Yanlin Li

①

31

(1). Let u_1 and u_2 be the solution to the problem

Let $w = u_1 - u_2$ $\Rightarrow$ $\begin{cases} w_t - k w_{xx} = 0 \\ w(x,0) = 0 \\ w(0,t) = 0 \\ w(L,t) = 0 \end{cases}$

$E[w](t) = \int_0^L w^2(x,t)\,dx$

$$\frac{d}{dt}E[w](t) = \int_0^L \frac{d}{dt} w^2(x,t)\,dx$$

$$= \int_0^L 2w(x,t)\cdot w_t(x,t)\,dx$$

$$= \int_0^L 2w(x,t)\cdot k\,w_{xx}\,dx \qquad \begin{matrix} u \leftrightarrow w & du = w_x \\ dv = k w_{xx} & v = k w_x \end{matrix}$$

$$= \left(2w(x,t)\cdot k w_x\right)\Big|_0^L - k\int_0^L (w_x)^2\,dx$$

$w(L,t) = w(0,t) = 0$

$\therefore = 0 - k\int_0^L (w_x)^2\,dx < 0$ since $(w_x)^2 > 0$ and bounded by $(0, L)$ $k > 0$

$E[w](0) = \int_0^L w^2(x,0)\,dx = \int_0^L (0)^2\,dx = 0$

$\frac{d}{dt}E[w](t) < 0 \Rightarrow$ decreasing $\quad E[w](0) = 0$ initial point

$\Rightarrow E[w](t)$ is 0 at all points. $\Rightarrow w(x,t) = 0$

$\Rightarrow u_1 = u_2$

$\Rightarrow$ unique

10

(2) $\begin{cases} u_t - k u_{xx} = 0 \\ u(x,0) = 0 = \phi \\ u(0,t) = 1 = \psi \end{cases}$ $\quad 0 < x < \infty \quad t > 0$ $\quad v = u - 1$ $\quad \begin{cases} v_t - k v_{xx} = 0 \\ v(x,0) = -1 = \phi(x) \\ v(0,t) = 0 \end{cases}$

half-line

Yanlin Li

half-line

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}}\left[\int_0^\infty \left(e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}}\right)\phi(y)\,dy\right]$$

$$= \frac{1}{\sqrt{4\pi kt}}\left[\int_0^\infty \left(e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}}\right)(-1)\,dy\right]$$

$$= \frac{1}{\sqrt{4\pi kt}}\left[\int_0^\infty e^{-\frac{(x+y)^2}{4kt}} - e^{-\frac{(x-y)^2}{4kt}}\,dy\right]$$

$$s = \frac{x+y}{\sqrt{4kt}} \quad ds = \frac{dy}{\sqrt{4kt}} \qquad r = \frac{x-y}{\sqrt{4kt}} \quad dr = \frac{-dy}{\sqrt{4kt}}$$

$$= \frac{1}{\sqrt{\pi}}\int_{\frac{x}{\sqrt{4kt}}}^{\infty} e^{-s^2}\,ds + \frac{1}{\sqrt{\pi}}\int_{\frac{x}{\sqrt{4kt}}}^{-\infty} e^{-r^2}\,dr$$

$$= \frac{1}{\sqrt{\pi}}\int_{\frac{x}{\sqrt{4kt}}}^{\infty} e^{-s^2}\,ds + \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\frac{x}{\sqrt{4kt}}} e^{-r^2}\,dr$$

$$= \frac{1}{\sqrt{\pi}}\left(\int_0^\infty e^{-s^2}\,ds - \int_0^{\frac{x}{\sqrt{4kt}}} e^{-s^2}\,ds\right) + \frac{1}{\sqrt{\pi}}\left(\int_{-\infty}^0 e^{-r^2}\,dr + \int_0^{\frac{x}{\sqrt{4kt}}} e^{-r^2}\,dr\right)$$

$$= \frac{1}{\sqrt{\pi}}\left(\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2}\,\mathrm{Erf}\left(\frac{x}{\sqrt{4kt}}\right)\right) + \frac{1}{\sqrt{\pi}}\left(\frac{\sqrt{\pi}}{2} + \frac{\sqrt{\pi}}{2}\,\mathrm{Erf}\left(\frac{x}{\sqrt{4kt}}\right)\right)$$

$$= \cancel{\frac{1}{2} + \frac{1}{2}}$$

$$= 1$$

$$u(x,t) = v(x,t) + 1 = 2.$$

9

Yanlin Li ③

(3).

$$u(x,t) = \frac{1}{2}(\phi(x-ct)+\phi(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)\,ds + \frac{1}{2}\iint f(x,t)\,dx\,dt$$

$$\phi(x) = \psi(x) = 0$$

$$u(x,t) = \frac{1}{2}\iint e^{2x}\,dx\,dt$$

$$= \frac{1}{2}\int_0^{t_0}\int_{x_0-ct_0+ct}^{x_0+ct_0-ct} e^{2x}\,dx\,dt$$

$$= \frac{1}{2}\int_0^{t_0} e^{2x}\Big|_{x_0-c(t_0-t)}^{x_0+c(t_0-t)}\,dt$$

$$= \frac{1}{2}\int_0^{t_0} e^{2x_0+2ct_0+2ct} - e^{2x_0-2ct_0+2ct}\,dt$$

$$= \frac{e^{2x_0}}{2}\int_0^{t_0} e^{2c(t_0+t)} - e^{-(2c(t_0+t))}\,dt$$

$$= \frac{e^{2x_0}}{2}\int_0^{t_0} 2\sinh(2c(t_0+t))\,dt$$

$$= e^{2x_0}\cdot\frac{1}{2c}\cdot\cosh(2c(t_0+t))\Big|_0^{t_0}$$

$$= \frac{e^{2x_0}}{2c}\left[\cosh(4ct_0) - \cosh(2ct_0)\right]$$

9

I change

Yanlin Li ④

If maximum M were to occur in the interior, then we have

$$\begin{cases} u_t = 0 \\ u_{xx} \leq 0. \end{cases}$$

We can rewrite the inequality as

$u_t - u_{xx} \leq -u$. but $-u$ doesn't have a sign.

However, since u is continuous on R.

and have $M \geq 0$ on R, we can

set u as a positive number A.

On the left hand side,

$u_t - u_{xx} \geq 0$ because $u_t = 0$, $u_{xx} \leq 0$

However, $-u \Rightarrow -A$ is a negative number.

So, contradiction. Therefore, M cannot be interior

of R, but must be on the sides $x=0$ or $x=L$ or

on the bottom $t=0$ of R.

Daniel Connell
7251101

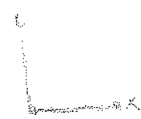
1) $\begin{cases} u_t - K u_{xx} = f(x,t), & 0<x<L, \quad t>0 \\ u(x,0) = \phi(x) \\ u(0,t) = g(t) \\ u(L,t) = h(t) \end{cases}$

$$E_{[u]}(t) = \int_0^L u^2(x,t)\,dx$$

→ Let $v(x,t)$ also be a solution.

Let $w(x,t) = u(x,t) - v(x,t)$. We want to show that $w(x,t) = 0$.

Since $w(x,t) = u(x,t) - v(x,t)$, we have $\begin{cases} w_t - K w_{xx} = 0 & \text{for } 0<x<L, \quad t>0 \\ w(x,0) = 0 \\ w(0,t) = 0 \\ w(L,t) = 0 \end{cases}$



$$E_{[w]}(t) = \int_0^L (w(x,t))^2\,dx$$

$$\frac{d}{dt} E_{[w]}(t) = 2\int_0^L w(x,t)\cdot w_t(x,t)\,dx = 2K\int_0^L w(x,t)\,w_{xx}(x,t)\,dx$$

$u = w(x,t)$ $\quad v = w_x(x,t)$
$du = w_x(x,t)dx$ $\quad dv = w_{xx}(x,t)dx$

$$= 2K\left[w(x,t)w_x(x,t)\Big|_0^L - \int_0^L (w_x(x,t))^2\,dx \right]$$

first term: $w(x,t)w_x(x,t)\Big|_0^L$
$w(L,t)w_x(L,t) - w(0,t)w_x(0,t) = 0$

$$= -2K\int_0^L (w_x(x,t))^2\,dx \leq 0 \implies \frac{d}{dt}E_{[w]}(t) \leq 0$$

$$E_{[w]}(0) = \int_0^L (w(x,0))^2\,dx = 0$$

10

Since $\frac{d}{dt}E_{[w]}(t) \leq 0$ and $E_{[w]}(0) = 0$, $w(x,t) = 0$ must be true.

Thus $0 = u(x,t) - v(x,t) \implies \boxed{u(x,t) = v(x,t)}$

Daniel Connell
7254101

19

2) $\begin{cases} u_t - ku_{xx} = 0 & 0 < x < \infty, \quad t > 0 \\ u(x,0) = \phi(x) = 0 \\ u(0,t) = 1 \end{cases}$

$$\text{Erf}(s) = \frac{2}{\sqrt{\pi}} \int_0^s e^{-x^2} dx$$

Let $v(x,t) = u(x,t) - 1$

$V_t(x,t) = U_t(x,t)$ $\quad v(x,0) = u(x,0) - 1 = -1$

$V_{xx}(x,t) = u_{xx}(x,t)$ $\quad v(0,t) = u(0,t) - 1 = 0$

$V_t - kV_{xx} = 0$

half line

$$\frac{1}{\sqrt{4\pi kt}} \Big[\underbrace{\int_0^\infty e^{-(x-y)^2/4kt}}_{T1} - \underbrace{e^{-(x+y)^2/4kt}}_{T2} \Big] \phi(y)\,dy$$

Let $p = \frac{x-y}{\sqrt{4kt}} \Rightarrow p\sqrt{4kt} = x - y \Rightarrow y = x - p\sqrt{4kt}$

$dy = -\sqrt{4kt}\,dp$

$y = 0 \to p = \frac{x}{\sqrt{4kt}}$

$y = \infty \to p = -\infty$

$\boxed{T1}$

$$-\sqrt{4kt} \int_{\frac{x}{\sqrt{4kt}}}^{-\infty} e^{-p^2} dp = \sqrt{4kt} \int_{-\infty}^{x/\sqrt{4kt}} e^{-p^2} dp$$

$\boxed{T2}$ Let $q = \frac{x+y}{\sqrt{4kt}}$ $\quad q\sqrt{4kt} = x + y \Rightarrow y = q\sqrt{4kt} - x$

$dy = \sqrt{4kt}\,dq$

$y = 0 \to q = \frac{x}{\sqrt{4kt}}$

$y = \infty \to q = \infty$

$$\sqrt{4kt} \int_{x/\sqrt{4kt}}^{\infty} e^{-q^2} dq$$

$$\frac{1}{\sqrt{4\pi kt}} \left[\sqrt{4kt} \left(\int_{-\infty}^{x/\sqrt{4kt}} e^{-p^2} dp - \int_{x/\sqrt{4kt}}^{\infty} e^{-q^2} dq \right) \right]$$

$$= \frac{1}{\sqrt{\pi}} \left[\int_{-\infty}^{0} e^{-p^2} dp + \int_0^{x/\sqrt{4kt}} e^{-p^2} dp + \int_\infty^0 e^{-q^2} dq + \int_0^{x/\sqrt{4kt}} e^{-q^2} dq \right]$$

$$= \frac{1}{2} \text{erf}\left(\frac{x}{\sqrt{4kt}}\right) + \frac{1}{2} \text{erf}\left(\frac{x}{\sqrt{4kt}}\right) = \text{erf}\left(\frac{x}{\sqrt{4kt}}\right)$$

Continued →

Daniel Connell

9

2) (cont)

$$v(x,t) = \mathrm{erf}\left(\frac{x}{\sqrt{4kt}}\right)$$

$$u(x,t) = -\mathrm{erf}\left(\frac{x}{\sqrt{4kt}}\right) + 1$$

3) $\begin{cases} u_{tt} - c^2 u_{xx} = e^{2x} \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}$

$\int u_{tt} = \int c^2 u_{xx} e^{2x} \, dx$ $\quad u = e^{2x}$ $\quad du = 2e^{2x}$ $\quad v = c^2 u_x$ $\quad dv = c^2 u_{xx} \, dx$

$u_t = e^{2x} u_x - 2\int_{-\infty}^{\infty} u_x e^{2x} \, dx$ $\quad u = e^{2x}$ $\quad du = 2e^{2x}$ $\quad v = u$ $\quad dv = u_x$

$= e^{2x} u_x - 2$

1.) Let u_1 and u_2 be Solutions to $u_t - ku_{xx} = f(x,t)$

Let $w = u_1 - u_2$

Silvino Reyes Farha

So, $\begin{cases} w_t - kw_{xx} = 0, & 0 < x < L, \ t > 0 \\ w(x,0) = 0 \\ u(0,t) = 0 \\ u(L,t) = 0 \end{cases}$

30

because its a square term

Let $E[u](t) = \int_0^L u^2(x,t)\,dx \geq 0$

$$\frac{dE}{dt} = \int_0^L 2ww_t\,dx$$

$u = w \qquad v = w_x$

$du = w_x dx \qquad dv = w_{xx}\,dx$

$$= \int_0^L 2kw_{xx}w\,dx$$

w_{xx}

$$= \cancel{ww_x}\Big|_0^L \xrightarrow{} 0 \;-\; \int_0^L w_x^2\,dx \leq 0$$

$W(L,t)\,w$

which means energy is decreasing.

So, $0 \leq E \leq \int_0^L w^2(x,0)\,dx = 0$

$\therefore \quad w = 0 \quad$ and $\quad u_1 = u_2 \quad \square$

10

$\left(\begin{array}{c} w = 0 \\ u_1 - u_2 = 0 \\ u_1 = u_2 \end{array} \right)$

Silvano Reyes Zamora

2.) $\begin{cases} u_t - k u_{xx} = 0 & 0 < x < \infty, \ t > 0 \\ u(x,0) = 0 \\ u(0,t) = 1 \end{cases}$

$v = -1$

v_t

Let $v = u - 1$

So, $\begin{cases} v_t - k v_{xx} = 0 & 0 < x < \infty, \ t > 0 \\ v(x,0) = -1 \\ v(0,t) = 0 \end{cases}$

Let $s = \frac{x-y}{\sqrt{4kt}}$

$ds = \frac{-1}{\sqrt{4kt}} dy$

$$V(x,t) = \int_0^\infty S(x-y,t)\phi(y)\,dy - \int_0^\infty S(x+y,t)\phi(y)\,dy$$

$$= -\int_0^\infty \frac{1}{\sqrt{4k\pi t}} e^{-(x-y)^2/4kt}\,dy + \int_0^\infty \frac{1}{\sqrt{4k\pi t}} e^{-(x+y)^2/4kt}\,dy$$

$$= \frac{1}{\sqrt{\pi}} \int_{x/\sqrt{4kt}}^{-\infty} e^{-s^2}\,ds + \frac{1}{\sqrt{\pi}} \int_{-x/\sqrt{4kt}}^{\infty} e^{-r^2}\,ds$$

$$= -\frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-s^2}\,ds - \frac{1}{\sqrt{\pi}}\int_{-\infty}^{0} e^{-s^2}\,ds + \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-r^2}\,dr - \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-r^2}\,dr$$

$$-a \quad = -\frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-s^2}\,ds - \frac{1}{\sqrt{\pi}}\int_{-\infty}^{0} e^{-s^2}\,ds + \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-r^2}\,dr - \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-r^2}\,dr$$

$$= -\frac{1}{2} \operatorname{Erf}(x/\sqrt{4kt}) - \frac{1}{2} + \frac{1}{2} - \frac{1}{2}\operatorname{Erf}(x/\sqrt{4kt})$$

$$= -\operatorname{Erf}(x/\sqrt{4kt})$$

$$\boxed{u(x,t) = 1 - \operatorname{Erf}(x/\sqrt{4kt})}$$

✓ 10

3.) $\begin{cases} u_{tt} - c^2 u_{xx} = e^{2x} & , (x,t) \in \mathbb{R}^2 \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}$

Silvino Reyes Farina

$$u = \frac{1}{2c}\int_0^t \int_{x-ct}^{x+ct} f(y,s)\, dy\, ds$$

$2x + 2ct - 2cs$

$2x - 2ct + 2cs$

$$= \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} e^{2y}\, dy\, ds$$

$$= \frac{1}{2c}\int_0^t \frac{1}{2} e^{2y} \Big|_{x-ct+cs}^{x+ct-cs} ds$$

$$= \frac{1}{2c}\int_0^t \frac{1}{2} e^{2(x+ct-cs)} - \frac{1}{2} e^{2(x-ct+cs)}\, ds$$

$u = 2cs$

$du = -2c\, ds$

$$= -\frac{1}{2c} e^{2x-2ct} \int_0^t \frac{-e^{-2cs} + e^{2cs}}{2}\, ds$$

$$= -\frac{1}{4c^2} e^{2x-2ct} \int_0^{2ct} \underbrace{\frac{-e^{-u} + e^{u}}{2}}_{\sinh(u)} du$$

$$= -\frac{1}{4c^2} e^{2x-2ct} \Big[\cosh(u)\Big]_0^{2ct}$$

$$\boxed{= -\frac{1}{4c^2} e^{\cancel{2x-2ct}} \Big[\cosh(2ct) - 1\Big]}$$

9

Silvino Reyes Farias

4.) Let $L, T > 0$

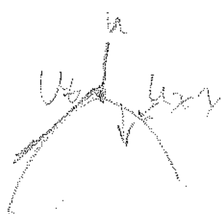
(*) $u_t - u_{xx} + u \leq 0$ $\qquad R = [0, L] \times [0, T]$

Suppose $M \geq 0$ is the max of (*)

If M is on the <u>inside</u>, then

$u_t = 0$ $\qquad u \geq 0$

$u_{xx} \leq 0$



So, $\overset{0}{u_t} - \overset{\geq 0}{u_{xx}} + \overset{\geq 0}{u} \geq 0$

which is a contradiction to

$\therefore$ M is on $x = L$, $x = 0$, or $t = 0$ $\square$

10

Midterm 2

Daniel C.
7424633

24

(1) Let $W = U_1 - U_2$ where U_1 and U_2 are solutions to the diffusion equation. Then multiplying W by itself we get: $0 = 0 \cdot W = (W_t - kW_{xx})(W) = (\frac{1}{2}W^2)_t - (2kW)_x + (\frac{1}{2}kW_x^2)_x$

Integrating over the interval $0 < x < L$ we get:

$$E[W](t) = \int_0^L (\tfrac{1}{2}W^2)_t - (2kW)_x + (\tfrac{1}{2}kW_x^2)_x \, dx$$

Taking the time derivative we arrive at (where the middle disappears because of the boundary conditions @ $x = L, 0$ and $t > 0$)

$$\frac{d}{dt}E[W](t) = \int_0^L W^2 \, dx + 2\int_0^L (kW_x)^2 \, dx = 0$$

So then we have the inequality

$$\int_0^L W^2(x,t) \, dx \leq -2k\int_0^L W(x,0)^2 \, dx \leq 0$$

Energy is decreasing as $t \to 0$. The right side disappears because both U_1 and U_2 have the same initial conditions. Thus,

$$\int_0^L W^2(x,t) = 0.$$

Energy decreasing as well as $E(0) = 0$ implies that $W = 0$ and $U_1 = U_2$, for all $t > 0$ on $0 < x < L$. Thus we have uniqueness.

6

(2) Let $v = u - 1$, So that $\begin{cases} v_t - k v_{xx} = 0 & \text{on } 0 < x < \infty,\ t > 0 \\ v(x,0) = -1 = \phi(x) \\ v(0,t) = 0 \end{cases}$

Now we can do an odd extension with $\phi(y)_{odd}$. Using the general formula, we get for the diffusion equation:

$$\frac{1}{\sqrt{4k\pi t}} \int_0^\infty e^{\frac{-(x-y)^2}{4kt}} \cdot \phi(y)_{odd} - \frac{1}{\sqrt{4k\pi t}} \int_0^\infty e^{-\frac{(x+y)^2}{4kt}} \cdot \phi(y)_{odd}$$

$\phi(y)_{odd} = -1$ so $$\underbrace{\frac{-1}{\sqrt{4k\pi t}} \int_0^\infty e^{\frac{-(x-y)^2}{4kt}} dy}_{(a)} + \underbrace{\frac{1}{\sqrt{4k\pi t}} \int_0^\infty e^{\frac{-(x+y)^2}{4kt}} dy}_{(b)}$$

let $P = (x-y)/\sqrt{4kt}$

Then, $x - \sqrt{4kt}\, P = y$

$-\sqrt{4kt}\, dp = dy$

Plugging in we get

$$\frac{-1}{\sqrt{4k\pi t}} \int_{x/\sqrt{4kt}}^{-\infty} e^{-p^2} \cdot -\sqrt{4kt}\, dp$$

$$= \frac{-1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{\sqrt{4kt}}} e^{-p^2} dp$$

(b) using the same method as (a) we get $$\frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^\infty e^{-q^2} dq$$

The Error function is $Erf(s) = \frac{2}{\sqrt{\pi}} \int_0^s e^{-x^2} dx$

So $Erf(s)/2$ will get us our form.

$$\boxed{v(x,t) = -\frac{1}{2} Erf(x,t) - \frac{1}{2} Erf(x,t)}$$

8

Daniel S.
7424633

(3) general formula is $u(x,t) = \frac{1}{2}[\phi_{even}(x+ct) + \phi_{even}(x-ct)] + \int_{x-ct}^{x+ct}\psi(y)\,dy + \iint_{\Delta} f$

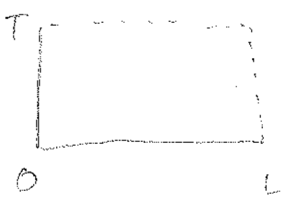
{general soln} + {particular soln}. Let's solve for the particular soln first.

$\int_0^t \int_{-\infty}^{\infty} e^{2x}\,dx\,dt$ let $u = 2x$ $du = 2\,dx$

$\int_0^t \int_{-\infty}^{\infty} e^{u} \cdot \frac{1}{2}\,du\,dt = \int_0^t \frac{\sqrt{\pi}}{2}\,dt$

$u(x,t) = \cosh(x+ct)$

3

(4) 

3

$u(x,t)_t - u(x,t)_{xx} + u(x,t) \le 0$

@ $x=0$

$u(0,t)_t - u(0,t) + u(0,t)$
$\downarrow$
0

So the sign is positive $\therefore M \ge 0$

@ $x = L$ $M \ge 0$

@ $t=0$ $M \ge 0$

in the interior we have M is negative but this is a contradiction

?

③ $\begin{cases} u_{tt} - c^2 u_{xx} = e^{2x}. \\ u(x,0) = 0. \\ u_t(x,0) = 0. \end{cases}$

chaohong cai

33

~~First we find a particular solution $u(x,t)$~~

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k t}} e^{-\frac{(x-y)^2}{4kt}} \phi(y)\,dy + \iint_T \frac{1}{\sqrt{4\pi k t}} e^{-\frac{(x-y)^2}{4kt}} f(y,s)\,dy\,ds .$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k t}} e^{-\frac{(x-y)^2}{4kt}} \cdot 0\,dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k t}} e^{-\frac{(x-y)^2}{4kt}} \cdot e^{2y}\,dy\,ds$$

~~$u(x,t) = \frac{1}{4c} e^{2x}$~~ .

7

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k t}} e^{-\frac{(x-y)^2}{4kt}} e^{2y}\,dy\,ds$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k t}} e^{-\frac{(x-y)^2}{4kt} + 2y}\,dy\,ds$$

$$= \iint \frac{1}{\sqrt{4\pi k t}} e^{\frac{-(x^2+y^2-2xy)+8kty}{4kt}}\,dy\,ds$$

3

$$e^{-(x^2+y^2-2xy)+8kty} = e^{-(x^2+y^2-2y(x+4kt))} = e^{-(x-y(x+4kt))^2 - \cdots}$$

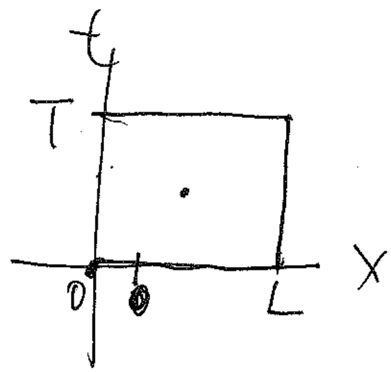
Let $x - y(4kt+x) = P$,

$= \frac{1}{2}\,\mathrm{erf}(\ \ )$

No time T.T. I can do it!

Chaohong Cai.

④ Let $V = u + \varepsilon$, $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$.



$V_t = u_t$, $V_{xx} = u_{xx}$, $u = V - \varepsilon$.

so, $V_t - V_{xx} + V - \varepsilon \le 0$,

$V_{max} = U_{max} + \varepsilon = M + \varepsilon$,

Suppose U_{max} takes place at (x_0, t_0) interior

$u(x_0, t_0) = M$, $V(x_0, t_0) = M + \varepsilon$.

17

$V_t(x_0, t_0) - V_{xx}(x_0, t_0) + V(x_0, t_0) - \varepsilon \le 0$.

However, if M is taken interior, maximum means the derivative is 0

then $V_t = 0$, $V_{xx} = 0$.,

Then $M \le 0$, contradiction.

So, M must take place on the boundary

chaohong cai.

(1) $\begin{cases} u_t - ku_{xx} = f(x,t) & 0<x<L,\ t>0 \\ u(x,0) = \phi(x) \\ u(0,t) = g(t) \\ u(L,t) = h(t) \end{cases}$

Suppose there're 2 solutions, u_1, u_2.

Let $W = u_1 - u_2$,

then $\begin{cases} W_t - kW_{xx} = 0 \\ W(x,0) = 0 \\ W(0,t) = 0 \\ W(L,t) = 0 \end{cases}$

According to the maximum principle,

the max of W is 0.

So, $W(x,t) \le 0$.

Let $W = u_2 - u_1$.

$\begin{cases} W_t - kW_{xx} = 0 \\ W(x,0) = 0 \\ W(0,t) = 0 \\ W(L,t) = 0. \end{cases}$, same, According to the max principle

$W \le 0$.

Hence, $\begin{matrix} u_1 - u_2 \le 0, \\ u_2 - u_1 \le 0, \end{matrix} \Rightarrow u_1 = u_2 \Rightarrow W = 0.$

Therefore, solution is unique.

10

~~② Use the odd extension.~~

$$\cancel{V(x,t)=\int_{-\infty}^{\infty} H(x-y,t)\,\phi_{odd}(y)\,dy}$$

② Let $V=U-1$,

then $\begin{cases} V_t - kV_{xx} = 0, & 0<x<\infty,\ t>0. \\ V(x,0) = -1. \\ V(0,t)=0. \end{cases}$, $\phi_{odd}(x) = \begin{cases} -1, & x>0 \\ \cancel{-1}, & x=0. \\ \cancel{-}1 & x<0 \end{cases}$

Use the odd extensions,

$$V(x,t) = \int_{-\infty}^{\infty} H(x-y,t)\,\phi_{odd}(y)\,dy$$

$$\cancel{= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} \cdot e^{\frac{-(x-y)^2}{4kt}} (-1)\,dy}$$

$$= \int_{0}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{\frac{-(x-y)^2}{4kt}} \cdot (-1)\,dy + \int_{-\infty}^{0} \frac{1}{\sqrt{4\pi kt}} e^{\frac{-(x-y)^2}{4kt}} (-1)\,dy.$$

$$\cancel{\int_0^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-(x-y)^2} = \int_0^{\infty} \frac{1}{\sqrt{4\pi kt}} \left(e^{\frac{(x-y)^2}{4kt}} - e^{\frac{-(x+y)^2}{4kt}}\right) dy}$$

$$\cancel{= \frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} \left(e^{\frac{(x-y)^2}{4kt}} - e^{\frac{-(x+y)^2}{4kt}}\right) dy}$$

Let $p = \frac{x-y}{\sqrt{4kt}}$, then $\int_0^{\infty} \frac{1}{\sqrt{4\pi kt}} \cdot e^{\frac{-(x-y)^2}{4kt}} (-1)\,dy$, $y \in (0,+\infty)$, ~~$t-y\in(x, x+\infty)$~~.

$dp = \frac{-1}{\sqrt{4kt}}\,dy$. $\quad = \int_{\frac{x}{\sqrt{4kt}}}^{-\infty} \frac{1}{\sqrt{\pi}} e^{-p^2}\,dp$.

$$\cancel{\int_0^{\frac{x}{\sqrt{4kt}}} \frac{1}{\sqrt{\pi}} e^{-p^2}\,dp + \int_{\frac{-x}{\sqrt{4kt}}}^{\infty} \frac{1}{\sqrt{\pi}} e^{-p^2}\,dp}$$

Following ②.   chaohong cai

$$\int_{\frac{x}{\sqrt{4kt}}}^{-\infty} \frac{1}{\sqrt{\pi}} e^{-p^2}\, dp$$

Let $p = -p$,

then $= -\int_{\frac{-x}{\sqrt{4kt}}}^{\infty} \frac{1}{\sqrt{\pi}} e^{-p^2}\, dp$.

$$= -\left(\int_{\frac{-x}{\sqrt{4kt}}}^{0} \frac{1}{\sqrt{\pi}} e^{-p^2}\, dp + \int_{0}^{\infty} \frac{1}{\sqrt{\pi}} e^{-p^2}\, dp \right).$$

$$= -\left(\frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{x}{\sqrt{4kt}}\right) \right),$$

$$\int_{-\infty}^{0} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-y)^2}{4kt}} (-1)\, dy$$

Let $q_e = \frac{x-y}{\sqrt{4kt}}$, $\int_{-\infty}^{0} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-y)^2}{4kt}} (-1)\, dy = \int_{\infty}^{\frac{-x}{\sqrt{4kt}}} \frac{1}{\sqrt{\pi}} e^{-q_e^2} (-1)\, dq_e$.

$$= \int_{\frac{-x}{\sqrt{4kt}}}^{\infty} \frac{1}{\sqrt{\pi}} e^{-q_e^2}\, dq_e$$

$$= \int_{\frac{-x}{\sqrt{4kt}}}^{0} \frac{1}{\sqrt{\pi}} e^{-q_e^2}\, dq_e + \int_{0}^{\infty} \frac{1}{\sqrt{\pi}} e^{-q_e^2}\, dq_e$$

$$= \frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{-x}{\sqrt{4kt}}\right).$$

Therefore, $v(x,t) = \frac{1}{2}\mathrm{erf}\left(\frac{-x}{\sqrt{4kt}}\right) - \frac{1}{2}\mathrm{erf}\left(\frac{x}{\sqrt{4kt}}\right)$.

$$u(x,t) = v(x,t) + 1 = \frac{1}{2}\mathrm{erf}\left(\frac{-x}{\sqrt{4kt}}\right) - \frac{1}{2}\mathrm{erf}\left(\frac{x}{\sqrt{4kt}}\right) + 1$$

Elizabeth Gutierrez-Lopez

31

1.) Show uniqueness of solution of

$$\begin{cases} u_t - Ku_{xx} = f(x,t) & \text{for } 0<x<L,\ t>0 \\ u(x,0) = \phi(x) \\ u(0,t) = g(t) \\ u(L,t) = h(t) \end{cases}$$

let $w = u - v$ and suppose u and v are two solutions to the heat Equation above.

$u_t - v_t = k(u_{xx} - v_{xx})$

$(u_t = k\, v_{xx})$

notice $w_{xx} = u_{xx} - v_{xx}$

$w_t = u_t - v_t$

so w solves $w_t - kw_{xx} = f(x,t) - f(x,t) = 0$

$w(x,0) = 0$

$w(0,t) = 0$

$w(L,t) = h(t)$

now consider $E(w) = \int_0^L w^2(x,t)\,dx$

$\frac{d}{dt}E(w) = \int_0^L \frac{d}{dt}[w^2]\,dx$

$\frac{d}{dt}E(w) = \int_0^L 2w\,w_t\,dx \qquad w_t = Kw_{xx}$

$\frac{d}{dt}E(w) = 2k\int_0^L w\,w_{xx}\,dx$

$u = w \qquad dv = w_{xx}$

$du = w_x \qquad v = w_x$

$= 2k\left[\underbrace{w w_x \Big|_0^L}_{0} - \int_0^L (w_x w_x)\,dx \right]$

$\frac{d}{dt}E(w) = -2k\int_0^L (w_x)^2\,dx \le 0 \Rightarrow$

So $\cancel{E(w)} \le 0$ but consider $E(w) = \int_0^L w^2(x,t)\,dx$ this must be ≥ 0

Thus, $0 \le E(w) \le 0$; This means $E(w) = 0$

but this $E(w) = 0$ is only possible if $w = 0$ b/c recall $E(w) = \int_0^L \underbrace{w^2(x,t)}_{0}\,dx$, in order for it to be zero inside must be zero.

- Hence, since $w = 0$ then $w = u - v = 0 \Rightarrow u = v$

So solution is unique.

9

2. Solve in terms of the error function

Elizabeth Gutierrez-Igreet

$$\begin{cases} u_t - K u_{xx} = 0 & 0 < x < \infty \quad t > 0 \text{ (half line)} \\ u(x,0) = 0 \\ u(0,t) = 1 \end{cases}$$

1 - Erf

Let $V := u - 1$ Since $V_t = u_t$, $V_{xx} = u_{xx}$

V satisfies
$$\begin{cases} V_t - K(V_{xx}) = u_t - Ku_{xx} = 0 \\ V(x,0) = -1 \\ V(0,t) = 0 \to \text{odd extension} \end{cases}$$

$$= \frac{1}{\sqrt{4\pi Kt}} \int_0^\infty \left(e^{-\frac{(x-y)^2}{4Kt}} - e^{-\frac{(x+y)^2}{4Kt}}\right)(-1)\,dy$$

$$= \frac{-1}{\sqrt{4\pi Kt}} \int_0^\infty e^{-\frac{(x-y)^2}{4Kt}}\,dy - \left[\frac{-1}{\sqrt{4\pi Kt}} \int_0^\infty e^{-\frac{(x+y)^2}{4Kt}}\,dy\right]$$

1. change variables: $S = \frac{x-y}{\sqrt{4Kt}}$ $-\sqrt{4Kt}\,ds = dy$

2. $- \left[\frac{-1}{\sqrt{4\pi Kt}} \int_0^\infty e^{-\frac{(x+y)^2}{4Kt}}\,dy\right]$

$S = x+y/\sqrt{4Kt}$ change variables

$\sqrt{4Kt}\,dS = dy$

$$+\frac{1}{\sqrt{\pi}} \int_{-\infty}^{x/\sqrt{4Kt}} -e^{-s^2}\,ds$$

$$- \frac{1}{\sqrt{\pi}} \int_{x/\sqrt{4Kt}}^\infty e^{-s^2}\,ds$$

$$= \frac{-1}{\sqrt{\pi}} \int_{-\infty}^{x/\sqrt{4\pi K}} e^{-s^2}\,ds$$

$$\frac{-1}{\sqrt{\pi}}\left[\int_{-\infty}^{0} e^{-s^2}\,ds - \int_0^{-x/\sqrt{4\pi Kt}} e^{-s^2}\,ds\right]$$

$$-\frac{1}{\sqrt{\pi}}\left[\int_0^\infty e^{-s^2}\,ds - \int_0^{x/\sqrt{4Kt}} e^{-s^2}\,ds\right]$$

$$\frac{-1}{\sqrt{\pi}}\left[\frac{\sqrt{\pi}}{2}\right] + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4\pi Kt}} e^{-s^2}\,ds$$

2. $-\frac{1}{2} + \frac{1}{2}\,\mathrm{Erf}(x/\sqrt{4Kt})$

$-1/2 + \frac{1}{2}\,\mathrm{Erf}(x/\sqrt{4Kt})$

$$-\left[-\frac{1}{2} + \frac{1}{2}\,\mathrm{Erf}(x/\sqrt{4Kt})\right]$$

$V = -\frac{1}{2} - \frac{1}{2}\,\mathrm{Erf}(x/\sqrt{4Kt})$

$V = -1\,\mathrm{Erf}(x/\sqrt{4Kt})$

$u = V + 1$

$$\boxed{u(x,t) = 1 - \mathrm{Erf}(x/\sqrt{4Kt})}$$

10

Elizabeth Gutierrez-Lopez

3. Solve by finding explicit formula. integrate out solution Recall this:

$$\begin{cases} u_{tt} - c^2 u_{xx} = e^{2x} \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \quad \cosh(x) = e^x + e^{-x}$$

$$[\sinh(x)]' = \cosh(x)$$

$$\underbrace{\frac{1}{2}\left(\phi(x+ct) + \phi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)}_{0} + \frac{1}{2c}\int_0^t \int_{x_0 - c(t_0-t)}^{x_0+c(t_0-t)} e^{2x}\, dx\, dt$$

$$= \frac{1}{2c} \int_0^t \int_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} e^{(2x)}\, dx\, dt$$

$$\int_0^t \left[\frac{1}{2}e^{2x}\right]_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} dt$$

$$\int_0^t \frac{1}{2}e^{2(x_0+c(t_0-t))} - \frac{1}{2}e^{2(x_0-c(t_0-t))}\, dt$$

$$\int_0^t \frac{1}{2}e^{2x_0+2ct_0-2ct} - \frac{1}{2}e^{2x_0-2ct_0+2ct}$$

$$e^{2x_0+2ct_0}\int_0^t \frac{1}{2}e^{-2ct} - e^{2x_0-2ct_0}\int \frac{1}{2}e^{2ct}$$

?

$$e^{2x_0+2ct_0} - e^{2x_0-2ct_0} \int_0^t \frac{1}{2}\left(e^{-2ct} + e^{2ct}\right) dt$$

$$\int_0^t \cosh(2ct)\, dt$$

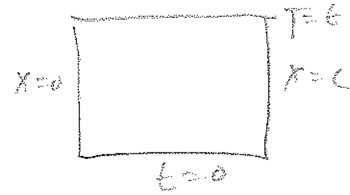
$$\boxed{e^{2x_0+2ct_0} \cdot -e^{2x_0-2ct_0} \cdot \left[\frac{1}{2ct}\sin(2ct)\right]}$$

8

Elizabeth Gutierrez-Lopez

4. Let $T > 0$. Suppose u is twice differentiable on open rectangle $(0,L) \times (0,T)$ and satisfies

$u_t - u_{xx} + u \leq 0$ $\Rightarrow$ ~~$u_t - u_{xx} \leq -u$~~



u continuous on $R = [0,L] \times [0,T]$

M max of u on R and $M \geq 0$. Show u attains value of M on sides $x=0$ or $x=L$ or $t=0$ of R

if u attains max at interior point (x_0, T)

$u_t(x_0, T) = 0$

$u_{xx}(x_0, T) \leq 0$

$u_t(x_0, T) - u_{xx}(x_0, T) + u \leq 0$

$\quad 0 \qquad\qquad \geq 0 \quad + u \geq 0$ So this is a contradiction

now consider u attains max at interior point (x_0, t_0)

$u_t(x_0, t_0) = 0$

$u_{xx}(x_0, t_0) < 0$

$u_t(x_0, t_0) - u_{xx}(x_0, t_0) + u \leq 0$

$\quad 0 \qquad\qquad < 0 \quad + u \geq 0$ So not ≤ 0 contradiction

Claudia Tsau
ID 3277134

22

$E[w](t) = \int_0^L u^2(x,t)\,dx$.

non-homogeneous diffusion on the half line.

① $\begin{cases} u_t - k u_{xx} = f(x,t) & \text{to } 0<x<L \quad t>0 \\ u(x,0) = \phi(x) \\ u(0,t) = g(t) \\ u(L,t) = h(t) \end{cases}$

let u_1 + u_2 be solutions of the PDE.

let $w = u_1 - u_2$.

First $f(x,t) = f(x,t)\,w = w(w_t - k w_{xx})$

$= \frac{1}{2}(w^2)_t + (-k w_x w)_x + k w_x^2$

integrate from 0 to L $\Rightarrow \frac{1}{2}\int_0^L w_t^2 - \cancel{k w_x w}\Big|_0^L + k\int_0^L w_x^2$

$k\int_0^L [w_t(x,t)]^2 = -k\int_0^L w_x(x,t)^2 + f(x,t)$.

since the solutions is controlled completely by boundary and initial conditions.

$w_t(x,t) \le w_x(x,0) + f(x,t)$

$w(x,t) = 0$

$\Rightarrow u_1 = u_2$ ☑

Why

7

5

Diffusion Homogeneous, Half Line

(2) $\begin{cases} u_t - k u_{xx} = 0 & 0 < x < \infty \quad t > 0. \\ u(x,0) = 0 = \phi(y) \\ u(0,t) = 1 \end{cases}$

$\begin{cases} v_t - k v_{xx} = 0 & 0 < x < \infty \quad t > 0 \\ v(x,0) = 1 \\ v(0,t) = 2 \end{cases}$

No

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_0^\infty \left[e^{-\frac{(x-y)}{4kt}} - e^{-\frac{(x+y)}{4kt}} \right] \phi(y)\, dy$$

(A) $\frac{1}{\sqrt{4\pi kt}} \int \left[e^{-\frac{(x-y)^2}{4kt}} \cancel{\phi(y)} \right]$ (B) $\frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x+y)}{4kt}} \cancel{\phi(y)}\, dy.$

$\boxed{s = \frac{x-y}{\sqrt{4kt}}}$

$ds = \frac{-1}{\sqrt{4kt}} dy$

$\sqrt{4kt}\, ds = dy$

$\boxed{u = \frac{x+y}{\sqrt{4kt}}}$

$du = \frac{1}{\sqrt{4kt}} dy$

$\sqrt{4kt}\, du = dy.$

$-\frac{1}{\sqrt{\pi}} \int e^{-u^2} \cancel{\phi(y)}\, du$

$\mathrm{Erf}(s) = \frac{2}{\sqrt{\pi}} \int_0^s e^{-x^2} dx.$

$-\frac{1}{\sqrt{\pi}} \int e^{-s^2} \cancel{\phi(y)}\, ds$

$-\frac{1}{\sqrt{\pi}} \int_0^{\frac{x-y}{\sqrt{4kt}}} e^{-s^2} - \frac{1}{\sqrt{\pi}} \int_0^{\frac{x+y}{\sqrt{4kt}}} e^{-u^2} du$

$-2\left[\mathrm{Erf}\left(\frac{x-y}{\sqrt{4kt}}\right) - \mathrm{Erf}\left(\frac{x+y}{\sqrt{4kt}}\right) \right]$

$\Rightarrow$ Since $u+1 : v$ is also a solution of u then this follows

6

CLAUDIA TSAU

$* \; c = \sqrt{T/\rho}$ speed

$$\left| \frac{e^{2x} e^{ct} e^{-cs}}{e^{cs}} - \frac{e^{2x} e^{cs}}{e^{ct}} \right.$$

$$\left| e^{2x}\left[\frac{e^{ct}}{e^{cs}} - \frac{e^{cs}}{e^{ct}} \right] \right.$$

wave nonhomo. whole line

③ $\begin{cases} u_{tt} - c^2 u_{xx} = e^{2x} & (x,t) \in \mathbb{R}^2 \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}$

$$\tfrac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \tfrac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

$$+ \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds \quad + h(t - \tfrac{x}{c})$$

$$0 + 0 + \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} e^{2y}\,dy\,ds \quad + 0$$

$$\Rightarrow \int_0^t \int_{x-ct+cs}^{x+ct-cs} e^{2y}\,dy\,ds. \qquad u = 2y \quad du = 2dy \quad \tfrac{1}{2}du = dy.$$

$$= \tfrac{1}{2}\int_0^t \int e^{u}\,dy\,ds = \tfrac{1}{2}\int_0^t e^{2(x+ct-cs)} - e^{2(x-ct+cs)}\,ds$$

$$= \tfrac{1}{2}\int_0^t e^{2x+2ct-2cs}\,ds - \tfrac{1}{2}\int_0^t e^{2x-2ct+2cs}\,ds.$$

$u = -2cs$
$du = -2c\,ds$
$\frac{1}{2c}du = ds$

8

$v = 2u \quad dv = 2du$

$$\tfrac{1}{4c}\int_0^t e^{2x+2ct} e^{2u}\,du \qquad -\tfrac{1}{4c}\int_0^t e^{2x-2ct} e^{2u}\,du$$

$u = 2cs$
$du = 2c\,ds$
$\frac{1}{2c}du = ds$
$v = 2u$

$$\tfrac{1}{2c}\int_0^t e^{2x+2ct} e^{v}\,dv \qquad -\tfrac{1}{2c}\int_0^t e^{2x-2ct} e^{v}\,dv$$

$$\tfrac{1}{2c}\left[e^{2x+2ct}e^{t} - e^{2x+2ct}\right] \quad -\tfrac{1}{2c}\left[e^{(2x-2ct)}e^{t} - e^{(2x-2ct)}\right]$$

$$\tfrac{1}{2c}\left[e^{2x+2ct+t} - e^{2x+2ct} - e^{2x-2ct+t} + e^{2x-2ct}\right]$$

$$\tfrac{1}{2c}\left[e^{2x+2ct}[e^{t}-1] - e^{2x-2ct}[e^{t}-1]\right]$$

$$= \tfrac{1}{2c}\left[\cancel{[e^{t}-1]}\left[e^{2x+2ct} - e^{2x-2ct}\right]\right]$$

(4) $u_t - u_{xx} + u \leq 0$

u cont. $R = [0,L] \times [0,T]$.

$M = \max\limits_{0<x<L} u$

$M \geq 0$

Since we know that $M \geq 0$ and is a max on u.

on $R\,[(0,L) \times (0,T]]$.

If $u_t - u_{xx} + u \leq 0$. $\Rightarrow$ $M \geq 0 \geq u(x,t)$

$\geq u(x,0) = 0$

$u_t - u_{xx} \leq -u$

$u_{xx} - u_t \geq u$

$u_{xx} \geq u + u_t$

$u_t \leq u_{xx} - u$

By the max principle.
since M is a max of u
it should be on boundary $(x=0, x=L)$
and bottom.
$(t=0)$

? 3

CLAUDIA TSAU

③ continued

$$\frac{1}{c}\left[\frac{(e^t-1)}{2}\,\frac{(e^{2x+2ct}-e^{2x-2ct})}{2}\right]$$

MIDTERM 2

Please write your name on each page of your answer sheet and **do not fold the pages together.**

(1) Show the uniqueness to the solution of

$$\begin{cases} u_t - ku_{xx} = f(x,t) & \text{for } 0 < x < L, \quad t > 0 \\ u(x,0) = \phi(x) \\ u(0,t) = g(t) \\ u(L,t) = h(t) \end{cases}$$

for sufficiently nice functions f, g, h, ϕ. You can use whatever method you want. A useful energy for the heat equation is the L^2 energy given by $E[u](t) = \int_0^L u^2(x,t)dx$.

(2) Solve in terms of the error function

$$\begin{cases} u_t - ku_{xx} = 0 & \text{on } 0 < x < \infty, t > 0 \\ u(x,0) = 0 \\ u(0,t) = 1. \end{cases}$$

Hint: First consider $v := u - 1$. What equation does v satisfy? Then solve that equation, keeping in mind that we are solving this on the half-line. The error function is given by

$$\mathcal{E}\text{rf}(s) = \frac{2}{\sqrt{\pi}} \int_0^s e^{-x^2} dx.$$

(3) Solve by finding an explicit formula. Make sure to integrate out the solution of

$$\begin{cases} u_{tt} - c^2 u_{xx} = e^{2x} & \text{on } (x,t) \in \mathbb{R}^2 \\ u(x,0) = 0 \\ u_t(x,0) = 0. \end{cases}$$

Note that $\sinh(x) = \frac{e^x - e^{-x}}{2}$, $\cosh(x) = \frac{e^x + e^{-x}}{2}$ and that $(\sinh(x))' = \cosh(x)$.

(4) Let $L, T > 0$. Suppose u is twice differentiable on the open rectangle $(0, L) \times (0, T)$ and satisfies the partial differential inequality

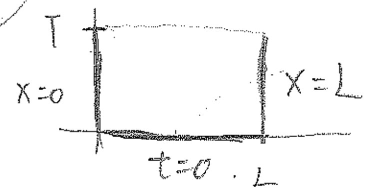
$$u_t - u_{xx} + u \leq 0.$$

Suppose further that u is continuous on $R = [0, L] \times [0, T]$. If M is the maximum on of u on R and $M \geq 0$, then show that u attains the value M on the sides $x = 0$ or $x = L$ or on the bottom $t = 0$ of R. Hint: Consider the sign or value of each quantity in the partial differential inequality at a maximum point if it were to occur in the interior. No $v = u + \varepsilon$ trick is necessary for this problem.

(4) maximum Principle

Junyue Han

19



$u_t - u_{xx} + u \le 0$

$M \geq 0$.

$u_t - u_{xx} + u \le 0$

3

$u \le u_{xx} - u_t$.

$$M = \max_{\substack{0 \le t \le T \\ 0 \le x \le L}} u = \max_{\substack{0 < t \le T \\ 0 < x < L}} u_{xx} - u_t.$$

if. u is - u_t is -

let $v_t = u_t$, $v_{xx} = u_{xx}$

$\Rightarrow v_t - v_{xx} + v + \varepsilon \le 0$

Junyue Han

in homogeneous

3) $\begin{cases} u_{tt} - c^2 u_{xx} = e^{2x} = f(x,t) \\ u(x,0) = 0 = \phi(x) \\ u_t(x,0) = 0 = \psi(x) \end{cases}$

Solution. $u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-y)^2}{4kt}} \phi(y)\, dy$

$+ \int_0^T \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{\frac{-(x-y)^2}{4kt}} f(y,s)\, dy\, ds$.

Since $\phi(x) = 0$ the 1st term vanishes.

$u(x,t) = \int_0^T \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{\frac{-(x-y)^2}{4kt}} e^{2y} dy\, ds$ 3

$= \frac{1}{\sqrt{4\pi kt}} \int_0^T \int_{-\infty}^{\infty} e^{-\frac{x^2+y^2-2xy+4ky}{2kt}/y} dy\, ds$.

$= \frac{1}{\sqrt{4\pi kt}} \int_0^T \int_{-\infty}^{\infty} e^{\frac{-x^2y+y^3-2xy^2+4ky^2}{2kt}} dy\, ds$.

$=$

Junyhe Han.

(2). half-line.

$$\begin{cases} u_t - k u_{xx} = 0 & 0 < x < \infty, \ t > 0. \\ u(x,0) = 0 \\ u(0,t) = 1 \end{cases}$$

$V = u - 1$.

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_0^\infty \left[e^{-\frac{(x-y)^2}{4kt}} + e^{-\frac{(x+y)^2}{4kt}} \right] \phi(y)\, dy.$$

but $\phi(y) = 0$.

$\Rightarrow$ $v(x,0) = -1 = \phi(y)$.

$u(0,t) = 0$.

$$v(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_0^\infty \left[e^{-\frac{(x-y)^2}{4kt}} + e^{-\frac{(x+y)^2}{4kt}} \right] (-1)\, dy$$

$$v(x,t) = \underbrace{-\frac{1}{\sqrt{4\pi k t}} \int_0^\infty e^{-\frac{(x-y)^2}{4kt}} dy}_{①} \ \underbrace{- \frac{1}{\sqrt{4\pi k t}} \int_0^\infty e^{-\frac{(x+y)^2}{4kt}} dy}_{②} = (*)$$

$r = \frac{x-y}{\sqrt{4kt}}$. $dr = \frac{dy}{\sqrt{4kt}}$. $\qquad r = \frac{x+y}{\sqrt{4kt}}$ $dr = \frac{dy}{\sqrt{4kt}}$

$$① = \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4kt}}} e^{-r^2} dr.$$

$$② = -\frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4kt}}} e^{-r^2} dr.$$

$$(*) = 2v(x,t) = \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4kt}}} e^{-r^2} dr - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4kt}}} e^{-r^2} dr.$$

$$v(x,t) = \frac{\mathcal{E}rf\left(\frac{x}{\sqrt{4kt}}\right) - \mathcal{E}rf\left(\frac{x}{\sqrt{4kt}}\right)}{2}$$

$$u(x,t) = \frac{\mathcal{E}rf\left(\frac{x}{\sqrt{4kt}}\right) - \mathcal{E}rf\left(\frac{x}{\sqrt{4kt}}\right)}{2} + 1.$$

Junyue Han

(1.) Uniqueness

$$\begin{cases} u_t - Ku_{xx} = f(x,t) & 0 < x < L, \quad t > 0. \\ u(x,0) = \phi(x) \\ u(0,t) = g(t) \\ u(L,t) = h(t) \end{cases}$$

let u_1 & u_2 be 2 solutions of the equation

let $w = u_1 + u_2$

$$\begin{cases} w_t - Kw_{xx} = 0 & 0 < x < L. \\ w(x,0) = \phi(x) \\ w(0,t) = g(t) \\ w(L,t) = h(t) \end{cases}$$

$0 = 0 \cdot W$

$= (W_t - KW_{xx}) \cdot (W)$

$= -\frac{1}{2}(W^2)_t - (KW_tW_x)_x + W_{tx}^2$

$= -\frac{1}{2}\int (W^2)_t \, dx - kW_tW_x \Big|_0^l + \int W_t^2 \, dx.$

$= -\frac{1}{2}\int (W^2)_t \, dx + \int W_t^2 \, dx \leq 0.$

$-\frac{1}{2}\int (W^2)_t \, dx = \int W_t^2 dx$

$\Rightarrow \int_0^l w(x,t)^2 dx = 0.$ vanishes.

$\Rightarrow W \equiv 0 \Rightarrow u_1 \equiv u_2. \ \forall x \geq 0$

$\forall x \geq 0.$

?

5

Raleigh L

1)

Assume that u and w are 2 independent solutions to the given equation and conditions.

Let $z = u - w$. We will try to use the energy method to show that $E[z](t) = 0$ for $t > 0$, which implies that $z = 0$, meaning that $u = w$.

26

$$E[z](t) = \int_0^L z^2(x,t)\,dx$$

$$\frac{d\,E[z](t)}{dt} = \frac{d}{dt}\int_0^L z^2(x,t)\,dx$$

Move the derivative inside the integral to get,

$$\frac{d\,E[z](t)}{dt} = \int_0^L \frac{\partial}{\partial t}\left(z^2(x,t)\right)dx$$

By the chain rule: $\displaystyle \frac{d\,E[z](t)}{dt} = 2\int_0^L z(x,t)\cdot z_t(x,t)$

Since $z = u - w$, z itself is a solution to

$$\begin{cases} z_t - k z_{xx} = 0 \\ z(x,0) = 0 \\ z(0,t) = 0 \\ z(L,t) = 0 \end{cases}$$

By the equation above we know that

$$z_t = k z_{xx}$$

So now we have

$$\frac{d\,E[z](t)}{dt} = 2\int_0^L z(x,t)\cdot k\, z_{xx}(x,t)$$

↓

1 (continued)

Recall L

Now apply the IBP formula, $\int u \, dv = uv - \int v \, du$

Here, $u = z(x,t)$ and $dv = z_{xx}(x,t)$

This leaves,
$$\int_0^L z(x,t) \cdot z_{xx}(x,t) = z(x,t) \Big|_{x=0}^{x=L} z_x(x,t) - \int_0^L z_x(x,t) \cdot z_x(x,t)$$

So now we have

$$\frac{dE[z](t)}{dt} = 2k \left[z(x,t) \Big|_{x=0}^{x=L} z_x(x,t) - \int_0^L z_{xx}(x,t) dx \right]$$

Since we know that $z(L,t) = 0$, then we diff. both sides with respect to x to get $z_x(L,t) = 0$. Similarly for $z(0,t) = 0$, this gives $z_x(0,t) = 0$. With this information, the first term in the brackets goes to 0.

Also, the 2nd term in brackets also goes to 0, since

$$\int_0^L z_{xx}(x,t) = z_x \Big|_{x=0}^{x=L}$$

which has shown to be 0. Then,

$$\frac{dE[z](t)}{dt} < 0$$

Since we know energy is constant, and that $z(x,t)$ is 0 at $x=0$, $x=L$, and $t>0$, since we know energy is constant, and that in general, since z^2 is always positive, $E(z) \geq 0$, we conclude that Energy is 0, AND that in general, $E[z](t) = 0$ for $t > 0$, meaning that likewise $z(x,t) = 0$ for $t > 0$, so then

$$u - w = 0 \rightarrow w = u$$

And so our 2 independent solutions are actually the same and thus uniqueness

4)

Rallyst L

This question asks us to prove the maximum principle. To prove this:

Assume by contradiction that the maximum M occurs somewhere in the interior of the region given.

Then, at this point, say (x_0, t_0), we know that

$u_t(x_0,t_0) = 0$ and $u_{xx}(x_0,t_0) \le 0$? 8

$u_t \le u_{xx} - u$

$u \le u_{xx}$

$u_t + u \le 0$

But since we found $u_t(x_0,t_0) = 0$ then, this means $u \le 0$.

Now assume the maximum instead occurs at the boundary of the region, so $(x_0,t_0) = (x,T)$.

Here, $u_t(x_0,t_0) \ge 0$ and $u_{xx}(x_0,t_0) \le 0$

From the PDE, we have $u_t + u \le 0$

$u_t \le u_{xx} - u$

However, this contradicts what we found earlier, since if $u_t \le u_{xx} - u$, and $u_t(x_0,t_0) \ge 0$, then $u \not\le 0$ (as it was earlier) can never be true and thus the maximum can only happen at the boundary.

2) Raleigh L

If $v(x,t) = u(x,t) - 1$

Recall then that the general solution for the diffusion equation on the half-line w/ Dirichlet conditions is:

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \left[\int_{-\infty}^{+\infty} e^{\frac{-(x-y)^2}{4kt}} - \int_{-\infty}^{+\infty} e^{\frac{-(x+y)^2}{4kt}} \right] \phi(y)\, dy$$

Since $v(x,t) = u(x,t) - 1$, this means

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \left[\int_{-\infty}^{+\infty} e^{\frac{-(x-y)^2}{4kt}} - \int_{-\infty}^{+\infty} e^{\frac{-(x+y)^2}{4kt}} \right] \phi(y)\, dy - 1$$

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{+\infty} e^{\frac{-(x-y)^2}{4kt}} \phi(y)\, dy - \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{+\infty} e^{\frac{-(x+y)^2}{4kt}} \phi(y)\, dy - 1$$

Let $r = \frac{-(x-y)}{\sqrt{4kt}}$ $\quad q = \frac{-(x+y)}{\sqrt{4kt}}$

$$= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{+\infty} e^{-r^2} \phi(y)\, dy - \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{+\infty} e^{-q^2} \phi(y)\, dy$$

Since $u(x,0) = \phi(y)$ and we're told $u(x,0) = 0$, then $\phi(x) = 0$

Also note that $y = x - r\sqrt{4kt} \to dy = \sqrt{4kt}\, dr$

$y = q\sqrt{4kt} - x \to dy = \sqrt{4kt}\, dq$

So the new bounds are

$$\frac{1}{\sqrt{4\pi kt}} \left(\int_{-\infty}^{x/4kt} e^{-r^2} + \int_{x/4kt}^{\infty} e^{-r^2}\, dr \right) + \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{x/4kt} e^{-q^2}$$

7

3)

Raleigh L

To find a solution to the nonhomogeneous wave equation we add a particular solution to our general solution.

Our general solution comes from D'Alembert's method, which states,

$$u_g(x,t) = \frac{1}{2}\left[\phi(x+ct) - \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

Our particular solution comes from the nonhomogeneous term, e^{2x}. We know that $u_{tt} = c^2 u_{xx}$ at e^{2x} if $c^2 = 4 \rightarrow c = 2$, so our particular solution term will be just $u_p = e^{2x}$

The whole solution then is:

$$u(x,t) = u_g + u_p$$

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) - \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds + e^{2x}$$

$\phi(x) = u(x,0)$ and so here $\phi(x) = 0$ because $u(x,0) = 0$.

Then,

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct}$$

3

ALEXANDER BAUER 94180S4

1) LET $u_1(x,t)$ AND $u_2(x,t)$ BE TWO SOLUTIONS OF $\begin{cases} u_t - ku_{xx} = f(x,t) \\ u(x,0) = \phi(x) & \text{FOR } 0<x<L, \\ u(0,t) = g(t) & t>0 \\ u(L,t) = h(t) \end{cases}$

LET $w(x,t) = u_1 - u_2$

THEN, $w_t = u_{1t} - u_{2t}$ AND $w_{xx} = u_{1xx} - u_{2xx}$

IT FOLLOWS THAT $w_t - kw_{xx} = u_{1t} - u_{2t} - ku_{1xx} + ku_{2xx}$

$$= (u_{1t} - ku_{1xx}) - (u_{2t} - ku_{2xx})$$

$$= f(x,t) - f(x,t) = 0$$

30

AND $w(x,0) = \phi(x) - \phi(x) = 0$

BY SIMILAR PROCESS $\to w(0,t) = 0$, $w(L,t) = 0$

BY THE MAXIMUM PRINCIPLE, $w(x,t) \leq 0$ AND BY THE MINIMUM PRINCIPLE $w(x,t) \geq 0$, SINCE $w_t - kw_{xx} = 0$.

IT FOLLOWS $w(x,t) \equiv 0$, SO $u_1(t) \equiv u_2(t)$ AND THEREFORE THE SOLUTION IS UNIQUE.

10

2) TAKE $v = u - 1$. THEN, $v_t = u_t$ AND $v_{xx} = u_{xx}$

SO $v_t - kv_{xx} = 0$

$v(x,0) = u(x,0) - 1 = -1$

$v(0,t) = u(0,t) - 1 = 0$

$$v(x,t) = \int_0^\infty \left(S(x-y,t) - S(x+y,t)\right) \cdot -1 \, dy$$

$$= \frac{-1}{\sqrt{4\pi kt}} \int_0^\infty e^{-(x-y)^2/4kt} - e^{-(x+y)^2/4kt} \, dy$$

TAKE $p = (y-x)/\sqrt{4kt}$ AND $q = (y+x)/\sqrt{4kt}$ $\quad dy = dp\sqrt{4kt}$

$$\Rightarrow \frac{-1}{\sqrt{\pi}} \int_{-x/\sqrt{4kt}}^{\infty} e^{-p^2} dp - \int_{x/\sqrt{4kt}}^{\infty} e^{-q^2} dq$$

$$\Rightarrow \frac{-1}{\sqrt{\pi}} \int_0^\infty e^{-p^2} dp + \int_0^{x/\sqrt{4kt}} e^{-p^2} dp - \left(\int_0^\infty e^{-q^2} dq - \int_0^{x/\sqrt{4kt}} e^{-q^2} dq\right)$$

$$= \frac{-1}{\sqrt{\pi}} \left(\left(\frac{\sqrt{\pi}}{2} + \frac{\sqrt{\pi}}{2} \operatorname{Erf}(x/\sqrt{4kt})\right) - \left(\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2} \operatorname{Erf}(x/\sqrt{4kt})\right)\right) = -\operatorname{Erf}\left(\frac{x}{\sqrt{4kt}}\right)$$

2) $-\mathrm{Erf}\left(x/\sqrt{4kt}\right) = v(x,t) = u(x,t) - 1$

$$\boxed{u(x,t) = 1 - \mathrm{Erf}\left(x/\sqrt{4kt}\right)}$$

10

3) $u(x,t) = \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} e^{2y}\,dy\,ds$

$$\int_0^t \frac{e^{2y}}{2}\Big|_{x-c(t-s)}^{x+c(t-s)} ds$$

$$\frac{e^{2x+2c(t-s)}}{2} - \frac{e^{2x-2c(t-s)}}{2} = e^{2x}\left(\frac{e^{2c(t-s)} - e^{-2c(t-s)}}{2}\right)$$

$$= e^{2x}\left(\sinh(2c(t-s))\right)$$

$2ct - 2cs$ $\quad -2ct + 2cs$

$$e^{2x}\left(\frac{e^{2ct}\,e^{-2cs}}{-2c} - \frac{e^{-2ct}\,e^{2cs}}{2c}\Big|_0^t\right)$$

$$\boxed{e^{2x}\left(\frac{e^{2ct}\,e^{-2ct}}{-2c} - \frac{1}{2c} + \frac{e^{2ct}}{+2c} + \frac{e^{-2ct}}{2c}\right)}$$

✓

10

MIDTERM 2

Please write your name on each page of your answer sheet and **do not fold the pages together.**

(1) Show the uniqueness to the solution of

$$\begin{cases} u_t - ku_{xx} = f(x,t) & \text{for } 0 < x < L, \quad t > 0 \\ u(x,0) = \phi(x) \\ u(0,t) = g(t) \\ u(L,t) = h(t) \end{cases}$$

for sufficiently nice functions f, g, h, ϕ. You can use whatever method you want. A useful energy for the heat equation is the L^2 energy given by $E[u](t) = \int_0^L u^2(x,t)dx$.

(2) Solve in terms of the error function

$$\begin{cases} u_t - ku_{xx} = 0 & \text{on } 0 < x < \infty, t > 0 \\ u(x,0) = 0 \\ u(0,t) = 1. \end{cases}$$

Hint: First consider $v := u - 1$. What equation does v satisfy? Then solve that equation, keeping in mind that we are solving this on the half-line. The error function is given by

$$\mathcal{E}\mathrm{rf}(s) = \frac{2}{\sqrt{\pi}} \int_0^s e^{-x^2} dx.$$

(3) Solve by finding an explicit formula. Make sure to integrate out the solution of

$$\begin{cases} u_{tt} - c^2 u_{xx} = e^{2x} & \text{on } (x,t) \in \mathbb{R}^2 \\ u(x,0) = 0 \\ u_t(x,0) = 0. \end{cases}$$

Note that $\sinh(x) = \frac{e^x - e^{-x}}{2}$, $\cosh(x) = \frac{e^x + e^{-x}}{2}$ and that $(\sinh(x))' = \cosh(x)$.

(4) Let $L, T > 0$. Suppose u is twice differentiable on the open rectangle $(0, L) \times (0, T)$ and satisfies the partial differential inequality

$$u_t - u_{xx} + u \leq 0.$$

Suppose further that u is continuous on $R = [0, L] \times [0, T]$. If M is the maximum on of u on R and $M \geq 0$, then show that u attains the value M on the sides $x = 0$ or $x = L$ or on the bottom $t = 0$ of R. Hint: Consider the sign or value of each quantity in the partial differential inequality at a maximum point if it were to occur in the interior. No $v = u + \varepsilon$ trick is necessary for this problem.

BRYAN
ORDONEZ

MIDTERM 2

Please write your name on each page of your answer sheet and **do not fold the pages together.**

(1) Show the uniqueness to the solution of

$$\begin{cases} u_t - ku_{xx} = f(x,t) & \text{for } 0 < x < L, \quad t > 0 \\ u(x,0) = \phi(x) \\ u(0,t) = g(t) \\ u(L,t) = h(t) \end{cases}$$

for sufficiently nice functions f, g, h, ϕ. You can use whatever method you want. A useful energy for the heat equation is the L^2 energy given by $E[u](t) = \int_0^L u^2(x,t)dx$.

(2) Solve in terms of the error function

$$\begin{cases} u_t - ku_{xx} = 0 & \text{on } 0 < x < \infty, t > 0 \\ u(x,0) = 0 \\ u(0,t) = 1. \end{cases}$$

Hint: First consider $v := u - 1$. What equation does v satisfy? Then solve that equation, keeping in mind that we are solving this on the half-line. The error function is given by

$$\mathcal{E}\text{rf}(s) = \frac{2}{\sqrt{\pi}} \int_0^s e^{-x^2} dx.$$

(3) Solve by finding an explicit formula. Make sure to integrate out the solution of

$$\begin{cases} u_{tt} - c^2 u_{xx} = e^{2x} & \text{on } (x,t) \in \mathbb{R}^2 \\ u(x,0) = 0 \\ u_t(x,0) = 0. \end{cases}$$

Note that $\sinh(x) = \frac{e^x - e^{-x}}{2}$, $\cosh(x) = \frac{e^x + e^{-x}}{2}$ and that $(\sinh(x))' = \cosh(x)$.

(4) Let $L, T > 0$. Suppose u is twice differentiable on the open rectangle $(0,L) \times (0,T)$ and satisfies the partial differential inequality

$$u_t - u_{xx} + u \leq 0.$$

Suppose further that u is continuous on $R = [0,L] \times [0,T]$. If M is the maximum on of u on R and $M \geq 0$, then show that u attains the value M on the sides $x = 0$ or $x = L$ or on the bottom $t = 0$ of R. Hint: Consider the sign or value of each quantity in the partial differential inequality at a maximum point if it were to occur in the interior. No $v = u + \varepsilon$ trick is necessary for this problem.

Brian Ordonez

29

1D

1. $u_t - ku_{xx} = f(x,t)$ $\quad 0 < x < L,\ t > 0$

$u(x,0) = \phi(x)$

$u(0,t) = g(t)$

$u(L,t) = h(t)$

$w = u_1 - u_2$

$w_t = (u_1)_t - (u_2)_t$

$w_{xx} = (u_1)_{xx} - (u_2)_{xx}$

$w_t - kw_{xx} = 0$

$w(x,0) = \phi(x) - \phi(x) = 0$

$w(0,t) = 0$

$w(L,t) = 0$

$E = \int_0^L u^2(x,t)\,dx$

$\frac{d}{dt}E = \int_0^L \frac{d}{dt} u^2(x,t)\,dx$

$\frac{d}{dt}E = \int_0^L 2u(x,t)u_t(x,t)$

note: $w_t = w_{xx}$

$\therefore \frac{d}{dt}E = \int_0^L 2u(x,t)\, w_{xx}(x,t)$

IBP

$u = w(x,t)$ $\quad dv = w_{xx}(x,t)$

$du = w_x(x,t)$ $\quad v = w_x(x,t)$

$= 2\left[[w(x,t)\cdot w_x(x,t)]_0^L - \int_0^L w_x(x,t)\, w_x(x,t) \right]$

Note due to initial conditions, this becomes 0

$\rightarrow 2\left[0 - \int_0^L w_x^2(x,t)\right]$

$\Rightarrow$ ~~$\frac{d}{dt}E$~~

$\frac{d}{dt}E = -2\int_0^L w_x^2(x,t)$

- Shows energy is decreasing for all time.
- Note $E \geq 0$
- Note; $w(x,0) = 0$
- Since at time $t = 0$, $E = 0$, and energy is decreasing for all time t, It must be that

$w(x,t) = 0$

$\therefore 0 = u_1 - u_2$

$\boxed{\therefore u_1 = u_2}$

Brian Ordonez

2 Solve in terms of the erf function

$$\begin{cases} u_t - k u_{xx} = 0 & \text{on } 0<x<\infty,\ t>0 \\ u(x,0) = 0 \\ u(0,t) = 1 \end{cases}$$

$$u(x,t) = \int_0^{\infty} H(x-y,t)\,\phi(y) - \int_{-\infty}^{0} H(x+y,t)\,\phi(-y)$$

$$\underbrace{\frac{1}{\sqrt{4k\pi t}}\int_0^{\infty} e^{-\frac{(x-y)^2}{4kt}}\cdot 1}_{A} \quad \underbrace{- \frac{1}{\sqrt{4k\pi t}}\int_{-\infty}^{0} e^{-\frac{(x+y)^2}{4kt}}(-1)}_{B}$$

Solving A

$s = \frac{x-y}{\sqrt{4kt}}$ $\quad y = x - s\sqrt{4kt}$

$ds = \frac{-dy}{\sqrt{4kt}}$ $\quad dy = -ds\sqrt{4kt}$

$$\frac{1}{\sqrt{4k\pi t}}\int_{\frac{x}{\sqrt{4kt}}}^{-\infty} e^{-s^2}\left(-ds\sqrt{4kt}\right)$$

$$= \frac{-1}{\sqrt{\pi}}\int_{\frac{x}{\sqrt{4kt}}}^{-\infty} e^{-s^2}\,ds$$

split integral

$$\frac{1}{\sqrt{\pi}}\Big[\underbrace{\int_{-\infty}^{0} e^{-s^2}ds}_{=\frac{\sqrt{\pi}}{2}} + \int_0^{\frac{x}{\sqrt{4kt}}} e^{-s^2}ds\Big]$$

looks like error function.

$$\frac{1}{2} + \frac{1}{2}\,\mathrm{Erf}\left(\frac{x}{\sqrt{4kt}}\right)$$

Solves A

Solving B

$$\frac{1}{\sqrt{4k\pi t}}\int_{-\infty}^{0} e^{-\frac{(x+y)^2}{4kt}}\,dy$$

$s = \frac{x+y}{\sqrt{4kt}}$ $\quad ds = \frac{dy}{\sqrt{4kt}}$

$y = s\sqrt{4kt} - x$ $\quad dy = ds\sqrt{4kt}$

$$\frac{1}{\sqrt{4k\pi t}}\int_{-\infty}^{x/\sqrt{4kt}} e^{-s^2}\,ds\,\sqrt{4kt}$$

$$\Rightarrow \frac{1}{\sqrt{\pi}}\Big[\underbrace{\int_{-\infty}^{0} e^{-s^2}ds}_{=\frac{\sqrt{\pi}}{2}} + \int_0^{x/\sqrt{4kt}} e^{-s^2}ds\Big]$$

looks like error function

$$\frac{1}{2} + \frac{1}{2}\,\mathrm{Erf}\left(\frac{-x}{\sqrt{4kt}}\right)$$

8

$$u(x,t) = \frac{1}{2} + \frac{1}{2}\,\mathrm{Erf}\left(\frac{x}{\sqrt{4kt}}\right) + \frac{1}{2} + \frac{1}{2}\,\mathrm{Erf}\left(\frac{x}{\sqrt{4kt}}\right)$$

$$= u(x,t) = 1 + \mathrm{Erf}\left(\frac{x}{\sqrt{\ldots}}\right.$$

Brian Ordonez

3.) Solve by finding an explicit formula, make sure to integrate out the sol. of

$$\begin{cases} u_{tt} - c^2 u_{xx} = e^{2x} \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}$$

$$u(x,t) = \frac{1}{2}\left(\phi(x+ct) - \phi(x-ct)\right) + \frac{1}{2c}\int \psi(w)\,dw + \frac{1}{2c}\iint f$$

note $u(x,0) = 0$

$u_t(x,0) = 0$

$$\therefore \quad \frac{1}{2c}\int_0^t \int_{x-c[t-s]}^{x+c[t-s]} e^{2y}\,dy\,ds$$

$$\begin{aligned} u &= 2y \\ du &= 2\,dy \\ \frac{du}{2} &= dy \end{aligned} \Rightarrow \frac{1}{2c}\int_0^t \int_{x-c[t-s]}^{x+c[t-s]} \frac{1}{2} e^{u}\,du$$

$$\Rightarrow \frac{1}{4c}\int_0^t e^{u}\Big]_{x-c[t-s]}^{x+c[t-s]} = \int_0^t e^{2y}\Big]_{x-c[t-s]}^{x+c[t-s]}$$

$$\Rightarrow \int_0^t \underset{A}{e^{2[x+ct-cs]}} - \underset{B}{e^{2[x-ct+cs]}}$$

$$\frac{1}{4c} \quad A \Rightarrow e^{2x+2ct} \cdot \int_0^t e^{t-2cs} \Rightarrow \frac{1}{4c} e^{2x+2ct} \cdot \frac{1}{-2c}\int_0^t e^{u}\,du$$

$$\begin{aligned} u &= t - 2cs \\ du &= -2c \\ \frac{du}{-2c} &= ds \end{aligned} \Rightarrow \frac{1}{-8c^2} e^{2x+2ct} \cdot e^{t-2cs}\Big]_0^t$$

$$\frac{1}{-8c^2} e^{2x+2ct} \cdot \left[e^{t-2ct} - e^{t}\right] \quad \leftarrow \text{Solves } A$$

$$\Rightarrow \boxed{\frac{1}{-8c^2}\left[e^{2x+t} - e^{2x+2ct+t}\right]}$$

8

3 Cont'd

Brian Ordonez

Solving B

$$\frac{-1}{4c}\int_0^t e^{2x-2ct+2cs}\,ds$$

$$\frac{-1}{4c}e^{2x-2ct}\int_0^t e^{2cs}\,ds$$

$u = 2cs$

$du = 2c$

$\frac{du}{2c} = ds$

$$\frac{-1}{4c}e^{2x-2ct}\int_0^t e^{u}\frac{1}{2c}\,du \Rightarrow \frac{-1}{8c^2}e^{2x-2ct}e^{u}\Big]_0^t$$

$$\Rightarrow \frac{-1}{8c^2}e^{2x-2ct}\left[e^{2ct}-1\right]$$

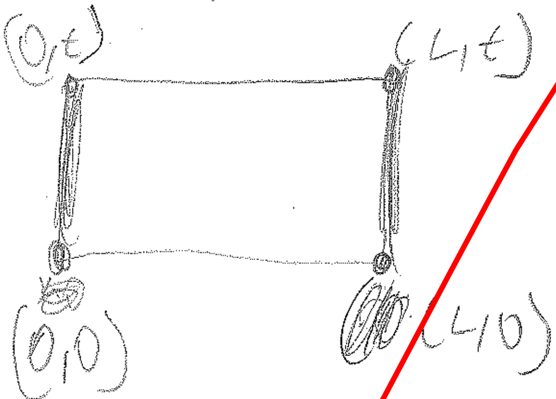
$$\Rightarrow \frac{-1}{8c^2}e^{2x}+\frac{1}{8c^2}e^{2x-2ct}$$

Combining A & B

$$u(x,t) = \frac{-1}{8c^2}\left[e^{2x+t}-e^{2x+2ct+t}+e^{2x}-e^{2x-2ct}\right]$$

Brian
Ordonez

4) By Max Principle, maximum will ALWAYS occur on the boundary.



3

Kevin wang

30

MIDTERM 2

Please write your name on each page of your answer sheet and **do not fold the pages together.**

(1) Show the uniqueness to the solution of

$$\begin{cases} u_t - ku_{xx} = f(x,t) & \text{for } 0 < x < L, \quad t > 0 \\ u(x,0) = \phi(x) \\ u(0,t) = g(t) \\ u(L,t) = h(t) \end{cases}$$

for sufficiently nice functions f, g, h, ϕ. You can use whatever method you want. A useful energy for the heat equation is the L^2 energy given by $E[u](t) = \int_0^L u^2(x,t)dx$.

(2) Solve in terms of the error function

$$\begin{cases} u_t - ku_{xx} = 0 & \text{on } 0 < x < \infty, t > 0 \\ u(x,0) = 0 \\ u(0,t) = 1. \end{cases}$$

Hint: First consider $v := u - 1$. What equation does v satisfy? Then solve that equation, keeping in mind that we are solving this on the half-line. The error function is given by

$$\mathcal{E}\text{rf}(s) = \frac{2}{\sqrt{\pi}} \int_0^s e^{-x^2} dx.$$

(3) Solve by finding an explicit formula. Make sure to integrate out the solution of

$$\begin{cases} u_{tt} - c^2 u_{xx} = e^{2x} & \text{on } (x,t) \in \mathbb{R}^2 \\ u(x,0) = 0 \\ u_t(x,0) = 0. \end{cases}$$

Note that $\sinh(x) = \frac{e^x - e^{-x}}{2}$, $\cosh(x) = \frac{e^x + e^{-x}}{2}$ and that $(\sinh(x))' = \cosh(x)$.

(4) Let $L, T > 0$. Suppose u is twice differentiable on the open rectangle $(0, L) \times (0, T)$ and satisfies the partial differential inequality

$$u_t - u_{xx} + u \leq 0.$$

Suppose further that u is continuous on $R = [0, L] \times [0, T]$. If M is the maximum on of u on R and $M \geq 0$, then show that u attains the value M on the sides $x = 0$ or $x = L$ or on the bottom $t = 0$ of R. Hint: Consider the sign or value of each quantity in the partial differential inequality at a maximum point if it were to occur in the interior. No $v = u + \varepsilon$ trick is necessary for this problem.

① $\begin{cases} u_t - ku_{xx} = f(x,t) \\ u(x,0) = \phi(x) \\ u(0,t) = g(t) \\ u(L,t) = h(t) \end{cases}$ $0 \le x \le L,\ t > 0$

$E[u](t) = \int_0^L u^2(x,t)\,dx$ (energy)

~~want~~ Let $w = u_1 - u_2$

show $\begin{cases} w_t - kw_{xx} = \cancel{f} \ 0 \\ w(x,0) = \cancel{\phi(x)} \ 0 \\ w(0,t) = \cancel{g(t)} \ 0 \\ w(L,t) = 0 \end{cases}$

$w_t = (u_1)_t - (u_2)_t$

$kw_{xx} = k(u_1)_{xx} - k(u_2)_{xx}$

$(u_1)_t - (u_1)_{xx} = (u_2)_t - (u_2)_{xx}$

$w(x,0) = u_1(x,0) - u_2(x,0)$

$\phi_1(x) - \phi_2(x) = 0 \Rightarrow \underline{\phi_1(x) = \phi_2(x)}$

$w(0,t) = u_1(0,t) - u_2(0,t)$

$g_1(t) - g_2(t) \Rightarrow \underline{g_1(t) = g_2(t)}$

$w(L,t) = u_1(L,t) - u_2(L,t)$

$h_1(t) - h_2(t) \Rightarrow \underline{h_1(t) = h_2(t)}$

$E[u](t) = \int_0^L u^2(x,t)\,dx$

$\frac{d}{dt} E[u](t) = \int_0^L \frac{d}{dt} u^2(x,t)\,dx$

$= \int_0^L 2u \cdot u_t(x,t)\,dx$

$2u \cdot k u_{xx}(x,t)\,dx$

$\int_0^L 2k\, u \cdot u_x (x,t)\,dx = \Big|_0^L$

$2k\left[\left(u(L,\cancel{t}) \cdot u_x(L,t)\right) - \left(u(0,t) \cdot u_x(0,t)\right)\right]$

$= 2k\left[h(t) \cdot u_x(L,t) - g(t) \cdot u_x(0,t)\right]$

$0 - \int 2k\, w_x \cancel{x}(0,t)$

8

(2)

$$\begin{cases} u_t - k u_{xx} = 0 & 0 < x < \infty, \quad t > 0 \\ u(x,0) = 0 \\ u(0,t) = 1 \end{cases}$$

half line odd ext.

$V = u - 1$

$$\begin{cases} v(x,0) = u(x,0) - 1 = -1 \Rightarrow \phi(x) \\ v(0,t) = u(0,t) - 1 = 0 \Rightarrow \\ v_t - k v_{xx} = u_t - k u_{xx} - 1 = -1 \end{cases}$$

$$\int_0^\infty H(x-y,t) - H(x+y,t)\, \phi(y)\, dy$$

$$\frac{1}{\sqrt{4k\pi t}}\left(\int_0^\infty \left(e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}}\right)(-1)\, dy\right)$$

$$\frac{1}{\sqrt{4k\pi t}}\left(\int_0^\infty -e^{-\frac{(x-y)^2}{4kt}} + e^{-\frac{(x+y)^2}{4kt}}\, dy\right)$$

$$\underbrace{\frac{1}{\sqrt{4k\pi t}}\int_0^\infty -e^{-\frac{(x-y)^2}{4kt}}}_{①} + \underbrace{\frac{1}{\sqrt{4k\pi t}}\int_0^\infty e^{-\frac{(x+y)^2}{4kt}}\, dy}_{②}$$

①
$s = \frac{x-y}{\sqrt{4kt}}$ $\quad y = x - s\sqrt{4kt}$

$ds = \frac{-dy}{\sqrt{4kt}}$

②
$s = \frac{x+y}{\sqrt{4kt}}$ $\quad y = s\sqrt{4kt} - x$

$ds = \frac{dy}{\sqrt{4kt}}$

$$\frac{-1}{\sqrt{4k\pi t}}\int_{-\infty}^{\frac{x}{\sqrt{4kt}}} e^{-s^2}\sqrt{4kt} \qquad = \frac{1}{\sqrt{4k\pi t}}\int_{\frac{x}{\sqrt{4kt}}}^{\infty} e^{-s^2}\sqrt{4kt}\, ds$$

$$\frac{-1}{\sqrt{\pi}}\int_{-\infty}^{\frac{x}{\sqrt{4kt}}} e^{-s^2}\, ds \qquad = \frac{1}{\sqrt{\pi}}\int_{\frac{x}{\sqrt{4kt}}}^{\infty} e^{-s^2}\, ds$$

10

$$= \frac{-1}{\sqrt{\pi}}\left(\underbrace{\int_{-\infty}^{0} e^{-s^2} ds} + \int_0^{\frac{x}{\sqrt{4kt}}} e^{-s^2} ds\right) \qquad \frac{1}{\sqrt{\pi}}\left(\underbrace{\int_0^\infty e^{-s^2}} - \underbrace{\int_0^{\frac{x}{\sqrt{4kt}}} e^{-s^2}}\right)$$

$$\left(\frac{\sqrt{\pi}}{2} + \frac{\sqrt{\pi}}{2}\operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right)\right) \qquad \frac{1}{\sqrt{\pi}}\left(\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2}\operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right)\right)$$

①+②

$$\cancel{-\frac{1}{2}} - \operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right) + \cancel{\frac{1}{2}} - \operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right) = \quad v(x,t) = -\operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right)$$

$$\begin{cases} u_t - k u_{xx} = 0 & 0 < x < \infty,\ t > 0 \\ u(x,0) = 0 \rightarrow \phi(x) & \text{half-line} \\ u(0,t) = 1 \rightarrow \psi(t) \end{cases}$$

$$u(x,t) = \int_0^\infty \left(H(x-y,t) + H(x+y,t) \right) \phi(y)\, dy +$$

$$\frac{1}{\sqrt{4k\pi t}} \left(\int_0^\infty \left(e^{-\frac{(x-y)^2}{4kt}} + e^{-\frac{(x+y)^2}{4kt}} \right) (0)\, dy \right)$$

$$u = v - 1 \Rightarrow (v-1)_t - k(v_{xx} - 1)$$

$v_t - k u_{xx} =$

③ $\begin{cases} u_{tt} - c^2 u_{xx} = e^{2x} & (x,t) \in \mathbb{R}^2 \\ u(x,0) = 0 = \phi(x) \\ u_t(x,0) = 0 = \psi(x) \end{cases}$

$\cancel{\frac{1}{2}(\phi(x+ct) + \phi(x-ct))}_{0} + \cancel{\frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds}_{0} + \frac{1}{2c}\iint F$

$\frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} e^{2y}\,dy\,ds$

$= \frac{1}{2c} \int_0^t 2e^{2y}\,ds \Big|_{x-c(t-s)}^{x+c(t-s)}$

$= \frac{1}{c} \int_0^t e^{2(x+c(t-s))}\,ds - e^{2(x-c(t-s))}\,ds$

① $r = x+c(t-s)$ ② $r = x-c(t-s)$

$dr = -c\,ds$ $dr = c\,ds$

$\underbrace{\frac{1}{c}\int_{x+ct}^{x} e^{2r}\left(-\frac{dr}{c}\right)} - \frac{1}{c}\int_{x-ct}^{x} e^{2r}\,\frac{dr}{c}$

$\frac{1}{c^2}\int_x^{x+ct} e^{2r}\,dr - \frac{1}{c^2}\int_{x-ct}^{x} e^{2r}\,dr$

$\frac{1}{c^2}\left(2e^{2r}\Big|_x^{x+ct} - \frac{1}{c^2}\left(2e^{2r}\Big|_{x-ct}^{x}\right.\right.$

$\frac{1}{c^2}\left(2e^{2(x+ct)} - 2e^{2(x)}\right) - \left(2e^{2x} - 2e^{2(x-ct)}\right)$

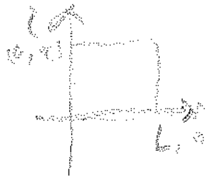
9

$$\frac{1}{c^2}\left(2e^{2(x+ct)} - 2e^{2x}\right) - \frac{1}{c^2}\left(2e^{2x} - 2e^{2(x-ct)}\right)$$

$$\cancel{2e^{2(x+ct)} \quad 2e^{2x}}$$

$$\Rightarrow \frac{1}{c^2}\left(2e^{2(x+ct)} - 4e^{2x} + \cancel{2e^{2(x-ct)}}\right)$$

$u(x,t)$



$u_t - u_{xx} + u \le 0$

$R = [0, L] \times [0, T]$

$m = \max u$

$v = u + \varepsilon$

$\forall \varepsilon > 0$ for $|x - t| < \delta$

3

$f(x$

Jordan Gonzalez

MIDTERM 2

Please write your name on each page of your answer sheet and **do not fold the pages together**.

(1) Show the uniqueness to the solution of

$$\begin{cases} u_t - ku_{xx} = f(x,t) & \text{for } 0 < x < L, \quad t > 0 \\ u(x,0) = \phi(x) \\ u(0,t) = g(t) \\ u(L,t) = h(t) \end{cases}$$

for sufficiently nice functions f, g, h, ϕ. You can use whatever method you want. A useful energy for the heat equation is the L^2 energy given by $E[u](t) = \int_0^L u^2(x,t)dx$.

(2) Solve in terms of the error function

$$\begin{cases} u_t - ku_{xx} = 0 & \text{on } 0 < x < \infty, t > 0 \\ u(x,0) = 0 \\ u(0,t) = 1. \end{cases}$$

Hint: First consider $v := u - 1$. What equation does v satisfy? Then solve that equation, keeping in mind that we are solving this on the half-line. The error function is given by

$$\mathcal{E}\mathrm{rf}(s) = \frac{2}{\sqrt{\pi}} \int_0^s e^{-x^2} dx.$$

(3) Solve by finding an explicit formula. Make sure to integrate out the solution of

$$\begin{cases} u_{tt} - c^2 u_{xx} = e^{2x} & \text{on } (x,t) \in \mathbb{R}^2 \\ u(x,0) = 0 \\ u_t(x,0) = 0. \end{cases}$$

Note that $\sinh(x) = \frac{e^x - e^{-x}}{2}$, $\cosh(x) = \frac{e^x + e^{-x}}{2}$ and that $(\sinh(x))' = \cosh(x)$.

(4) Let $L, T > 0$. Suppose u is twice differentiable on the open rectangle $(0, L) \times (0, T)$ and satisfies the partial differential inequality

$$u_t - u_{xx} + u \leq 0.$$

Suppose further that u is continuous on $R = [0, L] \times [0, T]$. If M is the maximum on of u on R and $M \geq 0$, then show that u attains the value M on the sides $x = 0$ or $x = L$ or on the bottom $t = 0$ of R. Hint: Consider the sign or value of each quantity in the partial differential inequality at a maximum point if it were to occur in the interior. No $v = u + \varepsilon$ trick is necessary for this problem.

Jordan Gomez Midterm 2 Perm: 705480

35

(1) $\begin{cases} u_t - ku_{xx} = f(x,t) & \text{for } 0<x<L,\ t>0 \\ u(x,0) = \phi(x) \\ u(0,t) = g(t) \\ u(L,t) = h(t) \end{cases}$

Let $u(x,t)$ and $v(x,t)$ be solutions $\longrightarrow$

Aside $w(x,t) = u(x,t) - v(x,t)$

$w_t(x,t) = u_t - v_t$

$w_x = u_x - v_x$

$w_{xx} = u_{xx} - v_{xx}$

let $w(x,t) = u(x,t) - v(x,t)$

so $\begin{cases} w_t - kw_{xx} = 0 & \text{for } 0<x<L,\ t>0 \\ w(x,0) = 0 = \phi(x) \\ w(0,t) = 0 \\ w(L,t) = 0 \end{cases}$

then $w(x,t) = \int_0^L H(x-y,t)\, \phi(y)\, dy \quad \frac{1}{\sqrt{4\pi kt}}\int_0^L e^{-\frac{(x-y)^2}{4kt}}\, (0)\, dy$

$= 0$

~~we know $w(x,t) \geq 0$ by maximum principle and~~

~~when $\tilde{w} = v - u$ then $w(x,t) \leq 0$ so $w(x,t) = 0$.~~

~~so $u(x,t) - v(x,t) = 0 \rightarrow u(x,t) = v(x,t)$ so unique solution~~

$w = u - v$.

Energy method: $E(t) \int_0^L w^2(x,t)\, dx$

$\frac{\partial}{\partial t} E(t) \int_0^L \frac{\partial}{\partial t}(w^2(x,t))\, dx = \int_0^L 2ww_t\, dx$

and we know $w_t = kw_{xx}$

so $= 2k\int_0^L w\, w_{xx}\, dx$ then do by parts $\begin{matrix} u = w & v = w_x \\ du = w_x & dv = w_{xx} \end{matrix}$

now we have

8

$= 2k\left[ww_x \Big|_0^L - \int_0^L w_x^2\, dx \right]$

$= 2k\left(w(L,t)\, w_x(L,t) - w(0,t)\, w_x(0,t)\right) - 2k\int_0^L w_x^2\, dx$

using the boundary conditions

$= 0 - 2k\int_0^L w_x^2\, dx$ which is negative so $-2k\int_0^L w_x^2 \leq 0$

notice that $W(x,0) = 0$ so $w(x,t) = u(x,t) - v(x,t) = 0$

$\therefore u(x,t) = v(x,t)$ so

Jordan Gomez   Perm 7654005

(2) Error function $\quad Erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2} \, ds \qquad \frac{Erf(x)}{2} = \frac{1}{\sqrt{\pi}} \int_0^x e^{-s^2} \, ds$

$$\begin{cases} u_t - k u_{xx} = 0 \\ u(x,0) = 0 = \phi(x) \\ u(0,t) = 1 \end{cases} \quad \text{on } 0 < x < \infty, \ t > 0 \qquad \text{[half line homogeneous]}$$

Let $V(x,t) = u(x,t) - 1$ so $\begin{cases} V_t - k V_{xx} = 0 \\ V(x,0) = -1 = \phi(x) \\ V(0,t) = 0 \end{cases}$

since half line homogeneous the solution formula is $V(x,t) = \int_0^\infty [H(x-y,t) - H(x+y,t)] \, \phi(y) \, dy$

so

$$V(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_0^\infty \left[e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}} \right] (-1) \, dy$$

$$= \frac{1}{\sqrt{4k\pi t}} \int_0^\infty e^{-\frac{(x+y)^2}{4kt}} - e^{-\frac{(x-y)^2}{4kt}} \, dy$$

$$= \frac{1}{\sqrt{4k\pi t}} \left[\int_0^\infty e^{\frac{-(x+y)^2}{4kt}} dy - \int_0^\infty e^{-\frac{(x-y)^2}{4kt}} dy \right]$$

* we can sub

$$\begin{cases} p = \left[\frac{x+y}{\sqrt{4kt}}\right] \ \& \ dp = \frac{dy}{\sqrt{4kt}} \\ \qquad dy = dp\,(\sqrt{4kt}) \\ q = \left[\frac{x-y}{\sqrt{4kt}}\right] \ \& \ dq = \frac{-dy}{\sqrt{4kt}} \\ \qquad dy = -dq\,(\sqrt{4kt}) \end{cases}$$

substitute p and q in

$$= \frac{1}{\sqrt{\pi}} \left[-\int_{\frac{x}{\sqrt{4kt}}}^\infty e^{-p^2} dp + \int_{-\frac{x}{\sqrt{4kt}}}^\infty e^{-q^2} dq \right]$$

$$= \frac{1}{\sqrt{\pi}} \left[\left(\int_0^\infty e^{-p^2} dp - \int_0^{\frac{x}{\sqrt{4kt}}} e^{-p^2} dp \right) + \left(\int_0^\infty e^{-q^2} dq + \int_0^{\frac{x}{\sqrt{4kt}}} e^{-q^2} dq \right) \right]$$

$$= \frac{1}{\sqrt{\pi}} \left[\frac{\sqrt{\pi}}{2} - \frac{1}{2} Erf\left(\frac{x}{\sqrt{4kt}}\right) + \frac{\sqrt{\pi}}{2} + \frac{1}{2} Erf\left(\frac{x}{\sqrt{4kt}}\right) \right]$$

so Erf cancel out

$$= \left[\frac{1}{2} + \frac{1}{2}\right] = \boxed{1}$$

If I am correct, it would come out to a

If I made a ~~mistake~~ I believe function $\cancel{\frac{1}{2} - Erf\left(\frac{x}{\sqrt{4kt}}\right)}$.

9

Jordan Gomez Perm 7054805

(3) $\begin{cases} u_{tt} - c^2 u_{xx} = e^{2x} & \text{on whole real line } (x,t) \in \mathbb{R} \text{ [nonhomogeneous]} \\ u(x,0) = 0 = \phi(x) \\ u_t(x,0) = 0 = \psi(x) \end{cases}$

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds$$

Since $\phi(x) = \psi(x) = 0$
we only need $u(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds$

and $= \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} e^{2y}\,dy\,ds$

$$= \frac{1}{2c}\int_0^t [e^{2(x+c(t-s))} - e^{2(x-c(t-s))}]\,ds$$

$$= \frac{1}{c}\int_0^t \left[\frac{e^{2(x+ct-cs)} - e^{2(x-ct+cs)}}{2}\right] ds$$

$$= \frac{1}{c}\int_0^t \left[\frac{e^{2x+2c(t-s)} - e^{2x-2c(t-s)}}{2}\right] ds \Leftarrow \text{trying to turn this into } \sinh(2c(t-s))$$

$$= \frac{1}{c}\int_0^t e^{2x}\left[\frac{e^{2c(t-s)} - e^{-2c(t-s)}}{2}\right] ds$$

$$= \frac{1}{c}\int_0^t e^{2x}[\sinh(2c(t-s))]\,ds = \frac{1}{c}\int_0^t e^{2x}[\sinh(2ct-2cs)]\,ds$$

aside $(\sinh(x))' = \cosh(x)$ & $\int \sinh(x) = -\cosh(x)$?? or $\cosh(x)$

$$= \frac{1}{c} \quad e^{2x}\left[\cosh(2ct-2cs)\cdot\left(-\frac{2cs^2}{2}\right)\right]\Big|_0^t$$

$$= \frac{1}{c}\cdot e^{2x}\left[\cosh(2c(t-t))\cdot\left(-\frac{2ct^2}{2}\right) + \cosh(2c(t-0))\cdot\left(\cancel{\frac{2c(0)}{2}}\right)\right]$$

$$= -e^{2x}t^2\cosh(0) + \frac{e^{2x}}{c}\cosh(2ct)$$

If I wasn't supposed to take out e^{2x} then it would look like

$\frac{1}{c}\int_0^t \sinh(2x+2c(t-s)) = -\cosh(2x)\cdot s^2 + \frac{\cosh(2x+2ct)}{c}$

Jordan Gomez

$(0,L) \times (0,T)$ = rectangle   7854805

$R = [0,L] \times [0,T]$

(4) $u_t - u_{xx} + u \leq 0$ & $L, T > 0$

M is the maximum of u on R & $M \geq 0$

normally allow $v(x,t) = \underset{= M}{u(x,t)} + \varepsilon x^2$ for $u_t - u_{xx} = 0$.

but $u_t - u_{xx} + u \leq 0$ (*) & $u_t - u_{xx} \leq -u$

Also $u_t + u \leq u_{xx}$ & $u \leq u_{xx} - u_t$

so $u \geq -(u_{xx} - u_t)$. We want to show that the maximum points lets name (x_0, t_0) are not interior points and that with a $\delta > 0$ that

$$\lim_{\delta \to 0^+} \frac{u(x_0,t_0) - u(x_0, t_0 - \delta)}{\delta} = 0$$

so that the points lie on the boundaries.

$v_t = u_t$

$v_x = u_x + 2\varepsilon x$

$v_{xx} = u_{xx} + 2\varepsilon$ $u_{xx} = v_{xx} - 2\varepsilon$

$0 \geq u_t - u_{xx} + u = v_t - (v_{xx} - 2\varepsilon) + u$

$= v_t - v_{xx} + 2\varepsilon + v - \varepsilon x^2$

We know that $M \geq 0$ and we can have $v(x,t) = u(x,t) \leq M$ so $v(x,t) \geq 0$ and if this point was the maximum then $v_t = 0$ and $v_{xx} \leq 0$

plugging this into $u_t - u_{xx} + u \leq 0$ we get that $\underset{=0}{v_t} - \underset{-(\leq 0)}{v_{xx}} + \underset{\geq 0}{v}$ so then $-v_{xx} + v \geq 0$

but this is a contradiction to (*) and thus the maximum cannot be an interior point.

maximum lies on the boundary point. $\delta > 0$

$$\lim_{\delta \to 0^+} \frac{v(x,t) - v(x, t-\delta)}{\delta} = 0.$$

10

Yizhi Lyu 4075586

32

(1) Assume there are 2 solutions u_1, u_2 and let $w = u_1 - u_2$.

So $\begin{cases} w_t - k w_{xx} = 0 \\ w(x,0) = 0 \\ w(0,t) = 0 \\ w(L,t) = 0. \end{cases}$

Since the maximum is in $t=0$ or $x = 0/L$. so the maximum is 0.

10

Then $w(x,t) \le 0$. Inversely let $w_2 = u_2 - u_1$, $w_2(x,t) \le 0$.

So $w = 0$ and $u_1 = u_2$.

So the solution is unique.

(2) Assume $v(x,t) = u(x,t) - 1$.

So $\begin{cases} v_t - k v_{xx} = 0 & \text{on } 0 < x < \infty,\ t > 0 \\ v(x,0) = -1 \\ v(0,t) = 0. \end{cases}$

half line heat. $0 < x < \infty$, $0 < t < \infty$.

$$v(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_0^\infty \left[e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}} \right] \phi y \, dy$$

$$= \frac{1}{\sqrt{4k\pi t}} \int_0^\infty \left[e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}} \right] \cdot -1 \, dy$$

$$= \frac{1}{\sqrt{4k\pi t}} \int_0^\infty e^{-\frac{(x+y)^2}{4kt}} - e^{-\frac{(x-y)^2}{4kt}} \, dy = \frac{1}{\sqrt{4K\pi t}} \left(\int_0^\infty e^{-\frac{(x+y)^2}{4kt}} dy - \int_0^\infty e^{-\frac{(x-y)^2}{4kt}} dy \right)$$

set $p = \frac{x+y}{\sqrt{4kt}}$ $q = \frac{x-y}{\sqrt{4kt}}$. so $\frac{dy}{dp} = \sqrt{4kt}$ $\frac{dy}{dq} = -\sqrt{4kt}$

$$v(x,t) = \frac{1}{\pi} \int_{\frac{x}{\sqrt{4kt}}}^\infty e^{-p^2} dp + \frac{1}{\pi} \int_{\frac{x}{\sqrt{4kt}}}^\infty e^{-q^2} dq$$

$$v(x,t) = -\frac{1}{2} \operatorname{Erf}\left(\frac{x}{\sqrt{4kt}}\right) - \frac{1}{2} \operatorname{Erf}\left(\frac{x}{\sqrt{4kt}}\right) = -\operatorname{Erf}\left(\frac{x}{\sqrt{4kt}}\right)$$

So $u(x,t) = -\operatorname{Erf}\left(\frac{x}{\sqrt{4kt}}\right) + 1$

10

3. $u_{tt} - c^2 u_{xx} = e^{2x}$

Yizhi Lyu 4073586

$u(x,t) = 0 + 0 + \frac{1}{2c}\iint_\Delta f$

$= \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds$

$= \frac{1}{2c}\int_0^t \int_{x-ct+cs}^{x+ct-cs} e^{2y}\,dy\,ds$

$= \frac{1}{2c}\int_0^t \left[\frac{1}{2}e^{2y}\right]_{x-ct+cs}^{x+ct-cs} ds$

$= \frac{1}{2c}\int_0^t \left(\frac{1}{2}e^{2(x+ct-cs)} - \frac{1}{2}e^{2(x-ct+cs)}\right) ds$

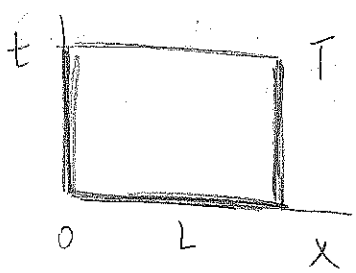
$= \frac{1}{2c}\left[-\frac{1}{4c}e^{2x+2ct-2cs} - \frac{1}{4c}e^{2x-2ct+2cs}\right]_0^t$

$= \frac{1}{2c}\left(-\frac{1}{4c}e^{2x+2ct-2ct} - \frac{1}{4c}e^{2x-2ct+2ct}\right)$

$= \frac{1}{2c}\left(-\frac{1}{2c}e^{2x}\right) = -\frac{1}{4c^2}e^{2x}$

9

4.



$u_t - u_{xx} + u \le 0$

if u is maximum.

$u \ge u_{xx} - u_t$

$u > 0$ means

$u_{xx} - u_t = 0$

~~If M is not on the sides x=0, x=L or t=0~~

Then ~~$M \in \{(x,t) \mid x \in (0,L),\ t \in (0,T)\}$~~

So it is on $x=0$ or $x=L$ or $t=0$

3

Guankui Qian 7961543

27

(1). $E[u](t)=\int_0^L u^2(x,t)\,dx$

$$\begin{cases} u_t - k u_{xx} = f(x,t) \\ u(x,0)=\phi(x) \\ u(0,t)=g(t) \\ u(L,t)=h(t) \end{cases}$$

5

the general solution is $f(x)+g(x) = f(x+ct)+g(x+ct)$

Assume $u(x,0)=0$. $u(0,t)=0$. $u(L,t)=0$.

$E(u)\int_0^L u^2(x,t)dx \Rightarrow \frac{d}{dt}\int_0^L u^2(x,t)dx$ $u^2\Big|_0^L = C$, the ~~time~~ only depends on L.

when $u(x,0)=0$. $u(0,t)=0$. $u(L,t)=0$, $E(u)=0$. and this is the unique solution. Hence proved

2). $u_t - k u_{xx}=0$ $0<x<\infty, t>0$ $Erf(s)=\frac{2}{\sqrt{\pi}}\int_0^s e^{-y^2}dy$

$u(x,0)=0$

$u(0,t)=1$ $V=u-1 \Rightarrow \begin{cases} V(x,0)=-1 \\ V(0,t)=0 \end{cases}$ this is exactly same to $\phi(x)=-1$

$u(x,t)=\frac{1}{\sqrt{4\pi kt}}\int_0^\infty \left[e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}}\right]\phi(y)\,dy$ $\phi(x)=-1$

$=\frac{1}{\sqrt{4\pi kt}}\int_0^\infty e^{-\frac{(x-y)^2}{4kt}}\phi(y) - \frac{1}{\sqrt{4\pi kt}}\int_0^\infty e^{-\frac{(x+y)^2}{4kt}}\phi(y)$

So $\frac{1}{\sqrt{4\pi kt}}\int_{-\frac{x}{\sqrt{4kt}}}^\infty e^{-p^2} - \frac{1}{\sqrt{4\pi kt}}\int_{\frac{x}{\sqrt{4kt}}}^{\infty} e^{-q^2}$

$=\frac{1}{\sqrt{4\pi kt}}\int_0^\infty e^{-\frac{(x-y)^2}{4kt}} - \frac{1}{\sqrt{4\pi kt}}\int_0^\infty e^{-\frac{(x+y)^2}{4kt}}$

$=-Erf\left(\frac{x}{\sqrt{4kt}}\right)$

$p=\frac{x-y}{\sqrt{4kt}}$ $q=\frac{x+y}{\sqrt{4kt}}$

$\begin{cases} y=0, \text{ the lower limit of } p \text{ is } \frac{x}{\sqrt{4kt}} \\ y\to\infty, \text{ the upper limit of } p \text{ is } \infty \\ y\to\infty, \text{ the lower limit of } q \text{ is } \frac{x}{\sqrt{4kt}} \\ y=0, \text{ the upper limit of } q \text{ is } -\infty \end{cases}$ $\Rightarrow$ Substitute $u=-Erf\left(\frac{x}{\sqrt{4kt}}-1\right)$

$u(x,t)=1-Erf\left(\frac{x}{\sqrt{4kt}}\right)$

10

(3). $\begin{cases} u_{tt}-c^2u_{xx}=e^{2x} & x,t\in\mathbb{R} \\ u(x,0)=0 \\ u_t(x,0)=0 \end{cases}$

$\sinh(x)=\frac{e^x-e^{-x}}{2}$ $\cosh(x)=\frac{e^x+e^{-x}}{2}$ $(\sinh(x))'=\cosh(x)$.

$u(x,t)=\int_0^t \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} e^{2s}\,ds\,d\tau = \frac{1}{2c}\int_0^t \left[2e^{2(x-c(t-\tau))} - 2e^{2(x+c(t-\tau))}\right]d\tau$

$=\frac{2e^{2(x-c(t-\tau))}}{c}\Big|_0^t + \frac{2e^{2(x+c(t-\tau))}}{c}\Big|_0^t = \frac{1}{2c}\left(2e^{2x}-2e^{2(x-ct)}+2e^{2x}-2e^{2(x+ct)}\right)$

9

$\gg$

4.

$$\begin{cases} U_t - U_{xx} + U \le 0 \\ R = [0, L] \times [0, T] \end{cases}$$

$U_t \le U_{xx} + U \le 0$

$U_t - U_{xx} \le -U$

$U_{xx} - U_t \ge U$

M is the maximum on of u on R and $M \ge 0$

If u attains M, we need max $(U_{xx} - U_t)$

If occure in the interior a. $0 < a < L$, $0 < a < T$

$\iint u_{xx} - \iint u_t \ge \underline{\underline{\iint u}} \Rightarrow$ the open rectangle

${}_0^L|x - \int t|_t^0 \ge u$, for $a < L$, $0 < a < T$, the sign is $\leq$

Any a between 0 and L, 0 and T. Can not let $x - \int t$ become largest

Only $x = 0$, $x = L$ or $t = 0$

3

Jordan Gonzalez
Math 124A

16

1. $u_t - ku_{xx} = f(x,t)$

$u(x,0) = \phi(x)$

$u(0,t) = g(t)$

$u(L,t) = h(t)$

$w(x,0) = 0$

$w(0,t) = 0$

$w(L,t) = 0$

Assuming that there are two possible solutions, we can deduce that they all equal 0 therefore they are the same

$$\int_0^L u_t^2 dt - ku_{xx}\, dx$$

$$\int_0^L 0^2\, dx$$

$$= \boxed{0}$$

3

7. $u_t - k u_{xx} = 0$

$u(x,0) = 0$

$u_t(x,0) = 0$

$\varphi_0 = \begin{cases} 0 & , x > 0 \\ 0 & , x = 0 \\ 1 & , x < 0 \end{cases}$

$\frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{\sqrt{4kt}}}$

$\frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{0} e^{-\frac{(x-y)^2}{\sqrt{4kt}}} \cdot 1 \, dy + \frac{1}{\sqrt{4\pi k t}} \int_{0}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \cdot 0 \, dy$

$1 = u - v \Rightarrow v = u - 1$

$s = \frac{(x-y)^2}{\sqrt{4kt}}$

$ds = \frac{dy}{\sqrt{4kt}}$

$dy = \sqrt{4kt}\, ds$

$\frac{1}{\sqrt{\pi}} \int_0^{\infty} 2 e^{-s^2} ds$

$\frac{2}{\sqrt{\pi}} \int_{\frac{1}{\sqrt{4kt}}}^{\infty} e^{-s^2} ds$

$\boxed{\operatorname{erf}\left(\frac{(x-y)^2}{\sqrt{4kt}}\right)}$

5

Jordan Gonzalez

3. $u_{tt} - c^2 u_{xx} = e^{2x}$

$u(x,0) = 0$

$u_t(x,0) = 0$

$$\int_0^t \left[\int_{x-c(t-s)}^{x+c(t+s)} e^{2x} \, ds \right] dx$$

$$\int_0^t \underset{v}{2e^{2x}} \cdot \underset{du}{c(t-s)} \, dx$$

$$2e^{2x} \cdot x c(t-s) - \int_0^t 2e^{2x} \, c(t-s) \, dx$$

$$2e^{2x} \cdot x c(t-s) - c(t-s) \int_0^t 2e^{2x} \, dx$$

$$2e^{2x} \cdot x c(t-s) - c(t-s) \left[e^{2x} \Big|_0^t \right]$$

$$2e^{2x} \cdot x c(t-s) - c(t-s) \left[e^{2t} - 1 \right]$$

$$\int_{x-c(t-s)}^{x+c(t+s)} e^{2x} \, ds$$

$$e^{2x} s \Big|_{x-c(t-s)}^{x+c(t+s)}$$

$$e^{2x}(x + c(t+s)) - e^{2x}(x - c(t-s))$$

$$e^{2x}\left[\cancel{x} + c(t+s)) - (\cancel{x} - c(t-s)\right]$$

$$2e^{2x} c(t-s)$$

$uv - \int v \, du$

$v = 2e^{2x}$ $\quad du = c(t-s)$

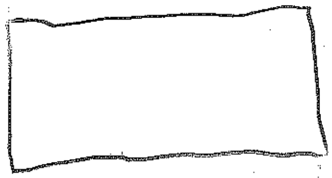
$dv = 4e^{2x}$ $\quad u = x c(t-s)$

5

#. $L, T > 0$

$u_t - u_{xx} + u \leq 0,$ $\quad R = [0, L] \times [0, T]$

Show that u attains the value M on the sides $x=0$ or $x=L$



3

Angela Ho
8343014

34

① $u_t - ku_{xx} = f(x,t)$ $\quad 0 < x < L,\ t > 0$

$u(x,0) = \phi(x)$

$u(0,t) = g(t)$

$u(L,t) = h(t)$

$E[u](t) = \int_0^L u^2(x,t)\,dx$

Let $w = u_1 - u_2$

so $w_t - kw_{xx} = f(x,t)$

$w(x,0) = \phi(x)$

$w(0,t) = g(t)$

$w(L,t) = h(t)$

Then : $E[w](t) = \int_0^L (w(x,t))^2\,dx$

$\frac{d}{dt} E[w](t) = 2\int_0^L w w_t\,dx = 2k\int_0^L w\, w_{xx} + f(x,t)\,dx$

$2k\int_0^L w\, w_{xx} + \int_0^L f(x,t)\,dx$

$\frac{(f(x,t))^2}{2} \Big|_0^L$

$\frac{1}{2}(f(L,t) - f(0,t))$

$u = w \quad dv = \int w_{xx}$

$du = dw \quad v = w_x$

$w w_x \Big|_0^L - \int_0^L w_x\,dw$

$(h(t) - g(t)) w_x \Big|_0^L - \int_0^L w_x\,dw$

This is decreasing and ≤ 0

$E[w](t) \quad 0$

so $w \leq 0$ and $w = u_1 - u_2 \leq 0$ ← then solution is unique since $u_1, u_2 \leq 0$

$E[w](0) = \phi(x)$

$\phi(x) \leq 0$

7

② $\begin{cases} u_t - k u_{xx} = 0 & 0 < x < \infty \ , t > 0 \\ u(x,0) = 0 \\ u(0,t) = 1 \end{cases}$

half line

$v = u - 1$

$H(x-y,t)\,\phi(y)\,dy$

$v(x,0) = u(x,0) - 1$

$v(x,0) = -1$

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \phi(y)\, dy$$

$$= \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} (-1)\, dy$$

$$\int_0^{\infty} + \int_{-\infty}^{0}$$

$$= -\left(\frac{1}{\sqrt{4\pi k t}} \int_0^{\infty} e^{-\frac{(x-y)^2}{4kt}} dy \right) - \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4kt}} dy$$

$S = \frac{x-y}{\sqrt{4kt}}$

$ds = \frac{-dy}{\sqrt{4kt}}$

$-\sqrt{4kt}\, ds = dy$

$$= \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-s^2} ds + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{0} e^{-s^2} ds$$

$$= \frac{1}{\sqrt{\pi}} \int_{\frac{x-y}{\sqrt{4kt}}}^{-\infty} e^{-s^2} ds + - \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-s^2} ds$$

$$= -\frac{1}{\sqrt{\pi}} \int^{\frac{x-y}{\sqrt{4kt}}} + \frac{1}{\sqrt{\pi}} \int_{\infty}^{\frac{x}{\sqrt{4kt}}}$$

(2) $\begin{cases} u_t - k u_{xx} = 0 & 0 < x < \infty, \; t > 0 \\ u(x,0) = 0 \\ u(0,t) = 1 \end{cases}$

Angela Ho

$V = u - 1$

$v(x,0) = u(x,0) - 1 = 0 - 1$

$v(x,0) = -1$

$$v(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_0^\infty \left(e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}} \right) \phi(y)\, dy$$

$$((H(x-y,t) - (H(x+y,t)$$

$$= -\frac{1}{\sqrt{4\pi k t}} \int_0^\infty e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}} \, dy$$

$S = \frac{(x-y)}{\sqrt{4kt}}$ $\quad$ $S = \frac{(x+y)}{\sqrt{4kt}}$

$ds = \frac{-dy}{\sqrt{4kt}}$ $\quad$ $ds = \frac{dy}{\sqrt{4kt}}$

$-\sqrt{4kt}\, ds = dy$

$$= -\frac{1}{\sqrt{4\pi k t}} \int_0^\infty e^{-\frac{(x-y)^2}{4kt}} dy + \frac{1}{\sqrt{4\pi k t}} \int_0^{\infty} e^{-\frac{(x+y)^2}{4kt}} dy$$

$$= \frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^{-\infty} e^{-s^2} dy + \frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^{\infty} e^{-s^2} dy$$

$$= \left(\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{\sqrt{4kt}}} = \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4kt}}} + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{0} \right) \quad -\frac{1}{\sqrt{\pi}} \int_{\infty}^{\frac{x}{\sqrt{4kt}}} e^{-s^2} dy$$

$$= \left(\frac{1}{\sqrt{\pi}} \operatorname{Erf}\left(\frac{x}{\sqrt{4kt}}\right) + \frac{\sqrt{\pi}}{2} \right) + -\frac{1}{\sqrt{\pi}} \left(\operatorname{Erf}\left(\frac{x}{\sqrt{4kt}}\right) \right)$$

$$\int_{-\infty}^{\infty} e^{-s^2} ds = \sqrt{\pi}$$

$$= \frac{\sqrt{\pi}}{2} = v(x,t)$$

$$\boxed{u(x,t) = \frac{\sqrt{\pi}}{2} - 1}$$

9

③ $\begin{cases} u_{tt} - c^2 u_{xx} = e^{2x} & (x,t) \in \mathbb{R}^2 \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}$

Angela Ho

$$u(x,t) = \frac{1}{2}(\phi(x+ct) + \phi(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds$$

$$= \frac{1}{2}(\underset{0}{\phi(x)} + \underset{0}{\phi(x)}) + \frac{1}{2c}\int_{x-ct}^{x+ct} 0\,ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} e^{2s}\,ds\,dt$$

$$= 0 + 0 + 0 + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} e^{2s}\,ds\,dt$$

$u = 2s$

$du = 2\,ds$

$\frac{du}{2} = ds$

$$= \left(\frac{1}{2c}\int_0^t dt\right) \frac{1}{2}\int_{x-c(t-s)}^{x+c(t-s)} e^u\,du$$

change bounds

$$= \frac{1}{2c}\, t \Big|_0^t$$

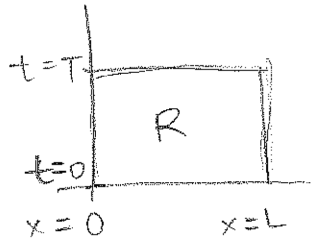
$$\cancel{\frac{1}{2}\int_{2(x-c(t-s))}^{2(x+c(t-s))} e^u\,du}$$

$$= \frac{t}{2c}$$

$$\cancel{\frac{1}{2} e^u \Big|_{2(x-c(t-s))}^{2(x+c(t-s))}}$$

$$u(x,t) = \frac{t}{4c}\left(e^{2x+2ct-2s} - e^{2x-2ct+2cs}\right)$$

4 $L, T > 0$

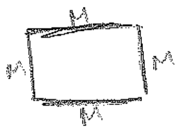


$u_t - u_{xx} + u \le 0$

Boundary Con.
for maximum
$u(x,0)=0$

$u(L,0)=0$

$M = \max(u) \ge 0$



$u_t = 0$

$u_{xx} \le 0$ concave down for a maximum

$u \le 0$

10

So $\underset{\substack{\uparrow \\ 0}}{u_t} - \underset{\substack{\uparrow \\ \le 0}}{u_{xx}} + \underset{\substack{\uparrow \\ \le 0}}{u} \le 0$

satisfying inequality which is ≤ 0

Jiayi Sun 9307901

35

1. $\begin{cases} u_t - k u_{xx} = f(x,t) \\ u(x,0) = \phi(x) \\ u(0,t) = g(t) \\ u(L,t) = h(t) \end{cases}$

Sol. Suppose u_1 and u_2 are 2 solutions for the heat equation

let $w = u_1 - u_2$

then we can get

$\begin{cases} w_t - k w_{xx} = 0 \\ w(0,t) = 0 \\ w(L,t) = 0 \end{cases}$

$w(x,0) = 0$

10

According to the maximum principle,

$w(x,t) \le 0$

According to the minimum principle

$w(x,t) \geq 0$

$\Rightarrow$ $w(x,t) = 0$

$\Rightarrow$ $u_1(x,t) = u_2(x,t)$

The solution is unique.

2. $\begin{cases} u_t - k u_{xx} = 0 & 0 < x < \infty, t > 0. \\ u(x,0) = 0 \\ u(0,t) = 1 \end{cases}$

Jiayi Sun
9307901

Half Line.

Sol let $v = u - 1$

Then $\begin{cases} v_t - k v_{tt} = 0 \\ v(x,0) = -1 \\ v(0,t) = 0 \end{cases}$

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty [H(x-y,t) - H(x+y,t)]\, \phi(y)\, dy$$

$$= \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left(e^{\frac{-(x-y)^2}{4kt}} - e^{\frac{-(x+y)^2}{4kt}} \right) \phi(y)\, dy$$

$$= \frac{1}{\sqrt{4\pi kt}} \int_0^\infty e^{\frac{-(x-y)^2}{4kt}} \phi(y)\, dy - \frac{1}{\sqrt{4\pi kt}} \int_0^\infty e^{\frac{-(x+y)^2}{4kt}} \phi(y)\, dy$$

$$= \underbrace{\frac{1}{\sqrt{4\pi kt}} \int_0^\infty e^{\frac{-(x-y)^2}{4kt}} (-1)\, dy}_{①} - \underbrace{\frac{1}{\sqrt{4\pi kt}} \int_0^\infty e^{-\frac{(x+y)^2}{4kt}} (-1)\, dy}_{②}$$

solve ①

let $r = \frac{x-y}{\sqrt{4kt}}$ then $dr = -\frac{1}{\sqrt{4kt}}$

$y = 0 = -\sqrt{4kt}\, r + x$

$r = \frac{x}{\sqrt{4kt}}$

$y = \infty = x - \sqrt{4kt}\, r$

$r = \frac{x - \infty}{\sqrt{4kt}} = -\infty$

so,

$$① = -\frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^{-\infty} e^{-r^2} dr$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{\sqrt{4kt}}} e^{-r^2} dr$$

Solve ②:

let $s = \frac{x+y}{\sqrt{4kt}}$ $\quad ds = \frac{1}{\sqrt{4kt}}\, dy \qquad y = \sqrt{4kt}\, s - x$

so,

$$② = \frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^{\infty} e^{-s^2}\, ds$$

$y = 0 \qquad \sqrt{4kt}\, s - x = 0$

$s = \frac{x}{\sqrt{4kt}}$

$y = \infty \qquad \sqrt{4kt}\, s - x = \infty$

$s = \frac{\infty + x}{\sqrt{4kt}} = \infty$

So, ① + ② = original $u(x,t)$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{\sqrt{4kt}}} e^{-r^2}\, dr + \frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^{\infty} e^{-s^2}\, ds.$$

$$= \frac{1}{\sqrt{\pi}} \left[\int_{-\infty}^{0} e^{-r^2}\, dr + \int_{0}^{\frac{x}{\sqrt{4kt}}} e^{-r^2}\, dr \right] + \frac{1}{\sqrt{\pi}} \left[\int_0^{\infty} e^{-s^2}\, ds - \int_0^{\frac{x}{\sqrt{4kt}}} e^{-s^2}\, ds \right]$$

$$= \frac{1}{\sqrt{\pi}} \left[\cancel{\frac{\sqrt{\pi}}{2} + Erf}\left(\frac{x}{\sqrt{4kt}}\right) \right] + \frac{1}{\sqrt{\pi}} \left[\frac{\sqrt{\pi}}{2} - Erf\left(\frac{x}{\sqrt{4kt}}\right) \right]$$

$$= \frac{1}{2} + \frac{1}{\sqrt{\pi}} Erf\left(\frac{x}{\sqrt{4kt}}\right) + \frac{1}{2} - \frac{1}{\sqrt{\pi}} Erf\left(\frac{x}{\sqrt{4kt}}\right)$$

$$= 1$$

9

3. $\begin{cases} u_{tt} - c^2 u_{xx} = e^{2x} & \text{on } (x,t) \in \mathbb{R}^2 \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}$

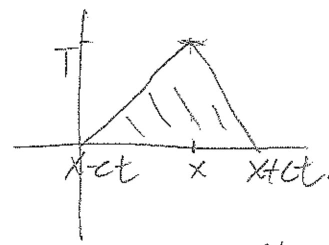
Jiayi Sun 9307901

Sol.

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) - \phi(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{ct+x} \psi(y)\,dy + \frac{1}{2}\iint_T f(x,t)\,dx\,dt$$

$$= \frac{1}{2}\cdot 0 + \frac{1}{2c}\int_{ct-x}^{ct+x} 0\,dy + \frac{1}{2}\iint_T f(x,t)\,dx\,dt.$$

$$= \frac{1}{2}\iint_T e^{2x}\,dx\,dt$$



$$\frac{x-ct+x+ct}{2} = x.$$

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2}$$

$$\cancel{\int_{0}^{x-ct} e^{2x}\,dx\,dt.}$$

$$\cancel{= \frac{1}{2}\int_0^t \frac{1}{2}e^{2x}\Big|_{x-ct}^{x+ct}\,dt}$$

$$\cancel{= \frac{1}{2}\int_0^t \frac{1}{2}\left(e^{2(x+ct)} - e^{2(x-ct)}\right)dt}$$

$$\cancel{= \frac{1}{2}\int_0^t \frac{e^{2x+2ct} - e^{-2x-2ct}}{2}\,dt}$$

$$\cancel{= \frac{1}{2}\int_0^t \sinh(x)\,dt.}$$

$$\cancel{= \frac{1}{2}}$$

$$= \frac{1}{2}\int_{x-ct}^{x+ct}\int^{t} e^{2x}\,dt\,dx$$

$$= \frac{1}{2}\int_{x-ct}^{x+ct} te^{2x}\Big|_0^t\,dx$$

$$= \frac{1}{2}\int_{x-ct}^{x+ct} te^{2x}\,dx$$

$u = t \qquad dv = e^{2x}\,dx$

$du = 0 \qquad v = \frac{e^{2x}}{2}.$

$$= \frac{1}{2}\cdot\frac{t}{2}e^{2x}\Big]_{x-ct}^{x+ct}$$

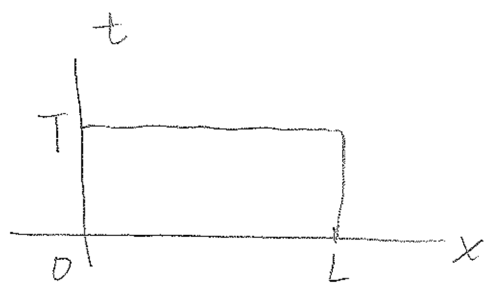
$$= \frac{t}{4}\left(e^{2(x+ct)} - e^{2(x-ct)}\right)$$

$$= \frac{t}{2}\cdot\frac{e^{2(x+ct)} - e^{2(x-ct)}}{2}$$

$$= \frac{t}{2}\cdot\sinh(x).$$

6

4. Jiayi Sun 9307901.



$R = [0,L] \times [0,T]$

M is maximum $M \geq 0$.

$u_t - u_{xx} + u \leq 0$.

WTS the maximum on the sides $x=0$, $x=L$ or on the bottom $t=0$.

Sol. Suppose the maximum point occur in the interior
then the first derivative vanish and the second derivative ≤ 0.

Then $u_t = 0$

$u_{xx} \leq 0 \Rightarrow -u_{xx} \geq 0$.

So $u_t - u_{xx} + u \leq 0$

$\Rightarrow 0 - u_{xx} + u \leq 0$.

10

But $-u_{xx} \geq 0$

and maximum of $u = M \geq 0$.

So $-u_{xx} + u \geq 0 \Rightarrow u_t - u_{xx} + u \geq 0$

contradiction.

The maximum cannot occur in the interior.

34

Keyu Liu

9732124

(1) let $w = u - v$ (u, v are two sols)

$$\frac{d}{dx} E[w](t) = 2\int_0^l w(x,t)\, w_t(x,t)\, dx$$

$$= w^2(x,t)\, w_x(x,t) - \int_0^l w^2(x,t)\, w_{xx}(x,t)\, dt$$

$$= h(t)\, h(t)\, w_x(0,t) - g(t)\, g(t)$$

$$\cdots$$

$$\leq 0$$

7

$\because$ $E[w](t)$ is decreasing $\forall\ t > 0$

$\therefore$ $w = 0$ ($u - v = 0$) there is no difference

$\therefore$ unique prove.

(2) Let $v = u - 1$

$$\begin{cases} v_t - k v_{xx} = 0 & \text{on } 0 < x < \infty,\ t > 0 \\ v(x,0) = -1 = \phi(x) \\ v(0,t) = 0 \end{cases}$$

$$v(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_0^\infty \left(e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}} \right) \phi(y)\, dy$$

$$= -\frac{1}{\sqrt{4\pi k t}} \int_0^\infty e^{-\frac{(x-y)^2}{4kt}} dy + \frac{1}{\sqrt{4\pi k t}} \int_0^\infty e^{-\frac{(x+y)^2}{4kt}} dy$$

Let $p = \frac{x-y}{\sqrt{4kt}}$ $\quad dp = -\frac{1}{\sqrt{4kt}} dy$ $\quad dy = -\sqrt{4kt}\, dp$

Let $q = \frac{x+y}{\sqrt{4kt}}$ $\quad dq = \frac{1}{\sqrt{4kt}} dy$ $\quad dy = \sqrt{4kt}\, dq$

$$= -\frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^{-\infty} e^{-p^2} dp + \frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^{\infty} e^{-q^2} dq$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{\sqrt{4kt}}} e^{-p^2} dp + \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-q^2} dq - \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4kt}}} e^{-q^2} dq$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4kt}}} e^{-p^2} dp + \frac{1}{\sqrt{\pi}} \int_{-\infty}^0 e^{-p^2} dp + \frac{1}{2} - \frac{1}{2} \operatorname{Erf}\left(\frac{x}{\sqrt{4kt}}\right)$$

9

$$= \frac{1}{2} \operatorname{Erf}\left(\frac{x}{\sqrt{4kt}}\right) + \frac{1}{2} + \frac{1}{2} - \frac{1}{2} \operatorname{Erf}\left(\frac{x}{\sqrt{4kt}}\right) = 1$$

$\therefore$ $u = v + 1 = 1 + 1 = 2$

Keyu Liu
9732124

(3) $u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)\,ds + \frac{1}{2c}\iint_{\Delta} f$

$= \frac{1}{2c}\iint_{\Delta} f$

$\frac{1}{2c}\int_0^{t_0}\int_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} e^{2x}\,dx\,dt$

$= \frac{1}{2c}\int_0^{t_0} \frac{1}{2}e^{2x}\Big]_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} dt = \frac{1}{4c}\int_0^{t_0} e^{2x_0+2c(t_0-t)} - e^{2x_0-2c(t_0-t)}\,dt$

$= \frac{1}{4c}\int_0^{t_0} e^{2x_0}e^{2ct_0}e^{-2ct}\,dt - \frac{1}{4c}\int_0^{t_0} e^{2x_0}e^{-2ct_0}e^{2ct}\,dt$

$= \frac{e^{2x_0}e^{2ct_0}}{4c}\,\frac{e^{-2ct}}{-(2c)}\Big]_0^{t_0} - \frac{e^{2x_0}e^{-2ct_0}}{4c}\,\frac{e^{2ct}}{2c}\Big]_0^{t_0}$

$= -\frac{e^{2x_0}e^{2ct_0}}{8c^2}\cdot e^{-2ct_0} - \frac{e^{2x_0}e^{-2ct_0}}{8c^2}\cdot e^{2ct_0}$

9

$= -\frac{e^{2x_0}}{8c^2} - \frac{e^{2x_0}}{8c^2}$

$= \frac{-2e^{2x_0}}{8c^2} = -\frac{e^{2x_0}}{4c^2}$

(4) $\because$ u has a maximum

$u_t = 0$

$u_{xx} \leq 0$

$u_t - u_{xx} \leq -u$

$\therefore$ $u_t - u_{xx} \geq 0$

$0 \leq -u$

this is impossible, since u is continuous on $R = [0, L] \times [0, T]$

$u > 0$ $\quad -u < 0$ in the interior of the rectangle

this is contradict

$\therefore$ the only choice is that $u = 0$

which means $u_t - u_{xx} = 0$ $\quad u_t - u_{xx} + u = 0$

when $u = 0$, it must be $x = 0$, $x = L$ or $t = 0$

to attains the max M, it can not be any points in the interior.

10

3. $\begin{cases} u_{tt} - c^2 u_{xx} = e^{2x} & \text{on } (x,t) \in \mathbb{R}^2 \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}$

$u(x,t) = \frac{1}{2}\left[g(x-ct) + g(x+ct)\right] + \frac{1}{2c}\int$ [illegible] $)\,ds$

3

$u(x,t) = 0$

4. $L, T > 0$

u is differentiable twice

and $u_t - u_{xx} + u \leq 0$

u is continuous on $R = [0,L] \times [0,T]$.

M is maximum on of u on R and $M \geq 0$

Show u attains value M on sides $x=0$ or $x=L$ or on bottom $t=0$ of R

0

Brian Sheung

8

1: $\begin{cases} u_t - ku_{xx} = f(x,t) \\ u(x,0) = \phi(x) \\ u(0,t) = g(t) \\ u(L,t) = h(t) \end{cases}$ for $0 < x < L, t > 0$

$u(x,t) =$ $=$ ~~[illegible]~~

Show Uniqueness

energy $E[u](t) = \int_0^L u^2(x,t)\,dx$

$\begin{cases} V_t - kV_{xx} \neq f(x,t) \\ v(x,0) \neq \phi(x) \\ v(0,t) \neq g(t) \\ v(L,t) \neq h(t) \end{cases}$ → Unique

0

2. $\begin{cases} u_t - ku_{xx} = 0 \\ u(x,0) = 0 \\ u(0,t) = 1 \end{cases}$ on $0 < x < \infty, t > 0$

$Erf(s) = \frac{2}{\sqrt{\pi}} \int_0^s e^{-x^2} dx$

$v = u - 1$

$V_t - kV_{xx} = 0$
$v(x,0) = -1$
$v(0,t) = 0$

$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty e^{\frac{-(x-y)^2}{4kt}} + e^{\frac{-(x+y)^2}{4kt}}$

$\rightarrow \frac{1}{\sqrt{4\pi kt}} \int_0^\infty e^{-p^2} dp$

let $p = \frac{x-y}{\sqrt{4kt}}$ $\quad q = \frac{x+y}{\sqrt{4kt}}$

$\rightarrow \frac{1}{2} Erf(s)$

$\frac{1}{\sqrt{4\pi kt}} \int_0^\infty e^{-p^2} dp + e^{-q^2} dq$

5

Let $w = u_1 - u_2$ $\quad \phi_1(x) - \phi_2(x) = 0$, $g_1(x) - g_2(x) = 0$, $h_1(x) - h_2(x) = 0$

Josh DiGino (1)
7137995

27

(1) To show uniqueness, we prove $\frac{d}{dt} E[u](w) = 0$

$\Rightarrow \frac{d}{dt} \int_0^L w^2(x,t)\,dx \Rightarrow \int_0^L 2ww_x$ t

$\Rightarrow 2kw\Big|_0^{L} - \int_0^L w_x \; dx \leq 0$

6

Following top solution $\Rightarrow \int_{-L}^{0} w^2(x,t)\,dx \geq 0$

So, $0 \leq \int_0^L 2ww_x\,dx \leq 0$

$\Rightarrow \int_0^L 2ww_x\,dx = 0$

$\Rightarrow \frac{d}{dt} E(u)(w) = 0$

$\Rightarrow w = 0.$

$\Rightarrow u_1 - u_2 = 0.$

$\Rightarrow$ $\boxed{\therefore \text{ unique } \checkmark}$

Homogenous Heat equation on half line

Josh DiFiore 71357995 ②

Let $v = u - 1$

② $$\begin{cases} u_t - ku_{xx} = 0 \\ u(x,0) = 0 \\ u(0,t) = 1 \end{cases} \qquad 0 < x < \infty,\ t > 0$$

$\Downarrow$

$$\begin{cases} v_t - ku_{xx} = 0 \\ u(x,0) = 0 \\ u(0,t) = 0 \end{cases}$$

Using odd extension, since it is on half-line

$$\Rightarrow v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left[e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}} \right] dy$$

$$\Rightarrow \frac{1}{\sqrt{4\pi kt}} \int_0^\infty e^{-\frac{(x-y)^2}{4kt}} dy - \frac{1}{\sqrt{4\pi kt}} \int_0^\infty e^{-\frac{(x+y)^2}{4kt}} dy$$

Let $p = \frac{x-y}{\sqrt{4kt}} \qquad q = \frac{x+y}{\sqrt{4kt}}$

$dp = \frac{-dy}{\sqrt{4kt}} \qquad dq = \frac{dy}{\sqrt{4kt}}$

$$\Rightarrow \left[-\frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^{-\infty} e^{-p^2} dp \right] - \left[\frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^{\infty} e^{-q^2} dq \right]$$

9

$$\Rightarrow \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{\sqrt{4kt}}} e^{-p^2} dp - \left[\frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^{\infty} e^{-q^2} dq \right]$$

$$1 - erf(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt \qquad \Rightarrow \left[\frac{1}{2} + \frac{erf\left(\frac{x}{\sqrt{4kt}}\right)}{2} \right] - \left[\frac{1}{2} - \frac{erf\left(\frac{x}{\sqrt{4kt}}\right)}{2} \right]$$

$$1 + erf(x) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^x e^{-t^2} dt$$

$$\Rightarrow erf\left(\frac{x}{\sqrt{4kt}}\right) = v(x,t)$$

$$\Rightarrow v(x,t) = u(x,t) - 1 \qquad \Rightarrow \boxed{erf\left(\frac{x}{\sqrt{4kt}}\right) + 1 = u(x,t)}$$

Josh DiGino ③
7137995

③ Inhomogeneous wave equation on whole line

$$\begin{cases} u_{tt} - c^2 u_{xx} = e^{2x} \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}$$

general solution formula is given by

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds + \frac{1}{2c}\iint_{\Delta T} f$$

plugging in $\phi(x)=0$, $\psi(x)=0$

$$\Rightarrow u(x,t) = \frac{1}{2c}\iint_{\Delta T} f \Rightarrow \frac{1}{2c}\int_0^{t_0}\int_{x_0-ct_0+ct}^{x_0+ct_0-ct} e^{2x}\,dx\,dt$$

$e^{2x}\left(e^{2ct_0-2ct} - e^{-2ct_0+2ct}\right) \Rightarrow e^{2x}\left(\sinh(2ct_0-2ct)\right)$

$$\Rightarrow \frac{1}{2c}\left(\int_0^{t_0}\left[\frac{e^{2x}}{2}\right]_{x_0-ct_0+ct}^{x_0+ct_0-ct} dt\right) = \frac{1}{2c}\int_0^{t_0}\left[\frac{e^{2(x_0+c(t_0-t))} - e^{2(x_0-c(t_0-t))}}{2}\right]dt$$

$$\Rightarrow \frac{1}{2c}\int_0^{t_0} e^{2x_0}\left(\sinh\left(2ct_0-2ct\right)\right)dt \Rightarrow \frac{e^{2x_0}}{2c}\left[\frac{-\cosh(2ct_0-2ct)}{-2c}\right]_0^{t_0}$$

$$\Rightarrow \frac{-e^{2x_0}}{4c^2}\left(-\cosh\left(2ct_0-2ct_0\right) + \cosh\left(2ct_0-0\right)\right)$$

−1

$$\Rightarrow \frac{-e^{2x}}{4c^2}\left[\cosh(2ct_0)\right] \Rightarrow u(x_0,t_0) = \frac{-e^{2x_0}}{4c^2}\left[\cosh(2ct_0)\right]$$

$$\Rightarrow \boxed{u(x,t) = \frac{-e^{2x}}{4c^2}\left[\cosh(2ct)\right]}$$

9

Josh DiGiro (4)
7137995

(4)

$u_t - u_{xx} + u \leq 0 \qquad L, T > 0$

on $\underline{(0, L)} \times \underline{(0, T)}$

By max principle, solution is maximized on boundaries, i.e. $x = L$, $t = T$, $x = 0$, $t = 0$

3

if M is max of u, $M \geq 0$,

we know $-(u_t - u_{xx}) \geq 0$

as well. $\Rightarrow u_{xx} - u_t \geq 0 \Rightarrow u_{xx} \geq u_t$

By max principle, $\frac{d}{dt} u_{xx} \geq M \geq 0$, $\frac{d}{dt} u_t \geq M \geq 0$

so therefore by squeeze thm and max principle,

$\frac{d}{dt} u$ is maximized on endpoints,

when $x \in \{0, L\}$, $t \in \{0, T\}$

Math124a
Midterm 2

Gabriela Flores
24 July 2017

22

(1) $\begin{cases} u_t - ku_{xx} = f(x,t) & \text{for } 0<x<L,\ t>0 \\ u(x,0) = \phi(x) \\ u(0,t) = g(t) \\ u(L,t) = h(t) \end{cases}$

Proof: Let $w = u - v$, where $v(x,t) = f(x,t)$.

Then $v_t(x,t) = u_t(x,t)$ and $v_{xx}(x,t) = u_{xx}(x,t)$.

$v(x,0) = v(0,t) = v(L,t) = 0$. Then $g(t) = h(t) = 0$.

Then $f(x,t)$ is a constant. But $u_t - ku_{xx} = f(x,t) \neq 0$. *

Then $u = v$. Therefore $u_t - ku_{xx} = f(x,t)$ has a unique solution for $0<x<L$, $t>0$.

3

Gabriela Flores

$* \; Erf(s) = \frac{2}{\sqrt{\pi}} \int_0^s e^{-x^2} dx$

(2) $\begin{cases} U_t - KU_{xx} = 0 & 0 < x < \infty, t > 0 \\ U(x,0) = 0 \\ U(0,t) = 1 \end{cases}$

$* \quad U(x,t) = \frac{1}{\sqrt{4\pi Kt}} \int_0^\infty \left(e^{-\frac{(x-y)^2}{4Kt}} + e^{-\frac{(x+y)^2}{4Kt}} \right) \phi(y)\, dy$

general solution for half-line on Heat Equ.

Let $V = u-1$. Then $\begin{cases} V_t - KV_{xx} = 0 & 0 < x < \infty, t > 0 \\ V(x,0) = -1 \\ V(0,t) = 0 \end{cases}$

$$V(x,t) = \frac{1}{\sqrt{4\pi Kt}} \int_0^\infty \left(e^{-\frac{(x-y)^2}{4Kt}} + e^{-\frac{(x+y)^2}{4Kt}} \right)(-1)\, dy$$

$$= \frac{1}{\sqrt{4\pi Kt}} \left[\int_0^\infty -e^{-\frac{(x-y)^2}{4Kt}} dy - \int_0^\infty e^{-\frac{(x+y)^2}{4Kt}} dy \right]$$

① ②

① Let $s = \frac{x-y}{\sqrt{4Kt}}$ $\quad - \frac{1}{\sqrt{\pi}} \int_0^\infty -e^{-s^2} dy$

$ds = \frac{-dy}{\sqrt{4Kt}}$

$(-1)s = \left(\frac{x+y}{\sqrt{4Kt}}\right)(-1)$

$(-1)ds = \frac{dy\,(-1)}{\sqrt{4Kt}}$ $\Rightarrow \frac{1}{\sqrt{\pi}} \int_0^\infty -e^{-s^2} dy$

② Let $s = \frac{x+y}{\sqrt{4Kt}}$ $\quad - \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-s^2} dy$

$ds = \frac{dy}{\sqrt{4Kt}}$

Put together: $+\frac{1}{\sqrt{\pi}} \int_0^\infty e^{-s^2} dy + \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-s^2} dy$

$= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-s^2} dy = Erf \int_0^\infty e^{-s^2} dy$ for $s = \frac{x-y}{\sqrt{4Kt}}$

6

(3) $\begin{cases} U_{tt} - c^2 U_{xx} = e^{2x} & \text{on } (x,t) \in \mathbb{R}^2 \\ U(x,0) = 0 & \phi \\ U_t(x,0) = 0 & \psi \end{cases}$

• $\sinh(x) = \frac{e^x - e^{-x}}{2}$ $\quad (\sinh)' = \cosh$

• $\cosh(x) = \frac{e^x + e^{-x}}{2}$

general solution for wave equation nonhomogeneous: $\frac{1}{2}(\phi(x-ct) + \phi(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds + \frac{1}{2c}\iint_T F(\dots)$

$x = \sin\theta$
$y = \cos\theta$

$u(x,t) = \frac{1}{2c}\iint e^{2x}\,dy\,dx$

$x = 2\sin\theta$
$dx = 2\cos\theta\,d\theta$

change of coordinates: $\frac{1}{2c}\int_0^{\pi}\int_0^1 e^{2\sin\theta}\, r\,dr\,d\theta = \frac{1}{2c}\int_0^{\pi} e^{2\sin\theta}\,d\theta \int_0^1 r\,dr$

5

$u_t - u_{xx} + u \leq 0$ 

Gabriela Flores

(4)

Let $L, T > 0$. We have $u_t - u_{xx} + u \leq 0$.

Suppose u is continuous on $R = [0,L] \times [0,T]$.

Assume M is maximum of u on R and $M \geq 0$.

Let M attain max value in the interior (x_0, t_0).

Then $u(x_0,t_0) \geq 0$. So $u_t(x_0,t_0) \leq 0$ and $u_{xx}(x_0,t_0) \geq 0$

Then $u(x_0,t_0) \leq \underbrace{u_{xx} - u_t}_{\leq 0}$ $\ast$.

So M not in the interior in R.

Then u attains value M on sides $x=0$, $x=L$, or $t=0$ of R.

8

Norbert Shao
3364957

Midterm 2

22

1. Suppose u_1 and u_2 are solutions are solutions to the given problem.

Let $u = u_1 - u_2$ then $f(x,t) = \phi(x) = g(t) = h(t) = 0$.

In addition, let $E[u](t) = \int_0^L u^2(x,t)\,dx$ which is a decreasing function then we

Show this

can construct the given inequality $0 \le \int_0^L u^2(x,t)\,dx \le \int_0^L u^2(x,0)\,dx = 0$

which leads to the conclusion that $u = 0$. Therefore $u_1 - u_2 = 0$

$\Rightarrow u_1 = u_2$ proving the solution is unique.

3

2. $\frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left[e^{-(x-y)^2/4kt} - e^{-(x+y)^2/4kt}\right] \phi(y)\,dy = u(x,t)$

Let $v = u - 1$:

$v_t - kv_{xx} = 0$

$v(x,0) = -1$

$v(0,t) = 0$

$$\Rightarrow -\frac{1}{\sqrt{4\pi kt}} \int_0^\infty e^{-(x-y)^2/4kt} - e^{-(x+y)^2/4kt}\,dy$$

$$\Rightarrow -\frac{1}{\sqrt{4\pi kt}} \left[\int_0^\infty e^{-(x-y)^2/4kt}\,dy - e^{-(x+y)^2/4kt}\,dy\right]$$

$p = -\frac{(x-y)}{\sqrt{4kt}}$ $\qquad q = \frac{(x+y)}{\sqrt{4kt}}$

$\frac{dp}{dy} = \frac{-1}{\sqrt{4kt}}$ $\qquad \frac{dq}{dy} = \frac{1}{\sqrt{4kt}}$

$dy = \sqrt{4kt}\,dq$

$-dy = \sqrt{4kt}\,dp$

$$\Rightarrow -\frac{1}{\sqrt{4\pi kt}} \left[-\sqrt{4kt}\int_{-\infty}^{\frac{x}{\sqrt{4kt}}} e^{-p^2}\,dp - \sqrt{4kt}\int_{\frac{x}{\sqrt{4kt}}}^{\infty} e^{-q^2}\,dq\right]$$

Norbert Shuo

$$\Rightarrow \frac{\sqrt{4\kappa t}}{\sqrt{4\pi\kappa t}}\left[\int_{-\infty}^{x/\sqrt{4\kappa t}} e^{-p^2}dp + \int_{\frac{x}{\sqrt{4\kappa t}}}^{\infty} e^{-q^2}dq\right]$$

$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{x/\sqrt{4\kappa t}} e^{-p^2}dp + \frac{1}{\sqrt{\pi}}\int_{\frac{x}{\sqrt{4\kappa t}}}^{\infty} e^{-q^2}dq$$

$$= \boxed{\frac{1}{2}\operatorname{Erf}(x/\sqrt{4\kappa t}) + \frac{1}{2}\operatorname{Erf}(x/\sqrt{4\kappa t}) + 1}$$

$u(0,t) = 1$

$\frac{1}{2}\operatorname{Erf}(0) = 0 + 0 + 1 = u(0,t)$

7

3. $u_{tt} - c^2 u_{xx} = f(x,t)$

$$u(x,t) = \frac{1}{2}[0] + \frac{1}{2c}\int 0 + \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} e^{2x}\,dx$$

$$= \frac{1}{2c}\left[\frac{e^{2x}}{2}\Big|_{x-c(t-s)}^{x+c(t-s)}\right]$$

$$= \frac{1}{2c}\left[\frac{e^{2(x+c(t-s))}}{2} - \frac{e^{2(x-c(t-s))}}{2}\right]$$

$$= \frac{1}{2c}\left[\frac{e^{2x}}{2}\left[\frac{e^{ct-cs}}{2} - \frac{e^{-ct+cs}}{2}\right]\right]$$

$$= \frac{1}{2c}\left[\frac{e^{2x}}{2}\left[\sinh(ct-cs)\right]\right]$$

$$= \boxed{\frac{e^{2x}}{4c}\left[\sinh(ct-cs)\right]}$$

$u_t = \frac{e^{2x}}{4c}(c)\left[\cosh(ct-cs)\right]$

$u_{tt} = e^{2x}c\left[\sinh(ct-cs)\right]$

$u_x = \frac{2e^{2x}}{4c}\sinh(ct-cs)$

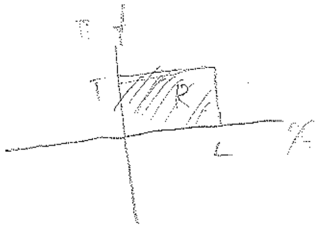
$u_{xx} = \frac{e^{2x}}{c}\sinh(ct-cs)$

$c^2 u_{xx} = ce^{2x}\sinh = u_{tt}$ ✓

9

4.

Norbert Shu
33531057



$u_t - u_{xx} + u \le 0$

according to the maximum principle the max of $u(x,t)$ should occur on either $x = 0, L$ or $t = 0$.

Suppose $u(x,t) = M$ when $0 < x < L$ or $t > 0$.

$u(L/2, t) = M$

3

Ernesto Sandoval
9948092

40

1. $\begin{cases} u_t - k u_{xx} = f(x,t) \\ u(x,0) = \phi(x) \\ u(0,t) = g(t) \\ u(L,t) = h(t) \end{cases}$

Let u_1 and u_2 be solutions to this PDE.

Let $w = u_1 - u_2$.

Then $\begin{cases} w_t - k w_{xx} = 0 \\ w(x,0) = 0 \\ w(0,t) = 0 \\ w(L,t) = 0 \end{cases}$

$E[w](t) = \int_0^L w^2(x,t)\,dx$

$$\frac{d}{dt} E[w](t) = \frac{d}{dt}\int_0^L w^2(x,t)\,dx$$

$$= \int_0^L \frac{d}{dt} w^2\,dx$$

$$= \int_0^L 2 w w_t\,dx$$

$$= \int_0^L 2k w w_{xx}\,dx$$

$$= \cancel{2k w w_x \Big|_0^L} - \int_0^L 2k w_x^2\,dx$$

$$= -\int_0^L 2k w_x^2 \le 0$$

10

$E > 0$ and is not increasing.

$E[w](0) = \int_0^L w^2(x,0) = 0$

Then $0 < E < 0$. Then $E = 0$

$\int_0^L w^2(x,t)\,dx = 0$ when $w = 0$.

$u_1 - u_2 = 0$

$u_1 = u_2$.

Then solutions are not unique.

Ernesto Sandoval 9948092

2. $\begin{cases} u_t - k u_{xx} = 0 & x>0,\ t>0 \\ u(x,0) = 0 \\ u(0,t) = 1 \end{cases}$

$$\cancel{u = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k t}} e^{-(x-y)^2/4kt}}$$

~~Let $v = u - 1$.~~

Then $\begin{cases} v_t - k v_{xx} = 0 \\ v(x,0) = -1 & , \quad x>0,\ t>0. \\ v(0,t) = 0 \end{cases}$

Let $\phi_{ODD} = \begin{cases} -1 & x>0 \\ 0 & x=0 \\ 1 & x<0 \end{cases}$

The solution to $\begin{cases} v_t - k v_{xx} = 0 \\ v(x,0) = \phi_{ODD}(x) \\ v(0,t) = 0 \end{cases}$

is $v = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k t}} e^{-(x-y)^2/4kt} \phi_{ODD}(y)\, dy$

$$= \int_{-\infty}^{0} \frac{1}{\sqrt{4\pi k t}} e^{-(x-y)^2/4kt}\, dy - \int_{0}^{\infty} \frac{1}{\sqrt{4\pi k t}} e^{-(x-y)^2/4kt}\, dy$$

Let $s = \frac{x-y}{\sqrt{4kt}}$. $ds = \frac{-1}{\sqrt{4kt}}\, dy$

$$v = \frac{1}{\sqrt{\pi}} \left(-\int_{-\infty}^{x/\sqrt{4kt}} e^{-s^2}\, ds + \int_{\frac{x}{\sqrt{4kt}}}^{\infty} e^{-s^2}\, ds \right)$$

$$v = \frac{1}{\sqrt{\pi}} \left(\cancel{-\int_{-\infty}^{0} e^{-s^2}\, ds} - \int_{0}^{x/\sqrt{4kt}} e^{-s^2}\, ds \cancel{+ \int_{0}^{\infty} e^{-s^2}} ds - \int_{0}^{\frac{x}{\sqrt{4kt}}} e^{-s^2} ds \right.$$

$$v = \frac{1}{\sqrt{\pi}} \left(-\int_{0}^{x/\sqrt{4kt}} e^{-s^2}\, ds - \int_{0}^{x/\sqrt{4kt}} e^{-s^2}\, ds \right)$$

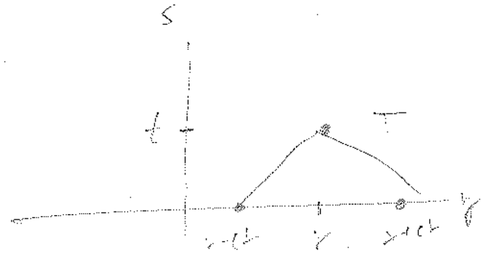
$$= -\frac{2}{\sqrt{\pi}} \left(\int_{0}^{x/\sqrt{4kt}} e^{-s^2}\, ds \right) = -\mathrm{Erf}\left(\frac{x}{\sqrt{4kt}}\right)$$

$$\boxed{u = -\mathrm{Erf}\left(\frac{x}{\sqrt{4kt}}\right) + 1}$$

0

Ernesto Sandoval

3. $$\begin{cases} u_{tt} - c^2 u_{xx} = e^{2x} \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}$$



$$u = \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} e^{2y}\, dy\, ds$$

$$u = \int_0^t \frac{e^{2y}}{2}\Big]_{x-c(t-s)}^{x+c(t-s)} ds$$

$$u = \frac{1}{2}\int_0^t e^{2(x+c(t-s))} - e^{2(x-c(t-s))}\, ds$$

$$u = \frac{1}{2}\int_0^t e^{2x+2ct-2cs} - e^{2x-2ct+2cs}\, ds$$

$$u = \frac{1}{2}\left(-\frac{e^{2x+2ct-2cs}}{2c} - \frac{e^{2x-2ct+2cs}}{2c}\Big]_0^t \right)$$

$$u = \frac{1}{4c}\left(e^{2x+2ct} + e^{2x-2ct} - e^{2x} - e^{2x} \right)$$

$$\boxed{u = \frac{1}{4c}\left(e^{2x+2ct} + e^{2x-2ct} - 2e^{2x} \right)}$$

10

4. $u_t - u_{xx} + u \le 0$

Ernesto Sandoval

Suppose the maximum M occurs on $R = (0,L] \times (0,T]$ and is ≥ 0.

$\therefore u(x_0, t_0) = M \ge 0$

Then $u_t(x_0, t_0) = 0$

and $u_{xx}(x_0, t_0) < 0$.

10

Then $u_t(x_0, t_0) - u_{xx}(x_0, t_0) + u(x_0, t_0) \le 0$

$-u_{xx}(x_0, t_0) + M \le 0$

$\therefore u_{xx}(x_0, t_0)$ is positive so this is a contradiction. Then the maximum must occur in the sides $x=0$ or $x=L$ or the bottom $t=0$ of R.

Name: Yi Liu

18

(1) $\begin{cases} u_t - k u_{xx} = f(x,t) \\ u(x,0) = \phi(x) \\ u(0,t) = g(t) \\ u(L,t) = h(t) \end{cases}$ $\quad E[u](t) = \int_0^L u^2(x,t)\,dx$

Sol: Suppose u_1, u_2 are 2 solutions of $u(x,t)$

Let's define $w = u_1 - u_2$

$$\frac{d}{dt}\int_0^L w^2(x,t)\,dx$$

$$= -2k\int_0^L w w_{xx} = 0$$

$\therefore E[w](t) = 0 \quad \therefore w = 0 \quad \therefore u_1 = u_2$

$\therefore u(x,t)$ has unique solution

Approach 2:

3

By max principle

$$\begin{cases} w_t - k w_{xx} = 0 \\ w(x,0) = \cancel{\phi(x)}\ 0 \\ w(0,t) = 0 \\ w(L,t) = 0 \end{cases}$$

$\therefore \frac{d}{dt}\int_0^L w^2(x,t)\,dx \le \frac{d}{dt} 0 = 0$

By min principle

$\therefore w^2(x,t) = 0$

$\therefore w = 0$

$\therefore u_1 = u_2$

$\therefore$ Unique

(2) $\begin{cases} u_t - k u_{xx} = 0 \text{ on } 0 < x < \infty,\ t > 0 \\ u(x,0) = 0 \\ u(0,t) = 1 \end{cases}$ $\quad \mathrm{Erf}(s) = \frac{2}{\sqrt{\pi}}\int_0^s e^{-x^2}\,dx$

$(v = u - 1) \quad \therefore \begin{cases} v_t - k v_{xx} = 0 \\ v(x,0) = -1 = \phi(x) \\ v(0,t) = 0 \end{cases}$

Sol: $v(x,t) = \frac{1}{\sqrt{4k\pi t}}\int_0^\infty \left(e^{\frac{-(x-y)^2}{4kt}} \cancel{+} e^{-\frac{(x+y)^2}{4kt}}\right)(-1)\,dy.$

Let's say $p = \frac{x-y}{\sqrt{4kt}} \qquad dp = -\frac{dy}{\sqrt{4kt}}$

$q = \frac{x+y}{\sqrt{4kt}} \qquad dq = \frac{dy}{\sqrt{4kt}}$

$$\therefore v(x,t) = \left[\frac{1}{\sqrt{\pi}}\int_{\frac{x}{\sqrt{4kt}}}^{-\infty} e^{-p^2}\,dp\right] - \frac{1}{\sqrt{\pi}}\int_{\frac{x}{\sqrt{4kt}}}^{\infty} e^{-q^2}\,dq$$

$$= \cancel{\tfrac{1}{\sqrt{\pi}}} \cdot \tfrac{1}{2}\mathrm{Erf}\left(\tfrac{x}{\sqrt{4kt}}\right) - \cancel{\tfrac{1}{\sqrt{\pi}}}\left(\tfrac{1}{2}\mathrm{Erf}\left(\tfrac{x}{\sqrt{4kt}}\right)\right)$$

$= 0.$

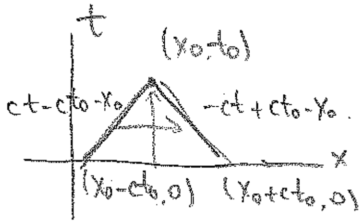
$\therefore v(x,t) = 0$

$\therefore u(x,t) = 1$

8

problem 3.

$$\begin{cases} u_{tt} - c^2 u_{xx} = e^{2x} & \text{(wave equation)} \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}$$



$u(x,t) = \frac{1}{2c}\iint f(y,s)\,dy\,ds$ (special) $\left(= e^{2x}\right)$

$$= \frac{1}{2c}\int_{ct-ct_0-x_0}^{-ct+ct_0-x_0\ T}\int_0^{t_0} e^{2y}\,dy\,ds$$

$$= \frac{1}{c}\int_{ct-ct_0-x_0}^{-ct+ct_0-x_0} e^{2y}\Big|_0^{t_0}\,ds$$

Calculation

$$= \frac{1}{c}\int_{ct-ct_0-x_0}^{-ct+ct_0-x_0} e^{2t_0}-1\,ds$$

$$= \frac{1}{c}(e^{2t_0}-1)s\Big|_{ct-ct_0-x_0}^{-ct+ct_0-x_0}$$

$$= \frac{1}{c}(e^{2t_0}-1)(-ct+ct_0-\cancel{x_0}-ct+ct_0+\cancel{x_0})$$

$$= (e^{2t_0}-1)(2ct_0-2ct)$$

$$\therefore u(x,t) = \frac{1}{2}\cancel{(\phi(x-ct)+(\phi(x+ct))} + \int_{x-ct}^{x+ct}\cancel{\psi(s)}\,ds +$$

$$(e^{2t_0}-1)(2ct_0-2ct)$$

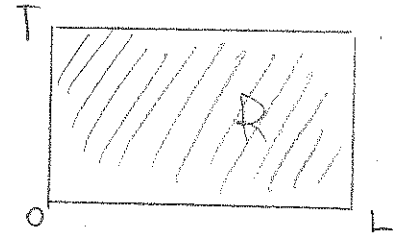
$$= (e^{2t_0}-1)(2ct_0-2ct)$$

[change variable $2t_0 = x$ $= (e^{x}-1)(2cx)$]

6

Problem 4

prove max principle



Suppose there is an (x_0,t_0) inside the rectangle.

Such that $0 < x_0 < L$

$0 < t_0 < T$

Let $\varepsilon > 0$. M is the max points.

as an max point, the first derivative of M should be concave down

means it's first derivative < 0

while if (x_0,t_0) interior, then exist Contridiction

$\therefore$ (x_0,t_0) can not be interior.

$\therefore$ (x_0,t_0) must be on the boundaries.

3

Ayrton Finton
8571564

MIDTERM 2

Please write your name on each page of your answer sheet and **do not fold the pages together.**

(1) Show the uniqueness to the solution of

$$\begin{cases} u_t - ku_{xx} = f(x,t) & \text{for } 0 < x < L, \quad t > 0 \\ u(x,0) = \phi(x) \\ u(0,t) = g(t) \\ u(L,t) = h(t) \end{cases}$$

energy.

for sufficiently nice functions f, g, h, ϕ. You can use whatever method you want. A useful energy for the heat equation is the L^2 energy given by $E[u](t) = \int_0^L u^2(x,t)dx$.

(2) Solve in terms of the error function

$$\begin{cases} u_t - ku_{xx} = 0 & \text{on } 0 < x < \infty, t > 0 \\ u(x,0) = 0 \\ u(0,t) = 1. \end{cases}$$

(half line)

Hint: First consider $v := u - 1$. What equation does v satisfy? Then solve that equation, keeping in mind that we are solving this on the half-line. The error function is given by

$$\mathcal{E}\mathrm{rf}(s) = \frac{2}{\sqrt{\pi}} \int_0^s e^{-x^2} dx.$$

(3) Solve by finding an explicit formula. Make sure to integrate out the solution of

$$\begin{cases} u_{tt} - c^2 u_{xx} = e^{2x} & \text{on } (x,t) \in \mathbb{R}^2 \\ u(x,0) = 0 \\ u_t(x,0) = 0. \end{cases}$$

wave w/ non-hom

$\frac{1}{2}\phi(x+ct) - \frac{1}{2}\phi(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy + \frac{1}{2}\iint f(x,t)\,dx\,dt$

Note that $\sinh(x) = \frac{e^x - e^{-x}}{2}$, $\cosh(x) = \frac{e^x + e^{-x}}{2}$ and that $(\sinh(x))' = \cosh(x)$.

(4) Let $L, T > 0$. Suppose u is twice differentiable on the open rectangle $(0,L) \times (0,T)$ and satisfies the partial differential inequality

$$u_t - u_{xx} + u \leq 0.$$

Suppose further that u is continuous on $R = [0,L] \times [0,T]$. If M is the maximum on of u on R and $M \geq 0$, then show that u attains the value M on the sides $x = 0$ or $x = L$ or on the bottom $t = 0$ of R. Hint: Consider the sign or value of each quantity in the partial differential inequality at a maximum point if it were to occur in the interior. No $v = u + \varepsilon$ trick is necessary for this problem.

Ayrton Finden
8571564

32

① $\begin{cases} u_t - k u_{xx} = F(x,t) & 0 < x < L \quad t > 0 \\ u(x,0) = \phi(x) \\ u(0,t) = g(t) \\ u(L,t) = h(t) \end{cases}$

Energy method

$$E[u](t) = \int_0^L u^2(x,t)\,dx$$

$$E[u] = \int_0^L \frac{d}{dt}[u^2]\,dx$$

$$\Rightarrow \int_0^L 2u\,u_t\,dx$$

we know $u_t = k u_{xx}$

$$\Rightarrow \int_0^L 2k\,u_{xx}u\,dx$$

$$\Rightarrow 2k[u_x]_0^L - \int_0^L u\,dx$$

$$\Rightarrow 2k(L) - 2k(0) - \int_0^L u\,dx$$

$2k(L) \geq 0$ from inference

we can infer $-\int_0^L u\,dx \leq 0$ from observation

$$\Rightarrow \quad 0 \leq 2k(L) - \int_0^L u\,dx \leq 0$$

by squeeze theorem

$$E[u](t) = 0$$

which by definition, the Energy equation equaling zero gives uniqueness (e.g. $u^2 = v_1 + v_2 \Rightarrow u^2 = 0 \Rightarrow v_1 = v_2$)

$\square$

9

Ayrton Finden
852-1564

(2) $\begin{cases} u_t - k u_{xx} = 0 \\ u(x,0) = 0 \\ u(0,t) = 1 \end{cases}$ $\quad 0 < x < \infty$, $t > 0$

let $v(x,t) = u(x,t) - 1$

$\Rightarrow v(x,t) + 1 = u(x,t)$

Then

$v(x,t)$:

$v(x,0) = -1$

$u(0,t) = 0$

Then

$$v(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_0^\infty \left(e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}} \right) \phi(y)\, dy$$

we know $\phi(y) = -1$

$$\Rightarrow \frac{-1}{\sqrt{4\pi k t}} \int_0^\infty e^{-\frac{(x-y)^2}{4kt}} dy + \frac{1}{\sqrt{4\pi k t}} \int_0^\infty e^{-\frac{(x+y)^2}{4kt}} dy$$

$\Rightarrow$ let $p = \frac{x-y}{\sqrt{4kt}}$ Jacobian: $-\sqrt{4kt}$ $\qquad$ let $q = \frac{x+y}{\sqrt{4kt}}$ Jacobian: $\sqrt{4kt}$

$$\Rightarrow \frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^{-\infty} e^{-p^2} dp + \frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^{\infty} e^{-q^2} dq$$

$$\Rightarrow -\left(\frac{1}{2} + \frac{1}{2} \operatorname{Erf}\left(\frac{x}{\sqrt{4kt}}\right) \right) + \left(\frac{1}{2} - \frac{1}{2} \operatorname{Erf}\left(\frac{x}{\sqrt{4kt}}\right) \right)$$

$$\Rightarrow v(x,t) = -\operatorname{Erf}\left(\frac{x}{\sqrt{4kt}}\right)$$

reverse the substitution i.e. $(u(x,t) = v(x,t) + 1)$

$$\Rightarrow \boxed{u(x,t) = 1 - \operatorname{Erf}\left(\frac{x}{\sqrt{4kt}}\right)}$$

$\square$

10

③

Ayrton Finden
85715864

$$\begin{cases} u_{tt} - c^2 u_{xx} = e^{2x} & \text{on } (x,t) \in \mathbb{R}^2 \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}$$

General solution:

$$u(x,t) = \tfrac{1}{2}\phi(x+ct) - \tfrac{1}{2}\phi(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)\,ds + \frac{1}{2}\int_0^{t_0}\int_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} f(x,t)\,dx\,dt$$

we know

$u(x,0) = \phi(x) = 0$

and

$u_t(x,0) = \psi(x) = 0$

$$\Rightarrow \tfrac{1}{2}(0) - \tfrac{1}{2}(0) + \frac{1}{2c}\int(0) + \frac{1}{2}\int_0^{t_0}\int_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} f(x,t)\,dx\,dt$$

$f(x,t) = e^{2x}$

$$\Rightarrow \tfrac{1}{2}\int_0^{t_0}\int_{x_0-ct_0+ct}^{x_0+ct_0-ct} e^{2x}\,dx\,dt$$

$$\Rightarrow \frac{1}{2}\int_0^{t_0} \frac{e^{2x}}{2}\Big|_{x_0-ct_0+ct}^{x_0+ct_0-ct} = \frac{1}{4}\int_0^{t_0} e^{2x}\Big|_{x_0-ct_0+ct}^{x_0+ct_0-ct}$$

$$\Rightarrow \tfrac{1}{4}\int_0^{t_0} e^{2x_0+2ct_0-2ct} - e^{2x_0-2ct_0+2ct}\,dt$$

$$\Rightarrow \tfrac{1}{4}\left[\left(\frac{e^{2x_0+2ct_0-2ct}}{-2c}\Big|_0^{t_0}\right) - \left(\frac{e^{2x_0-2ct_0+2ct}}{2c}\Big|_0^{t_0}\right)\right]$$

$$\Rightarrow \tfrac{1}{4}\left[\left(\frac{e^{2x_0+2ct_0-2ct_0}}{-2c} - \frac{e^{2x_0+2ct_0}}{-2c}\right) - \left(\frac{e^{2x_0-2ct_0+2ct_0}}{2c} - \frac{e^{2x_0-2ct_0}}{2c}\right)\right]$$

$$\Rightarrow \tfrac{1}{4}\left[\left(\frac{e^{2x_0}}{-2c} - \frac{e^{2x_0+2ct_0}}{-2c}\right) - \frac{e^{2x_0}}{2c} + \frac{e^{2x_0-2ct_0}}{2c}\right]$$

$$\Rightarrow \tfrac{1}{4}\left[-\frac{e^{2x_0}}{2c} + \frac{e^{2x_0+2ct_0}}{2c} - \frac{e^{2x_0}}{2c} + \frac{e^{2x_0-2ct_0}}{2c}\right]$$

$$\Rightarrow \tfrac{1}{4}\left[-\frac{e^{2x_0}}{2c} + \frac{e^{2x_0}e^{2ct_0}}{2c} - \frac{e^{2x_0}}{2c} + \frac{e^{2x_0}e^{-2ct_0}}{2c}\right]$$

$$\Rightarrow \frac{e^{2x_0}}{4c}\left[-\frac{1}{2} + \frac{e^{2ct_0}}{2} - \frac{1}{2} + \frac{e^{-2ct_0}}{2c}\right]$$

$$\Rightarrow \frac{e^{2x_0}}{4c}\left[-1 + \frac{e^{2ct_0} + e^{-2ct_0}}{2}\right]$$

Ayrton Finden
85715S4

10

$$\Rightarrow \frac{e^{2x_0}}{4c}\left[-1+\frac{e^{2ct_0}+e^{-2ct_0}}{2}\right]$$

$$\Rightarrow \frac{e^{2x_0}}{4c}\left[-1+\sinh(2ct_0)\right]$$

$$\Rightarrow -\frac{e^{2x_0}}{4c}+\frac{e^{2x_0}\sinh(2ct_0)}{4c}$$

$$\Rightarrow \boxed{u(x,t)=\frac{e^{2x_0}\sinh(2ct_0)}{4c}-\frac{e^{2x_0}}{4c}}$$

$\square$

(4) $L, T > 0$

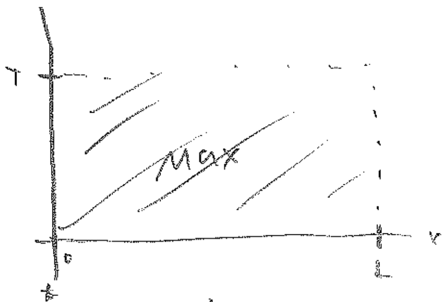
Ayrton Finden
857 1564

$u_t - u_{xx} + u \leq 0$

$R = [0, L] \times [0, T]$

Show max principle

can sub in gen eq for $V_t - V_{xx} = 0$ relation.



3

$$V(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_0^L \left(e^{\frac{-(x-y)^2}{4kt}} - e^{\frac{-(x+y)^2}{4kt}}\right] \phi(y)$$

let $\phi(y) = \alpha \in \mathbb{R}$ be an arbitrary value

$$\Rightarrow \frac{1}{\sqrt{4\pi k t}} \int_0^L \alpha e^{\frac{-(x-y)^2}{4kt}} - \alpha e^{\frac{-(x+y)^2}{4kt}} dy$$

$$\Rightarrow \quad p = \frac{x-y}{\sqrt{4kt}} \qquad q = \frac{x+y}{\sqrt{4kt}}$$

$$\Rightarrow \frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^{\frac{x-L}{\sqrt{4kt}}} \alpha e^{p^2} dp - \frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^{\frac{x+L}{\sqrt{4kt}}} \alpha e^{q^2} dq$$

if we take $\lim_{t \to \infty} v(x,t)$: $\frac{1}{\sqrt{\pi}} \int_0^0 \alpha e^{p^2} dp - \frac{1}{\sqrt{\pi}} \int_0^0 \alpha e^{q^2} dq$

which means $\lim_{t \to \infty} u(x,t) = 0$

If we take $\lim_{t \to 0} v(x,t)$: $\frac{1}{\sqrt{\pi}} \int_{\infty}^{\infty} \alpha e^{p^2} dp - \frac{1}{\sqrt{\pi}} \int_{\infty}^{\infty} \alpha e^{q^2} dq$

which means $\lim_{t \to 0} u(x,t) = 0$

Now if we take $\lim_{x \to \infty} v(x,t)$: $\frac{1}{\sqrt{\pi}} \int_{\infty}^{\infty} \alpha e^{p^2} dp - \frac{1}{\sqrt{\pi}} \int_{\infty}^{\infty} \alpha e^{q^2} dq$

which means $\lim_{x \to \infty} u(x,t) = 0$

Ayrbon Finden
052-1564

If we take $\lim_{x \to 0} v(x,t) = \frac{1}{\sqrt{\pi}} \int_0^{\frac{-l}{\sqrt{4kt}}} q e^{-p^2} \, dp - \frac{1}{\sqrt{\pi}} \int_0^{\frac{l}{\sqrt{4kt}}} q e^{-q^2} \, d\xi$

This means $\lim_{x \to 0} v(x,t) = M$, where M is a value in which $|M| \geq 0$.

Thus, the limits have shown that the maximum will only occur inside the rectangle $R\,[0,l] \times [0,T]$ as all values beyond this rectangle converge to 0. $\square$