MATH 124B: TAKE HOME FINAL

Any theorems covered during lecture or in Strauss' PDE textbook, including its appendix can be referenced. Any other theorems or nontrivial claims must be provided with proof. A claim is nontrivial if you do not know why its true., i.e. no "From Theorem 2 of Book X".

Due in class on Thursday or online Friday

(1) Derive the Laplacian in spherical coordinates.

(2) Compute the sum $\sum_{n=1}^{\infty} \frac{1}{n^8}$. You may use previously established sums. Then explain how one could obtain the sum for $\sum_{n=1}^{\infty} \frac{1}{n^{2k}}$ for any $k \in \mathbb{N}$.

(3) Derive Poisson's formula for $D = \{(r, \theta) \in \mathbb{R}^2 \mid r > a > 0\}$, given by

$$u(r,\theta) = (r^2 - a^2) \int_0^{2\pi} \frac{h(\phi)}{a^2 - 2ar\cos(\theta - \phi) + r^2} \frac{d\phi}{2\pi}$$

where u is a solution of

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{for } r > a \\ u = h(\theta) & \text{for } r = a \\ u \text{ is bounded as } r \to \infty. & \end{cases}$$

(4) A function $u(x, y)$ is subharmonic in D if $\Delta u \geq 0$ in D. Prove that its maximum value is attained on ∂D. Then show that it also satisfies the mean value inequality

$$u(0) \leq \frac{1}{2\pi} \int_0^{2\pi} u(r, \theta) d\theta$$

for all $r > 0$ such that $B(0, r) \subset D$.

(5) Show that the second smallest eigenvalue for the Neumann function is $\lambda_2 > 0$.

(6) Let $g(x)$ be a function on ∂D. Consider the minimum of the functional

$$\frac{1}{2} \iiint_D |\nabla w|^2 dV - \iiint_D f w dV$$

among all C^2 functions w for which $w = g$ on ∂D. Show that a solution of this minimum problem leads to a solution of the Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{in } D \\ u = g & \text{on } \partial D. \end{cases}$$

(7) The Neumann function $N(x, y)$ for a domain D is defined much like the Green's function, except for the boundary condition is replaced by

$$\frac{\partial N}{\partial n} = c, \qquad \text{on } \partial D,$$

for some constant c. Show that $c = \frac{1}{A}$ where A is the area of ∂D, ($c = 0$ if $A = \infty$). Then state and prove a theorem expressing the solution of the Neumann problem in terms of the Neumann function, (Theorem 7.3.1 in Strauss).

(8) Compute the eigenvalues of the 2 dimensional rectangle,

$$\begin{cases} \Delta u + \lambda u = 0 & \text{on } [0, a] \times [0, b] \\ u = 0 & \text{on } \partial D. \end{cases}$$

(Use separation of variables)

Solution

(1) One can directly compute using chain rule, however, there is an elegant way given in the textbook.

(2) There are multiple ways to do this, one way is the following: First compute the Fourier cosine series for x^3 $(0, \pi)$. Then for $n \neq 0$,

$$A_n = \begin{cases} \frac{6\pi}{n^2} & n \text{ even} \\ \frac{24}{n^4\pi} - \frac{6\pi}{n^2} & n \text{ odd}. \end{cases}$$

and

$$A_0 = \frac{\pi^3}{4},$$

keeping in mind that the A_0 term is not multiplied by 2 in using Parseval's identity. By Parseval's identity,

$$\sum_{n=0}^{\infty} |A_n|^2 \int_0^{\pi} \cos^2(nx)dx = \int_0^{\pi} x^6 dx$$

Therefore,

$$\sum_{n \text{ even}} \frac{36\pi^2}{n^4} + \sum_{n \text{ odd}} \left(\frac{(24)^2}{n^8\pi^2} - \frac{288}{n^6} + \frac{36\pi^2}{n^4} \right) = \frac{9\pi^6}{56}.$$

Note that $\displaystyle\int_0^{\pi} \cos^2(nx)dx = \frac{\pi}{2}$. Using the previously established sums:

$$\sum_{n \text{ odd}} \frac{1}{n^6} = \frac{\pi^6}{960}$$

$$\sum_{n \text{ odd}} \frac{1}{n^4} = \frac{\pi^4}{96}$$

$$\sum_{n \text{ even}} \frac{1}{n^4} = \frac{\pi^4}{1440}$$

Then

$$\frac{(24)^2}{\pi^2}\sum_{n \text{ odd}} \frac{1}{n^8} = \frac{9\pi^6}{56} - \frac{\pi^6}{40} + \frac{3\pi^6}{10} - \frac{3\pi^6}{8} = \frac{17\pi^6}{280}.$$

Let $S = \sum_{n=1}^{\infty} \frac{1}{n^8}$. Then

$$S = \sum_{n \text{ odd}} \frac{1}{n^8} + \sum_{n=1}^{\infty} \frac{1}{(2n)^8}$$

$$= \sum_{n \text{ odd}} + \frac{1}{256} S$$

so

$$S = \frac{256}{255}\frac{17\pi^8}{280(24)^2} = \frac{\pi^8}{9450}$$

In general, we compute the Fourier sine or cosine series of higher degree monomials and use Parseval's identity, or we can compute the Fourier cosine series of a sufficiently high power, then evaluate at 0 and inductively compute the sum.

Remark: We never discussed the Riemann zeta function $\zeta(s) = \sum_n \frac{1}{n^s}$ and some of you used the recurrence relation to compute. While mathematically correct, such an identity was not proved in class or given in the textbook, hence if you did not provide a proof of the recurrence relation, you will not receive full credit for the problem.

(3) After separating by variables, we have the radial solutions given by $r^n + r^{-n}$. Since we are considering the exterior of a circle with the extra condition that the solution be bounded as it tends to infinity, we keep the constant r^{-n} terms. Therefore,

$$u(r,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^{-n}(A_n \cos(n\theta) + B_n \sin(n\theta))$$

with Fourier coefficients

$$A_n = \frac{a^n}{\pi}\int_0^{2\pi} h(\phi)\cos(n\phi)d\phi$$

$$B_n = \frac{a^n}{\pi}\int_0^{2\pi} h(\phi)\sin(n\phi)d\phi.$$

Inserting these into the solution, we have

$$u(r,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} \frac{a^n}{r^n\pi}\int_{-\pi}^{\pi} h(\phi)(\cos(n\phi)\cos(n\theta) + \sin(n\phi)\sin(n\theta))d\phi.$$

Note that this is the same case as the interior of the circle case except we use a/r instead of r/a. Hence by changing variables $R = 1/r$ and $A = 1/a$, we have from the circle case,

$$\left(\frac{1}{a^2} - \frac{1}{r^2}\right)\int_0^{2\pi} \frac{h(\phi)}{a^{-2} - \frac{2}{ar}\cos(\theta-\phi) + r^{-2}} = r^2 - a^2\int_0^{2\pi} \frac{h(\phi)}{r^2 - 2ar\cos(\theta-\phi) + a^2}$$

(4) Let $\varepsilon > 0$ and define the function

$$v_\varepsilon(\mathbf{x}) := u(\mathbf{x}) + \varepsilon\|\mathbf{x}\|^2.$$

Suppose there is an interior maximum $\mathbf{x}_0$. Then at this point, by the second derivative test, we have $0 \geq \Delta v_\varepsilon(\mathbf{x}_0)$, however

$$\Delta v_\varepsilon = \Delta u + 4\varepsilon > 0,$$

which is a contradiction. Therefore,

$$u(\mathbf{x}) \leq v_\varepsilon(\mathbf{x}) \leq \max_D v_\varepsilon = \max_{\partial D} v_\varepsilon = \max_{\partial D} u + 4\varepsilon L^2$$

where $L = \operatorname{diam}(D)$. Letting $\varepsilon \to 0$ gives the result. Next to show the mean value inequality, first we use Green's first identity with $D = \{\|\mathbf{x}\| = r = a\}$ so that

$$\iint_{\partial D} \frac{\partial u}{\partial r} dS = \iiint_D \Delta u dV \geq 0.$$

In polar coordinates, dividing by the positive constant a,

$$0 \leq \int_0^{2\pi} u_r(a,\theta)d\theta = \frac{\partial}{\partial r}\left(\int_0^{2\pi} u(r,\theta)d\theta\right)\bigg|_{r=a}$$

So $\int_0^{2\pi} u(r,\theta)d\theta$ is an increasing function of r. When $r \to 0$, it is a minimum and equals $2\pi u(0)$. Therefore

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(r,\theta)d\theta = \frac{1}{2\pi r}\int_0^{2\pi} u(r,\theta)dS_\theta$$

where $dS_\theta = rd\theta$ is the surface area element. This is the proof in dimension 2, however the essential idea is the same for any dimension.

(5) The first eigenvalue for the Neumann problem is 0. By the minimum principle (or by energy methods), such eigenfunctions satisfy $\|\nabla w\| = 0$, so that the eigenfunctions are constant. Hence, there cannot be two linearly independent eigenfunctions for 0, so λ_1 is simple and $\lambda_2 > 0$.

(6) Define the energy functional

$$E[w] := \frac{1}{2}\iiint_D |\nabla w|^2 dV - \iiint_D fw dV.$$

Let u be a minimum and let w be a test function such that $w = 0$ on ∂D. Then $u + tw$ satisfies the boundary condition and

$$\begin{aligned}
E[u] &\leq E[u+tw] \\
&= \frac{1}{2}\iiint_D |\nabla(u+tw)|^2 - \iiint_D f(u+tw) \\
&= \frac{1}{2}\iiint_D |\nabla u|^2 + t\iiint_D \nabla u\cdot\nabla w + \frac{t^2}{2}\iiint_D |\nabla w|^2 - \iiint_D fu - t\iiint_D fw \\
&= E[u] - t\iiint_D (\Delta u + f)w + \frac{t^2}{2}\iiint_D |\nabla w|^2
\end{aligned}$$

Dividing by t and noting that the minimum occurs when $t = 0$, we have

$$\iiint_D (\Delta u + f)w = 0,$$

for any compactly supported test function, hence we $-\Delta u = f$ on D, with boundary condition $u = g$ on ∂D.

(7) By Green's first identity and the harmonic integral representation formula with $u = 1$,

$$\begin{aligned} cA &= \iint_{\partial D} \frac{\partial N}{\partial n} dS \\ &= \iiint_D \Delta N dV \\ &= \iiint_D \Delta\left(N + \frac{1}{4\pi\|\mathbf{x} - \mathbf{x}_0\|}\right) dV - \iiint_D \Delta\left(\frac{1}{4\pi\|\mathbf{x} - \mathbf{x}_0\|}\right) dV \\ &= -\iint_{\partial D} \frac{\partial}{\partial n}\left(\frac{1}{4\pi\|\mathbf{x} - \mathbf{x}_0\|}\right) dSv \\ &= 1, \end{aligned}$$

so $c = 1/A$.

We now will prove the following

Theorem. *If $N(\mathbf{x}, \mathbf{x}_0)$ is the Neumann function, then the solution of the Neumann problem is given by*

$$u(\mathbf{x}_0) = -\iint_{\partial D} \frac{\partial u(\mathbf{x})}{\partial n} N(\mathbf{x}, \mathbf{x}_0) dS + \frac{1}{A(\partial D)} \iint_{\partial D} u dS.$$

Note that the extra term comes from the fact that Neumann solutions are only unique up to a constant.

Proof. From the integral representation formula and definition of the (Neumann) Green's function we have

$$\begin{aligned} u(\mathbf{x}_0) &= \iint_{\partial D} \left(u\frac{\partial N}{\partial n} - \frac{\partial u}{\partial n} N\right) dS \\ &= c\iint_{\partial D} u dS - \iint_{\partial D} \frac{\partial u}{\partial n} N dS \end{aligned}$$

□

(8) This was done during lecture, the eigenfunction is

$$u(x, y) = \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right).$$

with eigenvalues

$$\lambda_{m,n} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2.$$