MIDTERM 2

Please write your name on each page of your answer sheet and **do not fold the pages together**.

(1) Show the uniqueness to the solution of

$$\begin{cases} u_t - ku_{xx} = f(x,t) & \text{for } 0 < x < L, \quad t > 0 \\ u(x,0) = \phi(x) \\ u(0,t) = g(t) \\ u(L,t) = h(t) \end{cases}$$

for sufficiently nice functions f, g, h, ϕ. You can use whatever method you want. A useful energy for the heat equation is the L^2 energy given by $E[u](t) = \int_0^L u^2(x,t)dx$.

(2) Solve in terms of the error function

$$\begin{cases} u_t - ku_{xx} = 0 & \text{on } 0 < x < \infty, t > 0 \\ u(x,0) = 0 \\ u(0,t) = 1. \end{cases}$$

Hint: First consider $v := u - 1$. What equation does v satisfy? Then solve that equation, keeping in mind that we are solving this on the half-line. The error function is given by

$$\mathcal{E}\text{rf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2} ds.$$

(3) Solve by finding an explicit formula. Make sure to integrate out the solution of

$$\begin{cases} u_{tt} - u_{xx} = e^{2x} & \text{on } (x,t) \in \mathbb{R}^2 \\ u(x,0) = 0 \\ u_t(x,0) = 0. \end{cases}$$

Note that $\sinh(x) = \frac{e^x - e^{-x}}{2}$, $\cosh(x) = \frac{e^x + e^{-x}}{2}$ and that $(\sinh(x))' = \cosh(x)$.

(4) Let $L, T > 0$. Suppose u is twice differentiable on the open rectangle $(0, L) \times (0, T)$ and satisfies the partial differential inequality

$$u_t - u_{xx} + u < 0.$$

Suppose further that u is continuous on $R = [0, L] \times [0, T]$. If M is the maximum on of u on R and $M \geq 0$, then show that u attains the value M on the sides $x = 0$ or $x = L$ or on the bottom $t = 0$ of R. Hint: Consider the sign or value of each quantity in the partial differential inequality at a maximum point if it were to occur in the interior. No $v = u + \varepsilon$ trick is necessary for this problem. **Bonus +2** Do the same problem but with $<$ replaced with $\leq$ in the PDE. (Make sure to justify each step).

1. Solutions

(1) Let u_1 and u_2 be two solutions. Let $w = u_1 - u_2$. Then w satisfies the PDE

$$\begin{cases} w_t - kw_{xx} = 0 \quad \text{for } 0 < x < L, \quad t > 0, \\ w(x,0) = 0 \\ w(0,t) = 0 \\ w(L,t) = 0. \end{cases}$$

Method 1 By the maximum principle, we have that $w(x,t) \leq 0$ for x, t in the domain. Since $-w$ satisfies the same equation with the same initial conditions, we know that $w(x,t) \geq 0$, hence $0 = w = u_1 - u_2$, hence $u_1 = u_2$.
Method 2 Taking the derivative of the energy gives us

$$\begin{aligned} \frac{d}{dt} E[w](t) &= \int_0^L 2uu_t dx \\ &= 2k \int_0^L uu_{xx} dx \\ &= -2k \int_0^L (u_x)^2 dx \leq 0. \end{aligned}$$

Hence energy is nonincreasing. By definition, we have $0 \leq E[w](t)$. When $t = 0$, we have $E[w](0) = \int_0^L w(x,0)^2 dx = 0$, so

$$E[w](t) = \int_0^L w(x,t)^2 = 0,$$

which implies that $w(x,t) = 0$.

(2) Let $v = u - 1$. Then v satisfies

$$\begin{cases} v_t - kv_{xx} = 0 \quad \text{for } 0 < x < \infty, \quad t > 0 \\ v(x,0) = -1 \\ v(0,t) = 0. \end{cases}$$

Since $v(0,t) = 0$, we can use the odd extension of the initial condition to obtain the half-line solution:

$$v(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_0^\infty e^{-\frac{(x+y)^2}{4kt}} dy - \frac{1}{\sqrt{4k\pi t}} \int_0^\infty e^{-\frac{(x-y)^2}{4kt}} dy.$$

Changing variables, we get

$$\begin{aligned} v(x,t) &= \frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^\infty e^{-s^2} ds - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{\sqrt{4kt}}} e^{-s^2} ds \\ &= \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-s^2} ds - \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4kt}}} e^{-s^2} ds - \frac{1}{\sqrt{\pi}} \int_{-\infty}^0 e^{-s^2} ds - \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4kt}}} e^{-s^2} ds \\ &= -\frac{2}{\sqrt{\pi}} \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4kt}}} e^{-s^2} = -\mathcal{E}\mathrm{rf}\left(\frac{x}{\sqrt{4kt}}\right) \end{aligned}$$

Therefore,

$$u(x,t) = 1 - \mathcal{E}\mathrm{rf}\left(\frac{x}{\sqrt{4kt}}\right)$$

(3) Using the nonhomogeneous solution to the wave equation on the whole line, we have

$$u(x,t) = \frac{1}{2} \iint_T e^{2y} dy ds$$

where T is the two dimensional characteristic triangle. The double integral can be parametrized and computing in the following way

$$\begin{aligned}
\iint_T e^{2y} dyds &= \int_0^t \iint_{x-(t-s)}^{x+(t-s)} e^{2y} dyds \\
&= \frac{1}{2}\int_0^t \left(e^{2y}\Big|_{x-(t-s)}^{x+(t-s)} \right) ds \\
&= \frac{e^{2x}}{2}\int_0^t (e^{2(t-s)} - e^{-2(t-s)} ds \\
&= e^{2x}\int_0^t \sinh(2(t-s))ds \\
&= -\frac{e^{2x}}{2}\cosh(2(t-s))\Big|_{s=0}^{s=t} \\
&= \frac{e^{2x}}{2}(\cosh(2t)-1).
\end{aligned}$$

Hence the solution is given by

$$u(x,t) = \frac{e^{2x}}{4}(\cosh(2t)-1)$$

(4) Suppose u attains a nonnegative maximum at some point in the interior, say (x_0, t_0) and first assume $t_0 < T$. Then

$$\begin{aligned}
u(x_0,t_0) &= M \geq 0 \\
u_t(x_0,t_0) &= 0 \\
-u_{xx}(x_0,t_0) &\geq 0.
\end{aligned}$$

Plugging in to the PDE, at the maximum point we have

$$0 \leq u_t(x_0,t_0) - u_{xx}(x_0,t_0) + u(x_0,t_0) < 0$$

which is a contradiction. If the maximum occurs at the interior of $(0, L)$ but at $t = T$, then the one-sided derivative at T of u_t is $u_t(x_0, T) \geq 0$ since this is a maximum. We get the same contradiction hence the maximum cannot occur in the interior. Since R is compact, u must obtain a maximum somewhere in R, which must be the sides or the bottom.

For the bonus, consider $v_\varepsilon = v - \varepsilon t$.