# MATH 124B: TAKE HOME FINAL

Any theorems covered during lecture or in Strauss' PDE textbook, including its appendix can be referenced. Any other theorems or nontrivial claims must be provided with proof. A claim is nontrivial if you do not know why its true., i.e. no "From Theorem 2 of Book X".

**Due in class on Thursday or online Friday**

(1) Derive the Laplacian in spherical coordinates.

(2) Compute the sum $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^8}$. You may use previously established sums. Then explain how one could obtain the sum for $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{2k}}$ for any $k \in \mathbb{N}$.

(3) Derive Poisson's formula for $D = \{(r, \theta) \in \mathbb{R}^2 \mid r > a > 0\}$, given by

$$u(r, \theta) = (r^2 - a^2) \int_0^{2\pi} \frac{h(\phi)}{a^2 - 2ar\cos(\theta - \phi) + r^2} \frac{d\phi}{2\phi}$$

where $u$ is a solution of

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{for } r > a \\ u = h(\theta) & \text{for } r = a \\ u \text{ is bounded as } r \to \infty. \end{cases}$$

(4) A function $u(x, y)$ is subharmonic in $D$ if $\Delta u \geq 0$ in $D$. Prove that its maximum value is attained on $\partial D$. Then show that it also satisfies the mean value inequality

$$u(0) \leq \frac{1}{2\pi r} \int_0^{2\pi} u(r, \theta) d\theta$$

for all $r > 0$ such that $B(0, r) \subset D$.

(5) Show that the second smallest eigenvalue for the Neumann function is $\lambda_2 > 0$.

(6) Let $g(x)$ be a function on $\partial D$. Consider the minimum of the functional

$$\frac{1}{2} \iiint_D |\nabla w|^2 dV - \iiint_D f w dV$$

among all $C^2$ functions $w$ for which $w = g$ on $\partial D$. Show that a solution of this minimum problem leads to a solution of the Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{in } D \\ u = g & \text{on } \partial D. \end{cases}$$

(7) The Neumann function $N(x, y)$ for a domain $D$ is defined much like the Green's function, except for the boundary condition is replaced by

$$\frac{\partial N}{\partial n} = c, \qquad \text{on } \partial D,$$

for some constant $c$. Show that $c = \frac{1}{A}$ where $A$ is the area of $\partial D$, ($c = 0$ if $A = \infty$). Then state and prove a theorem expressing the solution of the Neumann problem in terms of the Neumann function, (Theorem 7.3.1 in Strauss).

(8) Compute the eigenvalues of the 2 dimensional rectangle,

$$\begin{cases} \Delta u + \lambda u = 0 & \text{on } [0, a] \times [0, b] \\ u = 0 & \text{on } \partial D. \end{cases}$$

(Use separation of variables)