MATH 147A SAMPLE MIDTERM

April 28, 2016 The midterm will be 5 problems, some computational, some conceptual. Feel free to bring a reasonable sized paper for notes and formulas.

(1) Compute the curvature of
$$\gamma(t) = (\cos^3(t), \sin^3(t)).$$

(2) Compute the curvature and torsion (if it exists) of
$$\gamma(t) = \left(\frac{4}{5}\cos(t), 1 - \sin(t), -\frac{3}{5}\cos(t)\right)$$
Show that γ parametrizes a circle, find its center, radius and the plane in which it lies.

(3) Show that a curve unit speed curve with $\kappa(s) > 0$ for each $s \in [a, b]$ is a plane curve if and only if the torsion vanishes everywhere.

(4) State and prove the isoperimetric inequality, assuming Wirtinger's inequality and Green's theorem for area.

(5) Show that a reparametrization by arc-length gives a unit-speed curve.

(6) Derive the Frenet-Serret equation.

(7) Let $\gamma : (a, b) \to \mathbb{R}^2$ be a regular plane curve and let $a \in \mathbb{R}^2$ such that $\gamma(t) \neq a$ for all t. If there exists a $t_0 \in (a, b)$ such that
$$\|\gamma(t) - a\| \geq \|\gamma(t_0) - a\|$$
for all $t \in (a, b)$, show that the straight line joining the point a with $\gamma(t_0)$ is the normal line of γ at t_0. The same is true if we reverse the inequality. Draw a situation that illustrates both cases.

(8) Let $\gamma : (a, b) \to \mathbb{R}^2$ be a regular plane curve and let $[\alpha, \beta] \subset (a, b)$ be such that $\gamma(\alpha) \neq \gamma(\beta)$. Prove that there exists some $t_0 \in (\alpha, \beta)$ such that the tangent line of γ at t_0 is parallel to the segment of the straight line joining $\gamma(\alpha)$ with $\gamma(\beta)$. This is a curve version of mean value theorem. Hint: Consider $f(t) = \det(\gamma(t), \gamma(\beta) - \gamma(\alpha))$.

(9) Prove that a unit speed curve $\gamma : (a, b) \to \mathbb{R}^2$ is an arc of a circle if and only if all its normal lines pass through a given point.

(10) Let $\gamma : (a, b) \to \mathbb{R}^3$ be a unit speed curve with positive curvature. If $\|\gamma(s)\| = 1$ for all s, i.e. γ is a curve on a sphere, and it has constant torsion τ, prove that there exists $b, c \in \mathbb{R}$ such that
$$\kappa(s) = \frac{1}{b\cos(\tau s) + c\sin(\tau s)}.$$

(11) Let γ be a unit speed curve in $\mathbb{R}^3$ with constant curvature and zero torsion. Show that γ is a parametrization of a circle.