MATH 122B: MIDTERM

(1) Show that if f is holomorphic in D, then

$$f'(z_0) = \frac{df}{dz}(z_0) = \frac{\partial f}{\partial x}(z_0) = -i\frac{\partial f}{\partial y}(z_0)$$

(2) Compute the integral $\int_{-\infty}^{\infty} \frac{\cos(2x)dx}{x^4+1}$. Make sure to show that the answer you obtain is a real number.

(3) Compute $\int_{|z|=2} \frac{dz}{(z^{2016}+1)(z-3)(z-4)}$.

(4) Show that if f is holomorphic on $\mathbb{C}$ and $|f(z)| \leq |z|$ for all $z \in \mathbb{C}$, then f is a linear polynomial.

(5) Compute the Laurent series of $e^{z+\frac{1}{z}}$. Then compute the integral $\int_{|z|=1} e^{z+\frac{1}{z}}$. Show that your answer is a finite number.

Solutions

(1) Let $z_0 = x_0 + iy_0$ and $f = u + iv$ be holomorphic. Then the derivative by definition is

$$\lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} = \frac{df}{dz}(z_0)$$

which exists for any path $h \to 0$ and they are equal. Consider $h \in \mathbb{R}$, then

$$\lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(z_0)}{h} = \frac{\partial f}{\partial x}(z_0)$$

on the other hand,

$$\lim_{h\to 0} \frac{f(z_0+ih) - f(z_0)}{ih} = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(z_0)}{ih} = -i\frac{\partial f}{\partial y}(z_0).$$

(2) We use the semicircle contour C_R, with Γ_R denoting the upper half circle. Then

$$\int_{C_R} \frac{e^{i2z}dz}{z^4+1} = \int_{-R}^{R} \frac{e^{i2z}dz}{z^4+1} + \int_{\Gamma_R} \frac{e^{i2z}dz}{z^4+1}.$$

By residue theorem, the integral over the closed contour is

$$2\pi i(\text{Res}(\frac{e^{i2z}}{z^4+1}, e^{i\pi/4}) + \text{Res}(\frac{e^{i2z}}{z^4+1}, e^{3\pi/4}))$$

Computing, we have

$$\begin{aligned}\operatorname{Res}(\frac{e^{i2z}}{z^4+1}, e^{i\pi/4}) &= \frac{e^{-\sqrt{2}+i\sqrt{2}}}{4e^{3i\pi/4}}\\ \operatorname{Res}(\frac{e^{i2z}}{z^4+1}, e^{3\pi/4}) &= \frac{e^{-\sqrt{2}-i\sqrt{2}}}{4e^{9i\pi/4}}.\end{aligned}$$

We also have

$$\left|\int_{\Gamma_R} \frac{e^{i2z}dz}{z^4+1}\right| \leq \frac{\pi R}{R^4+1} \to 0$$

as $R \to \infty$.

Since $\operatorname{Re}\left(\int_{-R}^{R} \frac{e^{i2z}}{z^4+1} dz\right) = \int_{-R}^{R} \frac{\cos(2z)dz}{z^4+1}$, taking the real part of the sum of the residue gives us the answer.

(3) By residue theorem, the singularity inside $|z| = 2$ are the 2016-th roots of unity, denoted α_n, hence

$$\int_{|z|=2} \frac{dz}{(z^{2016}+1)(z-3)(z-4)} = 2\pi i \sum_{n=1}^{2016} \operatorname{Res}(\frac{1}{(z^{2016}+1)(z-3)(z-4)}, \alpha_n).$$

To find a closed form, we use the fact that the sum of the residues is equal to zero hence

$$\sum_{n=1}^{2016} \operatorname{Res}(f, \alpha_n) = -\operatorname{Res}(f,3) - \operatorname{Res}(f,4) - \operatorname{Res}(f,\infty),$$

where $f = \frac{1}{(z^{2016}+1)(z-3)(z-4)}$. Since

$$\begin{aligned}\operatorname{Res}(f,3) &= -\frac{1}{2016(3^{2015})}\\ \operatorname{Res}(f,4) &= \frac{1}{2016(4^{2015})}\\ \operatorname{Res}(f,\infty) &= 0,\end{aligned}$$

we get

$$\int_{|z|=2} \frac{dz}{(z^{2016}+1)(z-3)(z-4)} = 2\pi i \left(\frac{1}{2016(3^{2015})} - \frac{1}{2016(4^{2015})}\right).$$

(4) Let $z \in \mathbb{R}$. by Cauchy integral formula for the derivative, we have for any $R > 0$,

$$f'(z) = \frac{1}{2\pi i} \int_{|w-z|=R} \frac{f(w)}{(w-z)^2} dw.$$

Now

$$\begin{aligned}|f'(z)| &\leq \frac{1}{2\pi}\int_{|w-z|=R}\frac{|f(w)|}{|w-z|^2}dw\\ &\leq \frac{1}{2\pi R^2}(R+|z|)\int_{|w-z|=R}dw\\ &\leq \frac{R+|z|}{R}\leq 2,\end{aligned}$$

for $R>|z|$, here we use the inequality $|w|=|w-z+z|\leq|w-z|+|z|$ in the numerator. Since f' is bounded and holomorphic in $\mathbb{C}$, $f'\equiv A$, where A is a constant. This implies that $f(z)=Az+B$ for some constant B, however, $f(0)=0$ hence $B=0$. Therefore $f(z)=Az$.

(5) To compute the Laurent series, we compute the simpler terms and take their product. We have

$$\begin{aligned}e^z &= \sum_{n=0}^{\infty}\frac{z^n}{n!}\\ e^{\frac{1}{z}} &= \sum_{n=0}^{\infty}\frac{1}{n!z^n}.\end{aligned}$$

Taking their product, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{z^{n-k}}{n!k!}=\sum_{p=-\infty}^{\infty}\left(\sum_{n-k=p}\frac{1}{n!k!}\right)z^p.$$

Since there is an isolated singularity at $z=0$, the residue at 0 is given by the z^{-1} coefficient, namely

$$\operatorname{Res}(e^{z+\frac{1}{z}},0)=\sum_{n-k=-1}\frac{1}{n!k!}=\sum_{n=0}^{\infty}\frac{1}{n!(n+1)!}$$

hence by the residue theorem,

$$\int_{|z|=1}e^{z+\frac{1}{z}}dz=2\pi i\sum_{n=0}^{\infty}\frac{1}{n!(n+1)!}.$$

By comparison test, we have

$$\sum_{n=0}^{\infty}\frac{1}{n!(n+1)!}\leq\sum_{n=1}^{\infty}\frac{1}{n^2}<\infty$$