MATH 122B: COMPUTATIONAL FINAL

No calculator or notes. All contours are assumed to have positive orientation.

(1) Compute using any method $\int_{|z|=4} \frac{2z-4}{z(z-2)^2} dz$.

(2) Compute the residue of $\frac{1}{z^3 \sin(z)}$ at $z = 0$.

(3) Compute $\int_0^\infty \frac{\sqrt{x}}{x^3+1}$.

(4) Compute $\int_0^\infty \frac{dx}{x^5+1}$.

(5) Compute $\int_{-\infty}^\infty \frac{\sin^2(x)dx}{x^2}$.

Solution

(1) We can first simplify the expression as

$$\int_{|z|=4} \frac{2(z-2)}{z(z-2)^2} dz = 2\int_{|z|=4} \frac{dz}{z(z-2)}.$$

By straightforward application of the residue theorem, we have

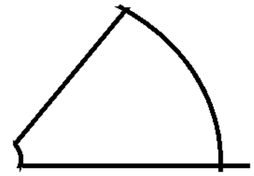
$$\begin{aligned} 2\int_{|z|=4} \frac{dz}{z(z-2)} &= 2\pi i \left(\text{Res}(\frac{1}{z(z-2)}, 0) + \text{Res}(\frac{1}{z(z-2)}, 2) \right) \\ &= 2\pi i(-1/2 + 1/2) = 0. \end{aligned}$$

(2) Since $\sin(0) = 0$ and $\sin'(0) = \cos(0) = 1 \neq 0$, it is a simple zero, hence the expression has a pole of order 4 at $z = 0$. One can then apply the residue formula, however in this case, it is easier to compute the Laurent series:

$$\begin{aligned} \frac{1}{z^3 \sin(z)} &= \frac{1}{z^3} \frac{1}{z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots} \\ &= \frac{1}{z^4} \frac{1}{1 - (\frac{z^2}{3!} - \frac{z^4}{5!} + \cdots)} \\ &= \frac{1}{z^4} \left(1 + \frac{z^2}{3!} - \frac{z^4}{5!} + O(z^4) \right) \end{aligned}$$

and since there is no $\frac{1}{z}$ term, the residue at $z = 0$ is 0.

(3) One way is to use the key hole contour. An alternative way is to use the **indented** wedge:



We need to do this because $z = 0$ is a branch point, hence not holomorphic there. The angle of the wedge is at $\theta = \frac{2\pi i}{3}$. Inside contains a simple pole at $z = e^{\pi i/3}$. Now for large $|z| = R$, we have

$$\left|\frac{\sqrt{z}}{z^3+1}\right| \leq \frac{M}{R^{5/2}}$$

and for small $|z| = r$, we have

$$\left|\frac{\sqrt{z}}{z^3+1}\right| \leq \sqrt{r}$$

hence accounting for the circumference, the integral tends to 0 as $R \to \infty$ and $r \to 0$. For the diagonal, it can be parametrized by $z = xe^{2\pi i/3}$, with x going from R to r, hence

$$\int_R^r \frac{(xe^{2\pi i/3})^{1/2} e^{2\pi i/3} dx}{x^3+1} = \int_r^R \frac{\sqrt{x}dx}{x^3+1}.$$

By residue theorem, the integral of the whole contour is $2\pi/3$. Computing by each piece, we have that the integral is $2\displaystyle\int_0^\infty \frac{\sqrt{x}dx}{x^3+1}$, hence the answer is $\pi/3$.

(4) This contour can be evaluated using a wedge since it is not a multi-valued function hence we do not need to worry about the branch points. A method is given in the take home exam, the answer is $\dfrac{\pi}{5\sin(\pi/5)}$.

(5) Integrating by parts, we have

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\sin^2(x)dx}{x^2} &= -\frac{\sin^2(x)}{x}\Bigg|_{-\infty}^{\infty} + 2\int_{-\infty}^{\infty} \frac{\sin(x)\cos(x)dx}{x} \\ &= 2\int_{-\infty}^{\infty} \frac{\sin(2x)dx}{2x} \\ &= \int_{-\infty}^{\infty} \frac{\sin(t)}{t} dt = \pi\end{aligned}$$

where we used the change of variables $t = 2x$. The boundary terms vanish by squeeze theorem since $|\sin^2(x)| \leq 1$, the last integral was done in lecture and in the practice final.