MATH 124B: MIDTERM

(1) Let $X \in C^2$ on (a, b). Assuming symmetric boundary conditions, i.e. $X'(x)X(x)\Big|_a^b = 0$, prove that the eigenvalue of $X'' + \lambda X = 0$ on (a, b) is non-negative.

(2) Solve the problem

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = x, \\ u_t(x,0) = 0. \end{cases}$$

(3) Find the sum $\displaystyle\sum_{n \text{ odd}} \frac{1}{n^2}$ using any method.

(4) Let

$$f_n(x) = \frac{n}{1 + n^2x^2} - \frac{n-1}{1 + (n-1)^2x^2}.$$

Show that $\displaystyle\sum_{n=1}^{\infty} f_n(x)$ converges point-wise to 0. Show that it does not converge uniformly and in L^2.

(5) Solve the equation $u_{xx} + u_{yy} = 1$ in the annulus $a < r < b$ with $u = 0$ on the boundary $r = a$ and $r = b$.

Solutions

(1) Multiplying both sides by X and integrating over (a, b), we have

$$\begin{aligned} -\lambda \int_a^b X^2 &= \int_a^b X'' X \, dx \\ &= (X'X)|_a^b - \int_a^b (X')^2 dx \\ &= -\int_a^b (X')^2 dx \end{aligned}$$

Isolating λ, we get

$$\lambda = \frac{\int_a^b (X')^2 dx}{\int_a^b X^2 dx} \geq 0$$

(2) The general solution for the wave equation with Dirichlet boundary condition is given by

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right) \sin\left(\frac{n\pi x}{L}\right).$$

Differentiating with respect to time

$$u_t(x,t) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} \left(-A_n \sin\left(\frac{n\pi ct}{L}\right) + B_n \cos\left(\frac{n\pi ct}{L}\right)\right) \sin\left(\frac{n\pi x}{L}\right).$$

With the initial condition $u_t(x,0) = 0$, we get

$$0 = \sum_{n=1}^{\infty} \frac{n\pi c}{L} B_n \sin\left(\frac{n\pi x}{L}\right)$$

so that all the $B_n = 0$. Using the initial condition $u(x,0) = x$, we have

$$x = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right).$$

This is a Fourier sine expansion of x on $(0,L)$, so we compute the coefficients:

$$\begin{aligned} A_n &= \frac{2}{L}\int_0^L x \sin\left(\frac{n\pi x}{L}\right) dx \\ &= -\frac{2x}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{2L}{n^2\pi^2}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_0^L = (-1)^{n+1}\frac{2L}{n\pi} \end{aligned}$$

hence the solution is

$$u(x,t) = \frac{2L}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)$$

(3) Using Parseval's identity with x and the Fourier sine series computed in the previous problem, we have

$$\sum_{n=1} \left(\frac{2L}{n\pi}\right)^2 \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx = \frac{2L^3}{\pi^2}\sum_{n=1}\frac{1}{n^2}$$

and

$$\int_0^L x^2 dx = \frac{L^3}{3}$$

Therefore,

$$\sum_{n=1}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

Now to obtain the odd sums, we split the sum into

$$\sum_{n \text{ odd}} \frac{1}{n^2} + \sum_{n \text{ even}} \frac{1}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

The even sum can be rewritten as

$$\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{24}$$

hence the odd sum is

$$\sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{6} - \frac{\pi^2}{24} = \frac{\pi^2}{8}.$$

(4) The partial sum is given by

$$\sum_{n=1}^{N} f_n(x) = \frac{N}{1+N^2x^2} = \frac{1}{N(\frac{1}{N^2}+x^2)}$$

which converges to 0 for $x > 0$ as $N \to \infty$.

Computing the L^2 norm, we have

$$\int_0^L \frac{N^2}{(1+N^2x^2)^2} dx = N \int_0^{NL} \frac{dy}{(1+y^2)^2} \to \infty$$

as $N \to \infty$ hence does not converge in L^2.

Furthermore, we have

$$\lim_{N\to\infty} \sup_{(0,L)} \frac{N}{1+N^2x^2} = \lim_{N\to\infty} N = \infty$$

hence does not converge uniformly.

(5) Solving for the rotationally symmetric solutions, the PDE becomes the ODE

$$u_{rr} + \frac{1}{r}u_r = 1.$$

Multiplying r to both sides,

$$(ru_r)_r = r.$$

Integrating twice, we have

$$u(r) = \frac{1}{4}r^2 + c_1 \ln(r) + c_2.$$

Inserting the boundary conditions, we have

$$0 = \frac{a^2}{4} + c_1 \ln(a) + c_2$$
$$0 = \frac{b^2}{4} + c_1 \ln(b) + c_2.$$

Subtracting one from the other, we get

$$c_1 = \frac{b^2 - a^2}{4(\ln(b) - \ln(a))}.$$

Inserting this and solving for c_2, we have

$$u(r) = \frac{r^2 - a^2}{4} - \frac{b^2 - a^2}{4} \left(\frac{\ln(r) - \ln(a)}{\ln(b) - \ln(a)} \right).$$