MATH 147A MIDTERM

Choose 5 of the 6. Each problem is worth 10 points for a total of 50 points. No extra credit for doing more problems. Please circle your choice of 5.

(1) Compute the curvature and torsion of

$$\gamma(t) = \left(2\cos\left(\frac{t}{\sqrt{5}}\right), 2\sin\left(\frac{t}{\sqrt{5}}\right), \frac{t}{\sqrt{5}}\right).$$

What curve is this.

(2) Compute the curvature and torsion (if it exists) of

$$\gamma(t) = (\alpha\cos(t), 1 - \sin(t), \beta\cos(t)),$$

where $\alpha^2 + \beta^2 = 1$. Show that γ parametrizes a circle, find its center, radius and the plane in which it lies.

(3) Show that a reparametrization by arc-length of a regular curve gives a unit-speed curve. Show this for the curve $\gamma(t) = (\cos(3t), \sin(3t))$.

(4) Let $\gamma : (a, b) \to \mathbb{R}^2$ be a regular plane curve and let $a \in \mathbb{R}^2$ such that $\gamma(t) \neq a$ for all t. If there exists a $t_0 \in (a, b)$ such that

$$\|\gamma(t) - a\| \leq \|\gamma(t_0) - a\|$$

for all $t \in (a, b)$, show that the straight line joining the point a with $\gamma(t_0)$ is the normal line of γ at t_0.

(5) Let $\gamma : (a, b) \to \mathbb{R}^3$ be a unit speed-curve such that $\kappa > 0$. Consider the Frenet-Serret frame $\{T, N, B\}$ centered at $\gamma(t_0)$ and use coordinates $\gamma(t) = (x(t), y(t), z(t))$ so that $T = (1, 0, 0)$, $N = (0, 1, 0)$ and $B = (0, 0, 1)$, and $\gamma(t_0) = (0, 0, 0)$. Prove that

$$\tau(t_0) = \lim_{s \to 0} \frac{6z(s)}{\kappa(t_0)s^3}.$$

(6) Let $\gamma : (a, b) \to \mathbb{R}^3$ be a unit-speed curve whose curvature κ and torsion τ do not vanish. Prove that if γ lies on a unit-sphere, then

$$\frac{1}{\kappa^2} + \left(\frac{\kappa'}{\kappa^2\tau}\right)^2 = 1$$

Hint: For an orthonormal frame $\{T, N, B\}$, $\gamma = (\gamma \cdot T)T + (\gamma \cdot N)N + (\gamma \cdot B)B$.
The converse is true if we assume that $\kappa' \neq 0$. Try to prove this for fun.

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