MATH 147A MIDTERM

Choose 5 of the 6. Each problem is worth 10 points for a total of 50 points. No extra credit for doing more problems. Please circle your choice of 5.

(1) Compute the curvature and torsion of

$$\gamma(t) = \left(2\cos\left(\frac{t}{\sqrt{5}}\right), 2\sin\left(\frac{t}{\sqrt{5}}\right), \frac{t}{\sqrt{5}}\right).$$

What curve is this.

(2) Compute the curvature and torsion (if it exists) of

$$\gamma(t) = (\alpha\cos(t), 1 - \sin(t), \beta\cos(t)),$$

where $\alpha^2 + \beta^2 = 1$. Show that γ parametrizes a circle, find its center, radius and the plane in which it lies.

(3) Show that a reparametrization by arc-length of a regular curve gives a unit-speed curve. Show this for the curve $\gamma(t) = (\cos(3t), \sin(3t))$.

(4) Let $\gamma : (a, b) \to \mathbb{R}^2$ be a regular plane curve and let $a \in \mathbb{R}^2$ such that $\gamma(t) \neq a$ for all t. If there exists a $t_0 \in (a, b)$ such that

$$\|\gamma(t) - a\| \leq \|\gamma(t_0) - a\|$$

for all $t \in (a, b)$, show that the straight line joining the point a with $\gamma(t_0)$ is the normal line of γ at t_0.

(5) Let $\gamma : (a, b) \to \mathbb{R}^3$ be a unit speed-curve such that $\kappa > 0$ and $0 \in (a, b)$. Consider the Frenet-Serret frame $\{T, N, B\}$ centered at $\gamma(0)$ and use coordinates $\gamma(t) = (x(t), y(t), z(t))$ so that $T = (1, 0, 0)$, $N = (0, 1, 0)$ and $B = (0, 0, 1)$, and $\gamma(0) = (0, 0, 0)$. Prove that

$$\tau(t_0) = \lim_{s\to 0} \frac{6z(s)}{\kappa(t_0)s^3}.$$

(6) Let $\gamma : (a, b) \to \mathbb{R}^3$ be a unit-speed curve whose curvature κ and torsion τ do not vanish. Prove that if γ lies on a unit-sphere, then

$$\frac{1}{\kappa^2} + \left(\frac{\kappa'}{\kappa^2\tau}\right)^2 = 1$$

Hint: For an orthonormal frame $\{T, N, B\}$, $\gamma = (\gamma \cdot T)T + (\gamma \cdot N)N + (\gamma \cdot B)B$.
The converse is true if we assume that $\kappa' \neq 0$. Try to prove this for fun.

Solution

1. The tangent vector is

$$\dot{\gamma}(t) = \left(-\frac{2}{\sqrt{5}}\sin\left(\frac{t}{\sqrt{5}}\right), \frac{2}{\sqrt{5}}\cos\left(\frac{t}{\sqrt{5}}\right), \frac{1}{\sqrt{5}}\right).$$

Since $\|\dot{\gamma}\| = 1$, $\gamma(t)$ is unit speed and $\dot{\gamma}(t) = T$, the unit tangent vector. Hence the curvature can be computed by $\|\ddot{\gamma}\|$, which is

$$\ddot{\gamma}(t) = \left(-\frac{2}{5}\cos\left(\frac{t}{\sqrt{5}}\right), -\frac{2}{5}\sin\left(\frac{t}{\sqrt{5}}\right), 0\right)$$

and so $\kappa = \|\ddot{\gamma}\| = \frac{2}{5}$. The principal normal is given by $N = \left(-\cos\left(\frac{t}{\sqrt{5}}\right), -\sin\left(\frac{t}{\sqrt{5}}\right), 0\right)$. Computing the binormal, we have

$$B = T \times N = \left(\frac{1}{\sqrt{5}}\sin\left(\frac{t}{\sqrt{5}}\right), -\frac{1}{\sqrt{5}}\cos\left(\frac{t}{\sqrt{5}}\right), \frac{2}{\sqrt{5}}\right)$$

and

$$\dot{B} = \left(\frac{1}{5}\cos\left(\frac{t}{\sqrt{5}}\right), \frac{1}{5}\sin\left(\frac{t}{\sqrt{5}}\right), 0\right) = -\frac{1}{5}\left(-\cos\left(\frac{t}{\sqrt{5}}\right), -\sin\left(\frac{t}{\sqrt{5}}\right), 0\right) = -\tau N.$$

so that $\tau = \frac{1}{5}$. The curve is a helix and hence the curvature and torsion can be computed using the formula provided in the book.

2. First we compute the tangent vector

$$\dot{\gamma}(t) = (-\alpha\sin(t), -\cos(t), -\beta\sin(t)).$$

Then $\|\dot{\gamma}\|^2 = 1$, hence is unit speed so $\dot{\gamma} = T$. The curvature is

$$\kappa = \|\ddot{\gamma}\| = \|(-\alpha\cos(t), \sin(t), -\beta\cos(t))\| = 1.$$

So we see that the principal normal is $\dot{T} = \kappa N = N$. The binormal is given by

$$B = T \times N = (\beta, 0, -\alpha).$$

So $\dot{B} = 0$, hence $\tau = 0$. Since the torsion vanishes and is constant curvature, γ parametrizes a circle, with radius $R = \frac{1}{\kappa} = 1$. The plane that it lies on is given by the binormal vector so $\beta x - \alpha z = 0$ and its center is given by

$$a = \gamma - \frac{1}{k}N = (0, 1, 0).$$

3. See solution to sample midterm for the first part. The arc-length formula is given by

$$s(t) = \int_0^t \|\dot{\gamma}\| du = \int_0^t 3du = 3t$$

and so its inverse is given by $t(s) = \frac{s}{3}$. Plugging this into the original curve, we get $\tilde{\gamma}(s) = \gamma(t(s)) = (\cos(s), \sin(s))$. Then $\frac{d}{ds}\tilde{\gamma} = (-\sin(s), \cos(s))$ hence is a unit speed curve.

4. See solution to sample midterm.

5. Consider the third order Taylor expansion

$$\gamma(s) = \gamma(0) + \gamma'(0)s + \gamma''(0)\frac{s^2}{2} + \gamma'''(0)\frac{s^3}{6} + O(s^4).$$

Now since γ is unit-speed, $\gamma'(0) = T(s)$, $\gamma''(s) = \kappa(s)N(s)$ and using the Frenet-Serret equation,

$$\begin{aligned}\gamma'''(s) &= \kappa'(s)N(s) + \kappa(s)\dot{N}(s)\\ &= \kappa'(s)N(s) + \kappa(s)(-\kappa(s)T(s) + \tau B(s)).\end{aligned}$$

Hence

$$\begin{aligned}\gamma(s) &= (x(s), y(s), z(s))\\ &= Ts + \kappa N\frac{s^2}{2} + \kappa' N\frac{s^3}{6} - \kappa^2 T\frac{s^3}{6} + \tau\kappa B\frac{s^3}{6} + O(s^4)\\ &= \left(s - \kappa^2\frac{s^3}{6} + O(s^4),\ \kappa\frac{s^2}{2} + \kappa'\frac{s^3}{6} + O(s^4),\ \tau\kappa\frac{s^3}{6} + O(s^4)\right)\end{aligned}$$

so

$$\tau = \lim_{s\to 0}\frac{6z(s)}{\kappa s^3}.$$

6. Since $\gamma \in S^2$, we have (assuming S^2 is centered at $(0,0,0)$) $\|\gamma\| = 1$ therefore, taking the derivative we have

$$\gamma' \cdot \gamma = T \cdot \gamma = 0$$

Taking the derivative once more, we have

$$\gamma'' \cdot \gamma + \gamma' \cdot \gamma' = 0$$

and since γ is unit speed, we have

$$N \cdot \gamma = -\frac{1}{\kappa}.$$

Taking the derivative once more,

$$\dot{N} \cdot \gamma + N \cdot \gamma' = \frac{\kappa'}{\kappa^2}$$

and using the Frenet-Serret equation and the fact that $\{T, N, B\}$ form an orthonormal basis,

$$\frac{\kappa'}{\kappa^2} = -\kappa T \cdot \gamma + \tau B \cdot \gamma = \tau B \cdot \gamma.$$

Now, writing γ as a linear combination of $\{T, N, B\}$, we have

$$\begin{aligned} \gamma &= (\gamma \cdot T)T + (\gamma \cdot N)N + (\gamma \cdot B)B \\ &= -\frac{1}{\kappa}N + \frac{\kappa'}{\kappa^2 \tau}B. \end{aligned}$$

Taking the dot product with γ, we have

$$\begin{aligned} 1 &= \gamma \cdot \gamma \\ &= -\frac{1}{\kappa}N \cdot \gamma + \frac{\kappa'}{\kappa^2 \tau}B \cdot \gamma \\ &= \frac{1}{\kappa^2} + \left(\frac{\kappa'}{\kappa^2 \tau}\right)^2. \end{aligned}$$