MATH 124B: COMPUTATIONAL FINAL

No calculator or notes.

(1) Compute the Fourier coefficients of $x^3 + x$ on $[-1, 1]$.

(2) Compute the Neumann eigenvalues and eigenfunctions of the one-dimensional problem

$$\begin{cases} y'' + \lambda y = 0 & \text{on } [0, L] \\ y'(0) = y'(L) = 0. \end{cases}$$

(3) Compute the Green's function for the 3 dimensional half plane $\{(x, y, z) \mid z > 0\}$. Note that the fundamental solution in 3 dimensions is $-\frac{1}{4\pi\|\mathbf{x}-\mathbf{x}_0\|}$. Then use this to compute the solution for the Dirichlet problem in the upper half space with boundary condition $u(x, 0) = h(x)$.

(4) Consider the Laplace eigenvalue problem with Dirichlet boundary conditions on the square $(0, \pi) \times (0, \pi)$. Compute the Rayleigh quotient with the trial function $xy(\pi - x)(\pi - y)$. Then repeat with the trial function $w = \sin(x)\sin(y)$.

(5) Find the harmonic function u in the semidisk $\{r < 1, 0 < \theta < \pi\}$ with u vanishing on $\theta = 0, \pi$ and $\frac{\partial u}{\partial r} = \sin\theta + \pi \sin(2\theta)$ on $r = 1$.

1. Solutions

(1) Since the function is odd, the cosine terms vanish. Calculating the sine terms one by one, we have

$$\begin{aligned} \int_{-1}^{1} x \sin(n\pi x)dx &= -2\frac{x}{n\pi}\cos(n\pi x)\Big|_0^1 + \frac{2}{n\pi}\int_0^1 \cos(n\pi x)dx \\ &= (-1)^{n+1}\frac{2}{n\pi} \end{aligned}$$

for $n = 1, 2, \ldots$, and

$$\begin{aligned} \int_{-1}^{1} x^3 \sin(n\pi x)dx &= 2\int_0^1 x^3 \sin(n\pi x)dx \\ &= -2\frac{x^3}{n\pi}\cos(n\pi x)\Big|_0^1 + \frac{6}{n\pi}\int_0^1 x^2\cos(n\pi x)dx \\ &= (-1)^{n+1}\frac{2}{n\pi} - \frac{12}{n^2\pi^2}\int_0^1 x\sin(n\pi x)dx \\ &= (-1)^{n+1}\frac{2}{n\pi} + (-1)^n\frac{12}{n^3\pi^3}. \end{aligned}$$

hence, combining we have

$$\begin{aligned} A_n &= 0 \\ B_n &= (-1)^{n+1}\frac{4}{n\pi} + (-1)^n\frac{12}{n^3\pi^3}. \end{aligned}$$

(2) Using the symmetric boundary conditions, we know that $\lambda \geq 0$, so that the solution to the constant coefficient second order equation is given by

$$y(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x).$$

To incorporate the Neumann boundary condition, we have

$$y'(x) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x),$$

so that $y'(0) = 0 = c_2\sqrt{\lambda}$. If $\lambda = 0$, then we have the constant solution $y = c$ (for $c \neq 0$) and if $\lambda \neq 0$, then $c_2 = 0$ so that

$$y'(L) = 0 = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}L).$$

If $c_1 = 0$, then we have the trivial solution which we exclude, so that $\lambda = \left(\frac{n\pi}{L}\right)^2$, for $n \in \mathbb{N}$. Hence the eigenvalues are given by $\lambda = \left(\frac{n\pi}{L}\right)^2$, for $n = 0, 1, 2, \ldots$, note that we include $\lambda = 0$, with corresponding eigenfunctions $y_n(x) = \cos(\sqrt{\lambda_n}x)$.

(3) We use the reflection principle to find a candidate Green's function and check that it satisfies the 3 properties. Define the reflection across the $z = 0$ plane by

$$\mathbf{x}^* = (x, y, z)^* = (x, y, -z).$$

Our candidate Green's function is

$$G(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi\|\mathbf{x} - \mathbf{x}_0\|} + \frac{1}{4\pi\|\mathbf{x} - \mathbf{x}_0^*\|}.$$

Since $1/r$ is a fundamental solution in 3 dimensions, G is harmonic when $\mathbf{x} \neq \mathbf{x}_0$ and $G + \frac{1}{4\pi\|\mathbf{x}-\mathbf{x}_0\|}$ is harmonic in the upper half space. When $\mathbf{x} \in \{z = 0\}$, we have $\|\mathbf{x} - \mathbf{x}_0\| = \|\mathbf{x} - \mathbf{x}_0^*\|$, so $G = 0$ on the boundary $\{z = 0\}$.

To compute the solution to the Dirichlet problem, we need to compute the normal derivative of G. We have

$$\nabla G(\mathbf{x}, \mathbf{x}_0) = \frac{\nabla(\|\mathbf{x} - \mathbf{x}_0\|)}{4\pi\|\mathbf{x} - \mathbf{x}_0\|^2} - \frac{\nabla(\|\mathbf{x} - \mathbf{x}_0^*\|)}{4\pi\|\mathbf{x} - \mathbf{x}_0^*\|^2},$$

and

$$\nabla(\|\mathbf{x} - \mathbf{x}_0\|) = \frac{\mathbf{x} - \mathbf{x}_0}{\|\mathbf{x} - \mathbf{x}_0\|}$$
$$\nabla(\|\mathbf{x} - \mathbf{x}_0^*\|) = \frac{\mathbf{x} - \mathbf{x}_0}{\|\mathbf{x} - \mathbf{x}_0\|}.$$

On the boundary, the unit outer normal is given by $\langle 0, 0, -1\rangle$, so

$$\frac{\partial G}{\partial n} = \nabla G(\mathbf{x}, \mathbf{x}_0) \cdot \langle 0, 0, -1\rangle = \frac{z_0 - z}{4\pi\|\mathbf{x} - \mathbf{x}_0\|^3} + \frac{z + z_0}{\|\mathbf{x} - \mathbf{x}_0^*\|^3}$$
$$= \frac{z_0}{2\pi((x - x_0)^2 + (y - y_0)^2 + z_0^2)^{3/2}},$$

hence the solution to the Dirichlet problem is given by

$$u(x_0, y_0, z_0) = \frac{z_0}{2\pi}\iint_{z=0} \frac{h(x, y)}{((x - x_0)^2 + (y - y_0)^2 + z_0^2)^{3/2}} dS$$

(4) Let $w(x,y) = xy(\pi - x)(\pi - y)$. The gradient is

$$\nabla(w) = \langle y(\pi - y)(\pi - 2x), x(\pi - x)(\pi - 2y)\rangle$$
$$|\nabla w|^2 = y^2(\pi - y)^2(\pi - 2x)^2 + x^2(\pi - x)^2(\pi - 2y)^2$$

and the L^2 norm is

$$\begin{aligned}\|\nabla w\|^2 &= \int_0^\pi \int_0^\pi |\nabla w|^2 dxdy \\ &= \int_0^\pi \int_0^\pi y^2(\pi - y)^2(\pi - 2x)^2 + x^2(\pi - x)^2(\pi - 2y)^2 dxdy \\ &= 2\int_0^\pi y^2(\pi - y)^2 dy \int_0^\pi (\pi - 2x)^2 dx \\ &= \frac{\pi^8}{45}\end{aligned}$$

and

$$\begin{aligned}\|w\|^2 &= \int_0^\pi \int_0^\pi x^2y^2(\pi - x)^2(\pi - y)^2 dxdy \\ &= \frac{\pi^{10}}{900}\end{aligned}$$

hence the Rayleigh quotient is given by

$$\frac{\|\nabla w\|^2}{\|w\|^2} = \frac{20}{\pi^2} = 2.03$$

For the test function $w = \sin(x)\sin(y)$, we compute

$$\begin{aligned}\nabla w &= \langle \cos(x)\sin(y), \sin(x)\cos(y)\rangle \\ |\nabla w|^2 &= \cos^2(x)\sin^2(y) + \sin^2(x)\cos^2(y) \\ \|\nabla w\|^2 &= \int_0^\pi \int_0^\pi \cos^2(x)\sin^2(y) + \sin^2(x)\cos^2(y) dxdy \\ &= \frac{\pi^2}{2}\end{aligned}$$

and

$$\begin{aligned}\|w\|^2 &= \int_0^\pi \int_0^\pi \sin^2(x)\sin^2(y) dxdy \\ &= \frac{\pi^2}{4}\end{aligned}$$

hence

$$\frac{\|\nabla w\|^2}{\|w\|^2} = 2$$

which is to be expected since we computed the Rayleigh quotient of the first eigenfunction.

(5) The domain is a semidisk (wedge-type), hence the solution is given by

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^n \sin(n\theta)$$

where

$$A_n = \frac{2}{n\pi}\int_0^\pi (\sin\theta + \pi\sin(2\theta))\sin(n\theta)d\theta.$$

For $n = 1$, we have

$$\begin{aligned} A_1 &= \frac{2}{\pi}\int_0^\pi \sin^2(\theta)d\theta + \frac{2}{\pi}\int_0^\pi \sin(2\theta)\sin(\theta)d\theta \\ &= 1 \end{aligned}$$

and for $n = 2$,

$$A_2 = \int_0^\pi \sin^2(2\theta)de\theta = \frac{\pi}{2}$$

and for $n > 2$, $m = 1, 2$

$$\int_0^\pi \sin(m\theta)\sin(n\theta)d\theta = 0,$$

hence

$$u(r,\theta) = r\sin(\theta) + \frac{\pi}{2}r^2\sin(2\theta).$$