MIDTERM 2 STUDY GUIDE

Study the homework problems too.

0.1. **Energy.** All of the energy problems boil down to something of this nature: $E[u](t)$ will be defined as some integral

$$E[u](t) = \int_a^b F(u, u_x, u_t)dx$$

where by $F(u, u_x, u_t)$ we mean some expression that uses those parts e.g. u^2 or $(u_x)^2 + (u_t)^2$ etc. Then one take the derivative with respect to t so that

$$\frac{d}{dt}E[u](t) = \int_a^b \frac{d}{dt}(F)dx$$

This will give you some expression involving u_t or u_{tt}. Then depending on the situation, you use the equation to substitute $u_t = ku_{xx}$ or $u_{tt} = c^2u_{xx}$ etc. This will change the inside expression (F_t) to something. Next you integrate by parts. This is where the boundary conditions will come and you can conclude things like $u = 0$ or $u \geq 0$ or the energy is decreasing etc. etc.

Example 0.1. Show the uniqueness to the solution of

$$\begin{cases} u_t - u_{xx} = f(x,t) & \text{for } 0 < x < L, t > 0 \\ u(x,0) = \phi(x) \\ u(0,t) = g(t) \\ u(L,t) = h(t). \end{cases}$$

For given functions f, g, h, ϕ. This can be done by considering the energy

$$E[w](t) = \int_0^L w(x,t)^2 dx$$

Example 0.2 (KdV equation)**.** The Korteweg-de Vries equation is given by

$$u_t + 6uu_x + u_{xxx} = 0$$

on $-\infty < x < \infty$ and $t > 0$. Suppose that u and all of its derivatives decay to 0 sufficiently rapidly as $x \to \pm\infty$. Show that the following energy

$$E[u](t) = \int_{-\infty}^{\infty} \frac{1}{2}(u_x)^2 - u^3 dx$$

is constant in time. This is equation is know as the KdV equation and is an active area of research, even today. See see KdV equations

0.2. **Maximum/Minimum Principle.** All this is saying is that the maximum and the minimum of solutions to the heat equation (on a bounded domain) occur on the parabolic boundary, i.e. $x = 0$ or $x = L$ or $t = 0$. (c.f. exercises in 2.4)

0.3. **Heat kernel.** The solution for the heat equation on the whole real line $-\infty < x < \infty$ with the initial condition $\phi(x)$, i.e.

$$\begin{cases} u_t - ku_{xx} = 0 \\ u(x,0) = \phi(x) \end{cases}$$

is given explicitly in integral form by the heat kernel:

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \phi(y)dy.$$

There are corresponding formulas for when you have non-homogenenous and half-lines

Example 0.3. Solve the following problem on the half-line

$$\begin{cases} u_t = ku_{xx} & \text{on } 0 < x < \infty, t > 0 \\ u(x,0) = -x \\ u_x(0,t) = 0, \end{cases}$$

in terms of the error function.

Example 0.4. Solve

$$\begin{cases} u_t - \frac{1}{4}u_{xx} = e^{-x} & \text{on } -\infty < x < \infty \\ u(x,0) = x^2. \end{cases}$$

Example 0.5. Suppose $u_t - u_{xx} = 0$ on the whole real line. If $u(x,0) = 1$, then what happens at each point x as $t \to \infty$? What about $u(x,0) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$?

Example 0.6. Write a formula for the solution to the problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = \sin(x), & \text{on } x, t \in \mathbb{R} \\ u(x,0) = u_t(x,0) = 0 \end{cases}$$

0.4. **Differentiating under the integral sign.** Verify directly that

$$u(x,t) = \frac{1}{2}(\phi(x+ct) + \phi(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} + \frac{1}{2c}\iint_T f$$

is the solution to the nonhomogeneous wave equation, where T is the characteristic triangle.