MATH 147A: HOMEWORK 6

Suggested due date Thursday, May 26.

(1) Provide details to example 6.3.5 in text, i.e. show that the stereographic projection is a conformal map between the sphere and the plane.

(2) (6.4.1) Determine the area of the part of the paraboloid $z = x^2 + y^2$ with $z \leq 1$ and compare with the area of the hemisphere $x^2 + y^2 + z^2 = 1$, $z \leq 0$.

(3) (6.4.2) A sailor circumnavigates Australia by a route consisting of a triangle whose sides are arcs of great circles. Prove that at least one interior angle of the triangle is greater than or equal to $\frac{\pi}{3} + \frac{10}{169}$ radians. Take the Earth to be a sphere of radius 6,500 km and assume that the area of Australia is 7.5 million square kilometers.

(4) (6.4.3) A spherical polygon on S^2 is the region formed by intersection of n hemispheres of S^2, where n is an integer greater than or equal to 3. Show that if $\alpha_1, \dots, \alpha_n$ are the interior angles of such a polygon, its area is equal to

$$\sum_{i=1}^{n} \alpha_i - (n-2)\pi.$$

(5) (6.4.4) Suppose that S^2 is covered by spherical polygons such that the intersection of any two polygons is either empty or a common edge or vertex of each polygon. Suppose that there are F polygons, E edges and V vertices (a common edge or vertex of more than one polygon is counted only once. Show that the sum of the angles of all the polygons is $2\pi V$. Deduce that $V - E + F = 2$. (Euler characteristic of a sphere).

(6) (6.4.5) Show that:
   (a) Every local isometry is an equiareal map
   (b) A map that is both conformal and equiareal is a local isometry.

   Give an example of an equiareal map that is not a local isometry.

(7) Let $\alpha : (a,b) \to \mathbb{R}^3$ be a regular parametrized curve, and $\beta : (a,b) \to mathbbR^3$ a smooth nonvanishing function. Given a surface S parametrized by $\sigma(u,v) = \alpha(u) + v\beta(u)$ with $\alpha' \neq 0$ and $\|\beta\| = 1$ (such a surface is called a **ruled surface**,
   (a) Check that we may assume that $\alpha'(u) \cdot \beta(u) = 0$ for all u. (Hint: replace $\alpha(u)$ with $\alpha(u) + t(u)\beta(u)$ for a suitable function t. (see exercise 5.3.4))
   (b) Suppose, moreover, that $\alpha'(u)$,$\beta(u)$, and $\beta'(u)$ are linearly dependent for every u. Conclude that $\beta'(u) = \lambda(u)\alpha'(u)$ for some function λ. Prove that
      (i) If $\lambda(u) = 0$ for all u, then S is a (generalized) cylinder.
      (ii) If λ is a nonzero constant, then S is a (generalized) cone.
      (iii) If λ and λ' are both nowhere zero, then S is a tangent developable (at least locally).