MIDTERM 2

Please write your name on each page of your answer sheet and **do not fold the pages together**.

(1) Show the uniqueness to the solution of

$$\begin{cases} u_t - ku_{xx} = f(x,t) & \text{for } 0 < x < L, \quad t > 0 \\ u(x,0) = \phi(x) \\ u(0,t) = g(t) \\ u(L,t) = h(t) \end{cases}$$

for sufficiently nice functions f, g, h, ϕ. You can use whatever method you want. A useful energy for the heat equation is the L^2 energy given by $E[u](t) = \int_0^L u^2(x,t)dx$.

(2) Solve in terms of the error function

$$\begin{cases} u_t - ku_{xx} = 0 & \text{on } 0 < x < \infty, t > 0 \\ u(x,0) = 0 \\ u(0,t) = 1. \end{cases}$$

Hint: First consider $v := u - 1$. What equation does v satisfy? Then solve that equation, keeping in mind that we are solving this on the half-line. The error function is given by

$$\mathcal{E}\text{rf}(s) = \frac{2}{\sqrt{\pi}} \int_0^s e^{-x^2} dx.$$

(3) Solve by finding an explicit formula. Make sure to integrate out the solution of

$$\begin{cases} u_{tt} - c^2 u_{xx} = e^{2x} & \text{on } (x,t) \in \mathbb{R}^2 \\ u(x,0) = 0 \\ u_t(x,0) = 0. \end{cases}$$

Note that $\sinh(x) = \frac{e^x - e^{-x}}{2}$, $\cosh(x) = \frac{e^x + e^{-x}}{2}$ and that $(\sinh(x))' = \cosh(x)$.

(4) Let $L, T > 0$. Suppose u is twice differentiable on the open rectangle $(0, L) \times (0, T)$ and satisfies the partial differential inequality

$$u_t - u_{xx} + u \leq 0.$$

Suppose further that u is continuous on $R = [0, L] \times [0, T]$. If M is the maximum on of u on R and $M \geq 0$, then show that u attains the value M on the sides $x = 0$ or $x = L$ or on the bottom $t = 0$ of R. Hint: Consider the sign or value of each quantity in the partial differential inequality at a maximum point if it were to occur in the interior. No $v = u + \varepsilon$ trick is necessary for this problem.