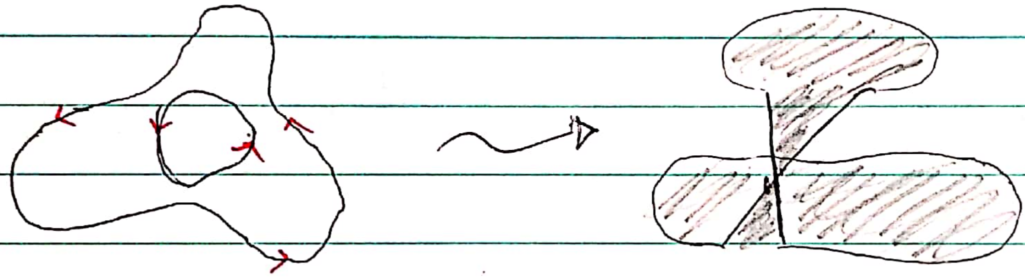
- How do we see $\tilde{E}_K^{ab}$?

$$\tilde{E}_K^{ab} \longmapsto E_K =$$



First construct a Seifert surface of K, an orientable surface S inside E_K with ∂S = a copy of K in ∂E_K.

How to see the Seifert surface?



First smooth every crossing $X =)($. We get a collection of circles, called Seifert circles. Now let each bound a disk and attach a band at each crossing. Above, we have done it for one crossing.

Note that just taking the disk the knot bounds, could give you an unorientable surface (in this case, the mobius band). This algorithm guarantees orientable.

- Now take the Seifert surface and a tubular nghbd of it, then glue the upper copies to lower copies. This gives you $E_n^{\mathbb{Z}ab}$ and

because of orientability you can do that unambiguously.

The deck transformations are taking any copy to another.