MIDTERM 1

(1) Solve the first order PDE

$$\begin{cases} -2u_x + u_y - yu = 0 \\ u(x,0) = x^2. \end{cases}$$

(2) Attempt to find a particular solution of

$$3yu_x - 2xu_y = 0.$$

Determine if there is a unique solution, no solution, or infinitely many solutions for auxiliary conditions

(a) $u(x,y) = x^2$ on the line $y = x$.
(b) $u(x,y) = 1 - x^2$ on the line $y = -x$.
(c) $u(x,y) = 2x$ on the ellipse $2x^2 + 3y^2 = 4$.

(3) Show that the general solution of the wave equation $u_{tt} - c^2 u_{xx} = 0$ is given by

$$u(x,t) = f(x - ct) + g(x + ct)$$

for sufficiently differentiable functions f and g by considering the change of variables

$$s = x - ct$$
$$r = x + ct.$$

(4) Solve the initial value problem

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{on } -\infty < x < \infty, t > 0 \\ u(x,0) = 0 \\ u_t(x,0) = \psi(x). \end{cases}$$

where

$$\psi(x) = \begin{cases} 1 & \text{on } |x| \leq 1 \\ 0 & \text{on } |x| > 1. \end{cases}$$

You may leave your solution in terms of ψ. Using this solution, answer the following

(a) Will the wave ever return to its original state at $x = 0$?
(b) What happens at each point x when $t \to \infty$?
(c) For some positive time $t > 0$, is there a location x such that $u(x,t) = 0$?

1. Solutions

(1) First we compute the characteristic curves.

$$\frac{dx}{dy} = -2.$$

Solving for this, we have

$$x = -2y + t$$

Changing variables so that

$$s = y$$
$$t = x + 2y$$

transforms the PDE to

$$u_s - su = 0.$$

This can be solved as a separable ODE so that

$$\frac{du}{u} = sds$$

so

$$\ln |u| = \frac{1}{2}s^2 + f(t)$$

where f is some function that depends on the other variable t. Exponentiating both sides yields

$$u(s,t) = C(t)e^{\frac{s^2}{2}}.$$

Plugging back x and y, we get

$$u(x,y) = C(x+2y)e^{\frac{y^2}{2}}.$$

Plugging in the auxiliary condition, we have

$$x^2 = u(x,0) = C(x),$$

which tells us what the function $C(t)$ is hence the particular solution is given by

$$u(x,y) = (x+2y)^2 e^{\frac{y^2}{2}}.$$

(2) First we find the general solution by the method of characteristic. Dividing by $3y$, we get $u_x - \frac{2x}{3y}u_y = 0$. Then we want to solve the ODE

$$\frac{dy}{dx} = -\frac{2x}{3y}.$$

This gives us

$$\frac{3}{2}y^2 = -x^2 + t.$$

For convenience, we multiply through by 2 and use the change of variables

$$s = x$$
$$t = 2x^2 + 3y^2.$$

The change of variables transforms the PDE to

$$u_s = 0,$$

hence the solution is given by

$$u(s,t) = f(t)$$

or

$$u(x,y) = f(2x^2 + 3y^2).$$

Now we check each condition.

(a) $x^2 = u(x,x) = f(5x^2)$, hence $f(s) = \frac{s}{5}$ would satisfy the initial condition so $u(x,y) = \frac{2x^2+3y^2}{5}$.

(b) $1 - x^2 = u(x,-x) = f(5x^2)$ so $f(s) = 1 - \frac{s}{5}$. Therefore the particular solution is $u(x,y) = 1 - \frac{2x^2+3y^2}{5}$.

(c) $2x = u(x,y) = f(4)$, since the right hand side is a constant and the left a variable, there does not exist an $f(s)$ that satisfies the equation, hence no solution.

(3) By applying Chain rule, the change of variables transforms the PDE to

$$u_{sr} = 0$$

This means that

$$u(s,r) = f(s) + g(r)$$

for some functions f and g. Plugging back the variables, we have

$$f(x,y) = f(x - ct) + g(x + ct).$$

(4) By d'Alembert's formula, the solution is given by

$$u(x,y) = \frac{1}{2}\int_{x-t}^{x+t} \psi(s)ds.$$

(a) No, see part b for details.

(b) Fix an arbitrary x. Then taking the limit as $t \to \infty$, we have

$$\lim_{t\to\infty} u(x,t) = \frac{1}{2}\int_{-\infty}^{\infty} \psi(s)ds = \frac{1}{2}\int_{-1}^{1} \psi(s)ds = 1.$$

So each point will converge to the height 1. Since the effect holds for any x, in particular, at $x = 0$, the height will converge to 1 and hence will not return to its initial height, $u(0,0) = 0$.

(c) Simply consider the point $x = 2$ and $t = 1$. Then

$$u(2,1) = \int_1^3 \psi(s)ds = 0.$$

In general, choosing a point $x > t + 1$ would yield 0 height. This comes from the fact that the solution to the wave equation experiences finite propagation.