Abstract. The initial value problem for the compressible Euler equations in two space dimensions is studied. Of interest is the lifespan of classical solutions with initial data that is a small perturbation from a constant state. The approach taken is to regard the compressible solution as a nonlinear superposition of an underlying incompressible flow and an irrotational compressible flow. This viewpoint yields an improvement for the lifespan over that given by standard existence theory. The estimate for the lifespan is further improved when the initial data possesses certain symmetry. In the case of rotational symmetry, a result of S. Alinhac is reconsidered. The approach is also applied to the study of the incompressible limit. The analysis combines energy and decay estimates based on vector fields related to the natural invariance of the equations.

0. Introduction. The compressible Euler equations comprise a nonlinear symmetric hyperbolic system of PDE’s that model ideal fluid flow. The characteristic wave speeds of this system are given by the fluid velocity and the local sound speed. The two wave families associated to these speeds may be considered separately. Hydrodynamical waves are described by the incompressible Euler equations, while acoustical waves are approximated by the linear wave equation, or more precisely, by the compressible Euler equations with irrotational fluid velocity. This paper attempts to use the nonlinear superposition of these two flows to study the long time behavior of small amplitude disturbances in two dimensional compressible fluid flow.

Consider the initial value problem for the compressible Euler equations in two or three space dimensions with smooth initial data which is a small perturbation of amplitude ε from a constant state. The life span is defined as the largest time interval on which there exists a classical solution to the initial value problem. From the theory of symmetric hyperbolic systems [6], [8], the life span is at least $O(1/\varepsilon)$. Results on formation of singularities show that the life span of a classical solution is no better than $O(1/\varepsilon^2)$ in 2D, [16], and $O[\exp(1/\varepsilon^2)]$ in 3D, [17]. In this paper, we extend the length of the life span under various assumptions on the initial data, in two space dimensions.

The simplest case is the one in which the initial fluid velocity is irrotational, that is, its curl is zero. The compressible Euler equations then behave much the same as scalar nonlinear wave equations. (This can readily be guessed by writing the velocity as the gradient of a potential, although we will not use this approach here.) We shall see in Theorem 2 that the life span of 2D irrotational
compressible flow is $O(1/\varepsilon^2)$. This fact will be built upon in the other results of this paper. For example, it can be perturbed slightly by taking nearly irrotational initial velocities, that is, one whose curl is $O(\varepsilon^2)$. The result for irrotational flow compares with results for the 2D scalar nonlinear wave equation (see [7], for example). The analogy between irrotational flow and nonlinear wave equations was previously studied in three space dimensions in [18], where there is almost global existence.

On the other hand, the incompressible Euler equations in two space dimensions have global smooth solutions ([22], [14], [8], [10]).

Viewing compressible flow as a superposition of irrotational flow and incompressible flow, one might suspect from these facts that some enhancement in the lower bound for the life span is valid. Of course, nonlinear interactions between the two separate components will play a crucial role. The situation is further complicated by the fact that although the incompressible flow is globally regular, upper bounds for its size grow rapidly in time. Nevertheless, some results are possible. The most general case is considered in Theorem 1 which states that for any $T > 0$, the life span is at least $T/\varepsilon$ for all $\varepsilon$ sufficiently small. The proof of this result uses the idea of superposition of the nonlinear flows. Any further improvement in the estimate for the life span requires a more detailed analysis of the first order approximates. Such a result, allowing for nonlinear interactions between first order components, can be obtained for rotationally symmetric flow (Theorem 3). In a series of papers [1], [2], Alinhac showed that the life span of rotationally symmetric flow is $O(1/\varepsilon^2)$. We give a new proof of his result (with a slightly shorter life span) in which the superposition of first order nonlinear flows replaces his asymptotic solution.

The chief techniques employed are energy and decay estimates using vector fields associated to the natural invariance of the equations. The Euler equations are non-relativistic equations, and so the recent methods of [13], [19], which apply to the case of 3D nonlinear elasticity, turn out to be relevant. However, the case of fluids is more complicated than nonlinear elasticity ([19]) due to the presence of the hydrodynamical waves which have no time decay. These waves do tend to remain localized, however. Acoustical waves, on the other hand, disperse and decay in time, so the nonlinear interaction between these two distinct types of waves is well-behaved, in principle. Further difficulties arise when the hydrodynamical waves self-interact. This is the essential feature of the incompressible case. One big advantage offered by the assumption of rotational symmetry is that the underlying incompressible flow is stationary, and so its effects are more easily assessed.

The estimates behind the proof of Theorem 1 also offer some insight into the incompressible limit. In [18], it was shown that life span estimates for the 3D small perturbation initial value problem can be rescaled to yield weak convergence in the incompressible (high Mach number) limit (see also [5], [12], [21]). The same is true in 2D. In the present work, thanks to the specific form of our
expansion based on an underlying incompressible flow, we are able to establish strong convergence in the incompressible limit on arbitrary time intervals using the rescaling idea, in Theorem 4.

Precise statements of the main results can be found in Section 2, after we briefly explain the notation to be used. A few preparatory sections then follow. In Section 3, we derive some basic calculus inequalities in the spirit of [13], and Section 4 deals with the commutation properties of the vector fields. The basis for the decay estimates is found in Section 5, where the equations of linearized acoustics are studied. Again the method is similar to [13], giving a somewhat different approach than the 3D study [18]. The proofs of the results are then presented, with the case of irrotational flow examined first, as everything else depends on that basic case.

1. Notation. Partial derivatives will be denoted by

\[ \partial_t = \partial / \partial t \quad \text{and} \quad \nabla = (\partial_1, \partial_2) = (\partial / \partial x_1, \partial / \partial x_2). \]

The radial derivative is defined by

\[ \partial_r = \frac{x}{r} \cdot \nabla, \quad r = \|x\|. \]

and the angular derivative in \( \mathbb{R}^2 \) is

\[ \Omega = x^\perp \cdot \nabla, \quad x^\perp = (-x_2, x_1). \]

It follows that

\[ \nabla = \frac{x}{r} \partial_r + \frac{1}{r} \frac{x^\perp}{r} \Omega. \]  \hfill (1.1)

The scaling operator is

\[ S = t \partial_t + r \partial_r. \]

Solutions to the Euler equations are invariant under simultaneous rotation of independent and dependent variables. Accordingly, it is natural to introduce the vector field \( \tilde{\Omega} \) which acts on vector functions \( u(x) \in \mathbb{R}^2 \) through the formula

\[ \tilde{\Omega} u = \Omega u - u^\perp. \]

We denote the collection of these vector fields by

\[ \Gamma = (\Gamma_0, \ldots, \Gamma_4) = (\partial_t, \nabla, \Omega, S) \]
In summary, the \( \tilde{\Gamma} = (\tilde{\Gamma}_0, \ldots, \tilde{\Gamma}_4) = (\partial_t, \nabla, \tilde{\Omega}, S) \).

For any multi-index \( a = (a_0, \ldots, a_4) \in \mathbb{Z}^4_+ \), we set
\[
\Gamma^a = \Gamma_0^{a_0} \cdots \Gamma_4^{a_4} \quad \text{and} \quad \tilde{\Gamma}^a = \tilde{\Gamma}_0^{a_0} \cdots \tilde{\Gamma}_4^{a_4}.
\]

In order to describe the solution space, we introduce the time-independent versions of these vector fields:
\[
\Lambda = (\Lambda_1, \ldots, \Lambda_4) = (\nabla, \Omega, r \partial_t)
\]
and
\[
\tilde{\Lambda} = (\tilde{\Lambda}_1, \ldots, \tilde{\Lambda}_4) = (\nabla, \tilde{\Omega}, r \partial_t).
\]

In summary, the \( \Gamma \) and \( \Lambda \) will be applied to scalar functions, and the \( \tilde{\Gamma} \) and \( \tilde{\Lambda} \), which simply replace \( \Omega \) by \( \tilde{\Omega} \), will be applied to vectors.

Associated to these vector fields, we define the Sobolev spaces
\[
H^m_\Lambda(\mathbb{R}^2) = \left\{ \rho \in L^2(\mathbb{R}^2) : \|\rho\|_{H^m_\Lambda}^2 = \sum_{|\gamma| \leq m} \|\Lambda^\gamma \rho\|_{L^2}^2 < \infty \right\},
\]
and
\[
\tilde{H}^m_\Lambda(\mathbb{R}^2) = \left\{ u \in L^2(\mathbb{R}^2)^2 : \|u\|_{\tilde{H}^m_\Lambda}^2 = \sum_{|\gamma| \leq m} \|\tilde{\Lambda}^\gamma u\|_{L^2}^2 < \infty \right\}.
\]

Solutions \((\rho, u)\) will be constructed in the space
\[
X^m(T) = \bigcap_{j=0}^m C^j \left( [0, T); H^{m-j}_{\Lambda} \times \tilde{H}^{m-j}_{\Lambda} \right),
\]
with the norm
\[
\|(\rho, u)\|_{X^m(T)}^2 = \sup_{0 \leq t \leq T} E_m[\rho(t), u(t)],
\]
given in terms of the generalized energy
\[
E_m[\rho(t), u(t)] = \sum_{|\gamma| \leq m} \left[ \|\Gamma^\gamma \rho(t)\|_{L^2}^2 + \|\tilde{\Gamma}^\gamma u(t)\|_{L^2}^2 \right].
An important role will be played by the $L^2$ orthogonal projections onto the irrotational and divergence free vectors, $P_1$ and $P_2$, respectively. They are most easily defined for $u \in L^2(\mathbb{R}^2)^2$ by

$$P_1 u = R(R \cdot u) \quad \text{and} \quad P_2 u = R^\perp(R^\perp \cdot u),$$

with $R = (R_1, R_2)$ being the pair of Riesz transformations, i.e. the operators with symbols $\hat{R}_j = \xi_j/|\xi|$. It is a simple matter to verify, using the Fourier transform, that the $P_j$ commute with the $\hat{\Gamma}$ and $\hat{\Lambda}$ and hence that

$$P_j : \tilde{H}^m_\Lambda \to \tilde{H}^m_\Lambda, \quad j = 1, 2.$$ 

Moreover, for all $u \in L^2(\mathbb{R}^2)^2$,

$$u = P_1 u + P_2 u, \quad \langle P_1 u, P_2 u \rangle_{L^2} = 0$$
$$\nabla \cdot u = \nabla \cdot P_1 u, \quad \nabla^\perp \cdot u = \nabla^\perp \cdot P_2 u.$$ 

2. Results. We will be concerned with the existence of smooth solutions $(\bar{\rho}(t,x), \bar{u}(t,x))$ to the two-dimensional compressible Euler with initial data that is a perturbation of order $\varepsilon$ from the constant state $(\bar{\rho}, 0)$. For simplicity, we will only consider polytropic fluids at constant entropy. The initial value problem takes the form

$$\partial_t \bar{\rho} + \nabla \cdot \bar{\rho} \bar{u} = 0$$
$$\bar{\rho} (\partial_t \bar{u} + \bar{u} \cdot \nabla \bar{u}) + \nabla (\bar{\rho}^\gamma) = 0, \quad (\gamma > 1)$$
$$\bar{\rho}(0,x) = \bar{\rho} + \epsilon \bar{u}(x), \quad (\bar{\rho} > 0)$$
$$\bar{u}(0,x) = \epsilon \bar{u}(x),$$

with $(\bar{\rho}_0, \bar{u}_0)$ given functions of order $\varepsilon$ in $H^m_\Lambda \times \tilde{H}^m_\Lambda$, for some $m$. Since the method involves energy estimates for smooth solutions, it is natural to rewrite the problem in symmetric hyperbolic form. At the same time, the sound speed at infinity, $c = (\gamma \bar{\rho}^{\gamma-1})^{1/2}$, will be normalized to be unity. This is achieved through the change of variables

$$u = \frac{1}{\bar{c}} u(t/\bar{c}, x)$$
$$\bar{u} = \frac{2}{\gamma - 1} \left[ \left( \frac{\bar{u}(t/\bar{c}, x)}{\bar{\rho}} \right)^{\frac{\gamma - 1}{2}} - 1 \right].$$
as was also used in [18]. The result of making this transformation is the following symmetric system

\[ \begin{align*}
\partial_t \vec{v} + \nabla \cdot \vec{u} &= -\vec{v} \cdot \nabla v - \frac{n-1}{2} \rho \nabla \cdot \vec{u}, \\
\partial_t \vec{u} + \nabla \cdot \vec{v} &= -\vec{u} \cdot \nabla \vec{u} - \frac{n-1}{2} \rho \nabla \vec{v},
\end{align*} \]

with initial data of order \( \varepsilon \). For notational convenience, throughout the remainder of the paper the superscript \( \varepsilon \) and the \( \text{tildes} \) will be dropped. The reader is cautioned that \( \rho \) no longer corresponds to the fluid density and that the solution \( (\rho, u) \) actually is a member of a family of solutions parametrized by \( \varepsilon \) through its dependence on the initial data.

We arrive at the initial value problem

\[ \begin{align*}
\partial_t \rho + \nabla \cdot u &= -u \cdot \nabla \rho - \frac{n-1}{2} \rho \nabla \cdot u, \\
\partial_t u + \nabla \rho &= -u \cdot \nabla u - \frac{n-1}{2} \rho \nabla \rho, \\
\rho(0, x) &= \rho_0(x), \quad u(0, x) = u_0(x), \quad (\varepsilon > 0).
\end{align*} \]

It will always be assumed that

\[ \rho_0, u_0 \in H^k_x \times \tilde{H}^k_x, \quad \|\rho_0\|_{H^k_x} + \|u_0\|_{\tilde{H}^k_x} \leq C_0 \varepsilon, \quad \text{for all } \varepsilon > 0, \]

for some integer \( k \) and constant \( C_0 \) independent of \( \varepsilon \).

The results are now summarized in the following theorems. The first result is valid for general smooth initial data. It improves the generic lower bound for the life span of smooth solutions of hyperbolic systems with initial data of amplitude \( \varepsilon \) from \( O(1/\varepsilon) \) to \( T/\varepsilon \) for an arbitrary constant \( T > 0 \), provided that \( \varepsilon \) is sufficiently small.

**Theorem 1.** Let \( T > 0 \) be arbitrary. Let \( m \geq 4 \) and assume that (2.4) holds with \( k = m + 3 \). Then there exists an \( \varepsilon_0 = \varepsilon_0(T) > 0 \) such that for all \( 0 < \varepsilon < \varepsilon_0 \), the initial value problem (2.1), (2.2), (2.3) has a unique solution \( (\rho, u) \in X^m(T/\varepsilon) \) which satisfies the uniform bound

\[ \| (\rho, u) \|_{X^m(T/\varepsilon)} \leq C \varepsilon, \quad 0 < \varepsilon < \varepsilon_0, \]

with a constant \( C \) independent of \( \varepsilon \).

The proof of this result is based on an expansion involving a solution to the compressible Euler equations with irrotational initial velocity \( P_1 u_0 \) plus a solution to the incompressible Euler equations with data \( P_2 u_0 \), with higher order corrections. The next results address the existence questions behind this expansion.

In the case of irrotational (and nearly irrotational) compressible flow, the life span of smooth solutions of size \( \varepsilon \) can be improved to \( O(1/\varepsilon^2) \), in analogy with the nonlinear wave equation in two space dimensions.
**Theorem 2.** (i) Let \( m \geq 4 \), assume that (2.4) holds with \( k = m \), and suppose that \( P_2 u_0^\varepsilon = 0 \). Then there exist constants \( \varepsilon_0, A > 0 \) such that for all \( 0 < \varepsilon < \varepsilon_0 \) the initial value problem (2.1), (2.2), (2.3) has a unique solution \( (\rho, u) \in X^m(A^2/\varepsilon^2) \) which satisfies

\[
\| (\rho, u) \|_{X^m(A^2/\varepsilon^2)} \leq C\varepsilon, \quad 0 < \varepsilon < \varepsilon_0,
\]

and

\[
P_2 u(t) = 0, \quad 0 < t < A^2/\varepsilon^2.
\]

(ii) Let \( m \geq 4 \), assume that (2.4) holds with \( k = m + 3 \), and suppose that \( \| P_2 u_0^\varepsilon \|_{\tilde{H}^m_N} \leq C_0 \varepsilon^2 \). Then the conclusions of part (i) hold.

The improvement in the irrotational case is due to the removal of the underlying incompressible flow. In the nearly irrotational case, the interaction between these two flows remains negligible, \( o(\varepsilon) \).

The second piece of the expansion in Theorem 1 involves the initial value problem for the incompressible Euler equation:

\[
\begin{align*}
\nabla \cdot v &= 0 \\
\partial_t v + v \cdot \nabla v + \nabla p &= 0 \\
v(0, x) &= v_0(x) \in \tilde{H}^m_N, \quad v_0 = P_2 v_0.
\end{align*}
\]

It is well-known that, in two space dimensions, the above initial value problem is globally well-posed for data in the standard Sobolev space \( H^m(\mathbb{R}^2) \). The following refinement generalizes this result to the space \( \tilde{H}^m_N(\mathbb{R}^2) \).

**Proposition 1.** If \( m \geq 4 \), then the initial value problem (2.5), (2.6), (2.7) has a unique global solution \( (p, v) \in X^m(T) \) for every \( T > 0 \). Moreover, \( \| (p, v) \|_{X^m(T)} \) depends continuously on \( T \) and \( \| v_0 \|_{\tilde{H}^m_N} \).

Although it is tangential to the main results, a few remarks on the proof of the proposition will be given in the last section.

Between the extremes of the general case and the irrotational case lies the case of rotational symmetry. Here the life span of smooth solutions of the compressible Euler equations with data of order \( \varepsilon \) can be extended to \( O \left[ 1/\varepsilon^2 \ln(1 + 1/\varepsilon^2) \right] \) (cf. Alinhac [2]). Interactions with the first order irrotational compressible and incompressible portions of the flow can only be assessed thanks to the strong symmetry restriction.

**Theorem 3.** Let \( m \geq 4 \), suppose that (2.4) holds with \( k = m + 3 \), and assume that the initial data is rotationally symmetric:

\[
\Omega_{\rho_0} = 0 \quad \text{and} \quad \Omega u_0 = 0.
\]
Also assume that

$$ \|(1 + r^2)P_2u_0^\varepsilon\|_{L^m} \leq C_0\varepsilon. $$

Then there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, the initial value problem (2.1), (2.2), (2.3) has a unique solution $(\rho, u) \in X^m(T_\varepsilon)$ with $T_\varepsilon = 1/\varepsilon^2 \ln (1 + 1/\varepsilon^2)$ satisfying the uniform bound

$$ \|(\rho, u)\|_{X^m(T_\varepsilon)} \leq C\varepsilon. $$

The solution remains rotationally symmetric.

The final result applies the bounds in Theorem 1 to the incompressible limit. By rescaling the time variable $t \rightarrow t/\varepsilon$ and the velocity $u \rightarrow u/\varepsilon$, the existence interval becomes fixed $[0, T]$, the velocity becomes $O(1)$, and the parameter $1/\varepsilon$ appears as a high Mach number in the equations. The rescaled problem is

(2.8) \[ \partial_t \rho + \nabla \cdot u = -u \cdot \nabla \rho - \frac{\rho - 1}{2} \nabla \cdot u \]

(2.9) \[ \partial_t u + \frac{\varepsilon}{\varepsilon^2} \nabla \rho = -u \cdot \nabla u - \frac{\rho - 1}{2} \nabla \rho. \]

(2.10) \[ \rho(0,x) = \rho_0^\varepsilon(x), \quad u(0,x) = \frac{1}{\varepsilon} u_0^\varepsilon(x). \]

**Theorem 4.** Assume that $m \geq 4$ and suppose that (2.4) holds for $k = m+3$. Take any $T > 0$. Then there exists $\varepsilon_0 = \varepsilon_0(T) > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the rescaled initial value problem (2.8), (2.9), (2.10) has a unique solution $(\rho, u) \in X^m(T)$.

Moreover, if $(p, v) \in X^m(T)$ is the solution of (2.5), (2.6), (2.7) with initial data $v(0) = \frac{1}{\varepsilon} P_2u_0^\varepsilon$, given by Proposition 1, then for any $0 < \mu < 1/2$ and for any $0 < \tau < T$, the time-homogeneous derivatives have the following bounds

$$ |\Gamma^\varepsilon(P_2u - v)(t)|_{L^2} \leq C\varepsilon^\mu, $$

$$ 0 \leq t < T, \quad |\alpha| \leq m, \quad a_0 = 0 $$

and

$$ \frac{1}{\varepsilon} |\Gamma^\varepsilon(\rho - p)(t,x)| + |\Gamma^\varepsilon(u - v)(t,x)| \leq C(\tau)\varepsilon^\mu, $$

$$ \tau \leq t < T, \quad x \in \mathbb{R}^2, \quad |\alpha| \leq m-2, \quad a_0 = 0. $$

Thus, if $u_0^\varepsilon = \varepsilon u_0$, for a fixed function $u_0$, and if $v$ is the solution of (2.5), (2.6), (2.7) with data $v_0 = P_2u_0$, then as $\varepsilon \rightarrow 0$, there is strong $L^2$ convergence of $P_2u$ to $v$ and uniform convergence of $u$ to $v$ except for an arbitrarily small initial layer. Previous work in 3D had established only weak convergence with no rate.

**3. Calculus inequalities.** In this section, a few elementary but useful inequalities will be prepared. Such inequalities capture decay at spatial infinity.
through the use of the vector field $\Omega$. Define the weight function

$$\sigma(\lambda) = (1 + \lambda^2)^{1/2}.$$ 

Notice that the derivatives of $\sigma$ are bounded.

**Lemma 1.** Let $f \in H^2_\Lambda$. Then for all $x \in \mathbb{R}^2$ and $t \geq 0$

\begin{align*}
\sigma(|x|)^{1/2} f(x) &\leq C \sum_{j=0}^{2n} \sum_{|\alpha| = 0}^{2n} \|\nabla^\alpha \Omega f\|_{L^2}, \\
\sigma(|x|)^{1/2} \sigma(t - |x|) f(x) &\leq C \sum_{j=0}^{2n} \sum_{|\alpha| = 0}^{2n} \|\sigma(t - |\cdot|) \nabla^\alpha \Omega f(\cdot)\|_{L^2}, \\
\sigma(|x|)^{1/2} \sigma(t - |x|)^{1/2} f(x) &\leq C \sum_{j=0}^{2n} \sum_{|\alpha| = 0}^{2n} \|\sigma(t - |\cdot|)^{1/2} \nabla^\alpha \Omega f(\cdot)\|_{L^2}, \\
\sigma(|x|)^{1/2} \sigma(t - |x|)^{1/2} f(x) &\leq C \sum_{j=0}^{2n} \left[ \|\Omega f\|_{L^2} + \sum_{|\alpha| = 1}^{2n} \|\sigma(t - |\cdot|) \nabla^\alpha \Omega f(\cdot)\|_{L^2} \right].
\end{align*}

**Proof.** By performing two integrations, we immediately have

$$|f(x)|^2 \leq \int \big| \partial_t \partial_2 \left[ f^2(y) \right] \big| \, dy.$$

From this follows (3.1), for $|x| \leq 1$. Apply (3.5) to $\sigma(t - |x|) f(x)$. Since the derivatives of $\sigma$ are bounded, we obtain

$$\sigma(t - |x|)^2 |f(x)|^2 \leq C \int \left[ |f|^2 + \sigma(t - |y|) |\nabla(f^2)| + \sigma(t - |y|)^2 \left( |\nabla^2(f^2)| \right) \right] \, dy \leq C \int \sigma(t - |y|)^2 \left[ |f|^2 + |\nabla f|^2 + |\nabla^2 f|^2 \right] \, dy.$$

This proves (3.2) in the case $|x| \leq 1$. An application of (3.5) to $\sigma(t - |x|)^{1/2} f(x)$ gives

$$\sigma(t - |x|) |f(x)|^2 \leq C \int \left[ |f|^2 + \sigma(t - |y|) \left( |\nabla f|^2 + |f||\nabla^2 f| \right) \right] \, dy,$$

leading to (3.3) and (3.4) when $|x| \leq 1$. 

On the other hand, if \( x = r\omega = (r \cos \theta, r \sin \theta) \), then

\[
2\pi |f(r\omega)| = \left| \int_\theta^{\theta+2\pi} \frac{d}{d\phi} \left[ (2\pi + \theta - \phi)f(\cos \phi, r \sin \phi) \right] d\phi \right|
\leq \int_0^{2\pi} \left[ 2\pi |\Omega f(\cos \phi, r \sin \phi)| + |f(\cos \phi, r \sin \phi)| \right] d\phi,
\]

and

\[
r|f(r\omega)|^2 = r \int_r^\infty \frac{d}{d\lambda} \left[ f(\lambda \omega)^2 \right] d\lambda \leq 2 \int_r^\infty |f(\lambda \omega)| |\nabla f(\lambda \omega)| \lambda d\lambda.
\]

Insertion of (3.6) into (3.7) completes the proof of (3.1).

If \( \sigma(t - r)|f(r\omega)| \) is substituted into (3.7), then we get

\[
2\pi |f(r\omega)| = \left| \int_\theta^{\theta+2\pi} \frac{d}{d\phi} \left[ (2\pi + \theta - \phi)f(\cos \phi, r \sin \phi) \right] d\phi \right|
\leq \int_0^{2\pi} \left[ 2\pi |\Omega f(\cos \phi, r \sin \phi)| + |f(\cos \phi, r \sin \phi)| \right] d\phi.
\]

Combine (3.8) with (3.6) to finish the proof of (3.2).

Last, go back to (3.7) once more and apply it now to \( \sigma(t - r)^{1/2}f(r\omega) \) to get

\[
r|\sigma(t - r)^{1/2}f(r\omega)|^2 \leq C \int_r^\infty \left[ \sigma(t - r)^2 + \sigma(t - r)^2 \right] f(\lambda \omega)^2 |\nabla f(\lambda \omega)| \lambda d\lambda.
\]

(3.3) and (3.4) follow now from (9) and (3.6). This completes the proof of the lemma.

The weights introduced in these inequalities are helpful in understanding the pointwise behavior of the solution to the Euler equation. \( \sigma(r) \) captures the localized character of incompressible flow, and \( \sigma(t - r) \) measures the decay of the irrotational compressible flow away from the light cone. A basic property that we will use often is

\[
\frac{\sigma(t)}{\sigma(r)} + \frac{\sigma(r)}{\sigma(t)} \leq C \sigma(t - r).
\]

This can be seen by considering the behavior of the weights in the three regions \( \sigma(t) \leq \sigma(r)/2, \sigma(r)/2 \leq \sigma(t) \leq 2\sigma(r), \) and \( 2\sigma(r) \leq \sigma(t) \).

**4. Commutation.** The vector fields defined in Section 1 do not commute. Indeed, we have

\[
[\partial, S] = \partial \quad \text{and} \quad \nabla, \Omega = \nabla^\perp.
\]
All other combinations commute. Thus in general, $\Gamma^a \Gamma^b \neq \Gamma^{a+b}$, and there will be additional derivatives of order less than $|a + b|$ on the right. We will use several times the observation that

$$|\Gamma^a \nabla \rho(x)| \leq |\nabla \Gamma^a \rho(x)| + \sum_{|b| < |a|} |\nabla \Gamma^b \rho(x)|$$

and

$$|\tilde{\Gamma}^a \nabla u(x)| \leq |\nabla \tilde{\Gamma}^a u(x)| + \sum_{|b| < |a|} |\nabla \tilde{\Gamma}^b u(x)|.$$

In performing energy estimates, it will be necessary to differentiate the equations with respect to the $\Gamma^a$ and $\tilde{\Gamma}^a$. The Leibnitz rule for derivatives of products still holds, and although the vector fields do not commute, the structure of the equations is such that when we differentiate with respect to a $\Gamma$ or a $\tilde{\Gamma}$ and commute, the lower order terms always drop out. A simple induction argument, which we will not reproduce here, shows that if $(\rho, u)$ is a solution of (2.1), (2.2), then

$$\partial_t \Gamma^a \rho + \nabla \cdot \tilde{\Gamma}^a u = - \sum_{b+c=a} \frac{a!}{b!c!} \left[ \tilde{\Gamma}^b u \cdot \Gamma^c \rho + \frac{\gamma-1}{2} \Gamma^b \rho \nabla \cdot \tilde{\Gamma}^c u \right] \equiv h_0^a,$$

$$\partial_t \tilde{\Gamma}^a u + \nabla \Gamma^a \rho = - \sum_{b+c=a} \frac{a!}{b!c!} \left[ \tilde{\Gamma}^b u \cdot \nabla \tilde{\Gamma}^c u + \frac{\gamma-1}{2} \Gamma^b \rho \nabla \Gamma^c \rho \right] \equiv h^a,$$

exactly as though (all of) the $\Gamma^a$ were usual partial derivatives.

A similar statement holds for the incompressible equations. Suppose that $(p, \nu)$ is a solution of (2.5), (2.6). Then

$$\nabla \cdot \tilde{\Gamma}^a \nu = 0,$$

$$\partial_t \tilde{\Gamma}^a \nu + \sum_{b+c=a} \frac{a!}{b!c!} \tilde{\Gamma}^b \nu \cdot \nabla \tilde{\Gamma}^c \nu + \nabla \Gamma^a p = 0.$$

We also record a pair of identities which will play an important part in the rotationally symmetric case. First, since $\rho \nabla \rho$ is a gradient, we have that $\rho \nabla \rho = P_1(\rho \nabla \rho)$. If this is successively differentiated then the lower order terms always cancel, and for any multi-index $a$ we get

$$\sum_{b+c=a} \frac{a!}{b!c!} \Gamma^b \rho \nabla \Gamma^c \rho = P_1 \left[ \sum_{b+c=a} \frac{a!}{b!c!} \Gamma^b \rho \nabla \Gamma^c \rho \right].$$

Now suppose that $u$ and $v$ are vectors such that

$$u \cdot \nabla v = P_j(u \cdot \nabla v),$$
for $j = 1$ or $j = 2$, then for every multi-index $a$

\begin{equation}
\sum_{b+c=a} \frac{a!}{b!c!} \tilde{\Gamma}^b u \cdot \nabla \tilde{\Gamma}^c v = P_j \left[ \sum_{b+c=a} \frac{a!}{b!c!} \tilde{\Gamma}^b u \cdot \nabla \tilde{\Gamma}^c v \right].
\end{equation}

Again, this can be proved inductively.

5. Decay. We now come to some pointwise inequalities which depend upon the form of the linear portion of the equations (2.1), (2.2). They are the foundation for the decay estimates.

**Lemma 2.** Let $(h_0, h) \in X^0(T)$, and let $(\rho, u) \in X^1(T)$ be a solution of the equations of linear acoustics:

\begin{align*}
\partial_t \rho + \nabla \cdot u &= h_0, \\
\partial_t u + \nabla \rho &= h.
\end{align*}

Set

\[ J(t, x) = \sum_{|\beta| \leq 1} \left[ |\tilde{\Gamma}^\beta \rho(t, x)| + |\tilde{\Gamma}^\beta P_1 u(t, x)| \right]. \]

Then for all $0 \leq t \leq T$ and $x \in \mathbb{R}^2$ we have

\begin{align*}
\sigma(t-r) |\nabla \rho(t,x)| &\leq CJ(t,x) + \frac{t^r}{t+r} |h_0(t,x)| + \frac{r^2}{t+r} |P_1 h(t,x)|, \\
\sigma(t-r) |\partial_t P_1 u(t,x)| &\leq CJ(t,x) + \frac{t^r}{t+r} |h_0(t,x)| + \frac{r^2}{t+r} |P_1 h(t,x)|, \\
\sigma(t-r) |\nabla \cdot u(t,x)| &\leq CJ(t,x) + \frac{t^r}{t+r} |h_0(t,x)| + \frac{t^r}{t+r} |P_1 h(t,x)|, \\
\sigma(t-r) |\partial_t \rho(t,x)| &\leq CJ(t,x) + \frac{r^2}{t+r} |h_0(t,x)| + \frac{tr}{t+r} |P_1 h(t,x)|.
\end{align*}

**Proof.** First observe that from the decomposition (1.1)

\[ r \partial_t u = x(\nabla \cdot u) + x^\perp (\nabla^\perp \cdot u) + \Omega u^\perp. \]

Apply this to $P_1 u$ to get

\begin{equation}
\sigma(t-r) |\partial_t P_1 u| = x(\nabla \cdot u) + \Omega (P_1 u)^\perp.
\end{equation}
From the equations (5.1) and (5.2), we have

\[ x \partial_t \rho + x \nabla \cdot u = \rho_0 \]
\[ t \partial_t P_1 u + r \nabla \rho = t P_1 h. \]

Add these and use (5.7) to obtain

\[ SP_1 u - \Omega(P_1 u)^\perp + L \rho = \rho_0 + t P_1 h, \tag{5.8} \]

in which

\[ L \rho \equiv x \partial_t \rho + r \nabla \rho. \]

Using (1.1), we find

\[ L \rho = \frac{x}{t} (S \rho - r \partial_t \rho) + r \nabla \rho \\
= \frac{x}{t} S \rho - \frac{r^2}{t} (\nabla \rho - \frac{1}{r} \Omega \rho) + r \nabla \rho. \]

If this formula is substituted into (5.8), upon rearrangement there results

\[ (t - r) \nabla \rho = \frac{x}{t + r} S \rho - \frac{x}{t + r} \Omega \rho - \frac{t}{t + r} SP_1 u + \frac{t}{t + r} \Omega P_1 u \\
+ \frac{t}{t + r} \rho_0 + \frac{r^2}{t + r} P_1 h. \tag{5.9} \]

Inequality (5.3) is an immediate consequence of this identity. Moreover, using (5.2) we have that

\[ (t - r) \partial_t P_1 u = -(t - r) \nabla \rho + (t - r) P_1 h, \]

and so (5.4) also comes from (5.9).

Going back to the equations (5.1) and (5.2) again, we have

\[ t \partial_t \rho + r \nabla \cdot u = \rho_0 \]
\[ x \cdot \partial_t P_1 u + r \partial_r \rho = x \cdot P_1 h. \]

Addition of these gives

\[ S \rho + \frac{x}{t} SP_1 u - \frac{r^2}{t} \frac{x}{r} \partial_r P_1 u + r \nabla \cdot u = \rho_0 + x \cdot P_1 h. \]
Hence, by (5.7)

\[
(t - r)\nabla \cdot u = \frac{t}{t + r}S\rho - \frac{x}{t + r} \cdot SP_1 u - \frac{r}{t + r} \cdot (\Omega P_1 u)_1
\]
\[
+ \frac{t^2}{t + r}b_0 + \frac{tx}{t + r} \cdot P_1 h.
\]

We now obtain (5.5) from (5.10), and if we use (5.1) to write

\[
(t - r)\partial_t \rho = -(t - r)\nabla \cdot u - (t - r)b_0,
\]
we also have (5.6).

Combining the formulas (4.2), (4.3) with this lemma gives:

**Lemma 3.** Let \((\rho, u) \in X^m(T)\) be a solution of (2.1), (2.2). For \(|a| \leq m - 1\), set

\[
J^a(t, x) = \sum_{|\alpha| \leq 1} \left[ |{\Gamma^b}_T a(t, x)| + |{\rceil \rceil}_a P_1 u(t, x)| \right].
\]

Then for all \(0 \leq t \leq T\) and \(x \in \mathbb{R}^2\) and all \(|a| \leq m - 1\)

\[
\sigma(t - r)|\nabla \Gamma^a \rho(t, x)| \leq C J^a(t, x) + \frac{tr}{t + r}|b_0^a(t, x)| + \frac{t^2}{t + r}|P_1 h^a(t, x)|
\]
\[
\sigma(t - r)|\partial_t T a \nabla u(t, x)| \leq C J^a(t, x) + \frac{tr}{t + r}|b_0^a(t, x)| + \frac{t^2}{t + r}|P_1 h^a(t, x)|
\]
\[
\sigma(t - r)|\nabla T a \nabla u(t, x)| \leq C J^a(t, x) + \frac{tr}{t + r}|b_0^a(t, x)| + \frac{t^2}{t + r}|P_1 h^a(t, x)|
\]
\[
\sigma(t - r)|\partial_t T a \nabla u(t, x)| \leq C J^a(t, x) + \frac{tr}{t + r}|b_0^a(t, x)| + \frac{t^2}{t + r}|P_1 h^a(t, x)|,
\]
where \(b_0^a\) and \(h^a\) are defined in (4.2), (4.3).

**6. Irrotational flow: Theorem 2(i).** In this, and all of the other existence proofs, we begin by assuming the existence of a local solution in the space \(X^m(T)\) for some \(T > 0\). See the remarks in the last section for such a construction. Thus, we shall devote our attention only to a priori estimates to extend the smooth local solution up to the life span in the statement of the theorem.

In the proofs, \(C\) will denote various constants which may change from line to line, but \(C\) will always be independent of the small parameter \(\varepsilon\).

We begin by observing that irrotational initial velocities are propagated. This can be seen by taking the curl of equation (2.2). The (two-dimensional) identity

\[
\nabla \perp \cdot (u \cdot \nabla u) = (u \cdot \nabla)(\nabla \perp \cdot u) + (\nabla \cdot u)(\nabla \perp \cdot u)
\]
implies that $\omega = \nabla^{1/2} \cdot u$ solves
\[
\partial_t \omega + u \cdot \nabla \omega + (\nabla \cdot u) \omega = 0.
\]

If $P_1 u_0^c = u_0^c$ then $\omega(0) = 0$, and so $\omega(t) = 0$, as long a smooth solution exists. It follows that $u(t) = P_1 u(t)$.

Now (6.1) can be rephrased as $P_2 (P_1 u \cdot \nabla P_1 u) = 0$, or equivalently, $P_1 u \cdot \nabla P_1 u = P_1 (P_1 u \cdot \nabla P_1 u)$. Thus if $u = P_1 u$, then
\[
(6.2) \quad u \cdot \nabla u = P_1 (u \cdot \nabla u).
\]

If we combine (6.2) with (4.6) and (4.7), we see that
\[
(6.3) \quad h^a = P_1 h^a.
\]

Now let $|a| \leq m - 1$. By (6.3), (5.11) and (5.13),
\[
\sigma(t - r) \left[ |\nabla \Gamma^a \rho| + |\nabla \cdot \Gamma^a u| \right] \leq C f^a + t \left[ |h_0^a| + |h^a| \right].
\]

If we take the $L^2$ norm of both sides and sum over $|a| \leq m - 1$, we obtain the bound
\[
(6.4) \quad \sum_{|a| \leq m - 1} \left[ \left| \sigma(t - r) \nabla \Gamma^a \rho \right|_{L^2} + \left| \sigma(t - r) \nabla \cdot \Gamma^a u \right|_{L^2} \right] \\
\quad \leq E_m(t)^{1/2} + \sum_{|a| \leq m - 1} t \left[ |h_0^a|_{L^2} + |h^a|_{L^2} \right],
\]
in which $E_m(t) \equiv \sum_{|a| \leq m} \left[ |\nabla \rho(t)|_{L^2}^2 + |\nabla^e u(t)|_{L^2}^2 \right]$.

Using (3.10) and then (4.2), we have
\[
|h_0^a|_{L^2} \leq \frac{C}{(1 + t)^{1/2}} \left| \sigma(r)^{1/2} \left( \sigma(t - r)^{1/2} h_0^a \right) \right|_{L^2} \\
\quad \leq \frac{C}{(1 + t)^{1/2}} \sum_{b + c = a} \left[ \left| \sigma(r)^{1/2} \left( \sigma(t - r)^{1/2} \Gamma^b \cdot \nabla \Gamma^c \rho \right) \right|_{L^2} + \left| \sigma(r)^{1/2} \left( \sigma(t - r)^{1/2} \Gamma^b \rho \nabla \cdot \Gamma^c u \right) \right|_{L^2} \right].
\]

Now thanks to the assumption that $m \geq 4$ (actually $m \geq 3$ would be enough here), we always have that at least $|b| + 2 \leq m$ or $|c| + 3 \leq m$ in the above sum.
We split up the sum accordingly:

\[
\|h^\alpha_0\|_{L^2} \leq \frac{C}{(1 + t)^{1/2}} \sum_{|\alpha| = m} \left[ \|\sigma(r)^{1/2}\tilde{\Gamma}^\alpha u\|_{L^\infty} \|\sigma(t - r) \nabla \Gamma^\alpha \rho\|_{L^2} \\
+ \|\sigma(r)^{1/2} \Gamma^\alpha \rho\|_{L^\infty} \|\sigma(t - r) \nabla \Gamma^\alpha u\|_{L^2} \right]
+ \frac{C}{(1 + t)^{1/2}} \sum_{|\alpha| \leq m} \left[ \|\tilde{\Gamma}^\alpha u\|_{L^2} \|\sigma(r)^{1/2} \sigma(t - r) \nabla \Gamma^\alpha \rho\|_{L^\infty} \\
+ \|\Gamma^\alpha \rho\|_{L^2} \|\sigma(r)^{1/2} \sigma(t - r) \nabla \Gamma^\alpha u\|_{L^\infty} \right].
\]

With the aid of (3.1) and (3.2), we get

\[
(6.5) \quad \|h^\alpha_0\|_{L^2} \leq \frac{C}{(1 + t)^{1/2}} E_m(t)^{1/2} Q(t),
\]

with

\[
Q(t) \equiv \sum_{|\alpha| \leq m-1} \left[ \|\sigma(t - r) \nabla \Gamma^\alpha \rho\|_{L^2} + \|\sigma(t - r) \nabla \Gamma^\alpha u\|_{L^2} \right].
\]

A similar argument also gives

\[
(6.6) \quad \|h^\alpha\|_{L^2} \leq \frac{C}{(1 + t)^{1/2}} E_m(t)^{1/2} Q(t).
\]

We are aiming for a bootstrap argument, but first it is necessary to pass from \(\nabla \cdot \Gamma^\alpha u\) to \(\nabla \Gamma^\alpha u\) on the left-hand side of (6.4). This can be done using integration by parts. Of course, we must use the fact that \(u\) is irrotational:

\[
\int \sigma(t - r)^2 (\nabla \cdot u)^2 \, dx = \int \sigma(t - r)^2 \partial_i u_i \partial_j u_j \, dx
= \int \sigma(t - r)^2 \partial_i u_i \partial_j u_j \, dx
- 2 \int \sigma(t - r) \partial_i [\sigma(t - r)u_i] \partial_j u_j \, dx
+ 2 \int \sigma(t - r) \partial_j [\sigma(t - r)u_i] \partial_i u_j \, dx.
\]

Note that \(\partial_i u_j = \partial_j u_i\), since \(\nabla \cdot u = 0\). The derivatives of \(\sigma(t - r)\) are uniformly bounded. Thus, the right-hand side of the last equation is bounded below by

\[
\int \sigma(t - r)^2 |\nabla u|^2 \, dx - C \int \sigma(t - r)|u||\nabla u| \, dx,
\]
from which follows the inequality

\[(6.7) \int \sigma(t-r)^2 |\nabla u|^2 \, dx \leq C \left( \int \sigma(t-r)^2 (\nabla \cdot u)^2 \, dx + \int |u|^2 \, dx \right). \]

Combining (6.7) and (6.4) with (6.5) and (6.6) gives the bound

\[Q(t) \leq C_1 \left[ E_m(t)^{1/2} + t^{1/2} E_m(t)^{1/2} Q(t) \right]. \]

By assumption, we have that \(E_m(0)^{1/2} \leq C_0 \varepsilon\). Define

\[T_1 = \sup \left\{ T > 0 : E_m(t)^{1/2} \leq 2C_0 \varepsilon, \ 0 < t < T \right\}, \]

put \(\delta_1 = 1/4C_0 C_1\), and set \(T_2 = \min(T_1, \delta_1/\varepsilon^2)\). Then for all \(t \leq T_2\),

\[Q(t) \leq C_1 \left[ E_m(t)^{1/2} + (\delta_1/\varepsilon)(2C_0 \varepsilon)Q(t) \right] \leq C_1 E_m(t)^{1/2} + Q(t)/2. \]

That is, we have shown

\[(6.8) \quad Q(t) \leq 2C_1 E_m(t)^{1/2}, \quad \text{for} \quad 0 \leq t \leq T_2. \]

This ends the first part of the proof.

Now we are ready to do energy estimates. For \(|a| \leq m\), multiply (4.2), (4.3) by \(\Gamma^a \rho, \tilde{\Gamma}^u\), respectively, and integrate with respect to \(x\). This results in

\[(6.9) \quad \frac{1}{2} \frac{d}{dt} E_m(t) = \sum_{|a| \leq m} \left[ \langle h_0^a, \Gamma^a \rho \rangle + \langle h^a, \tilde{\Gamma}^u \rangle \right] \]

\[= - \sum_{|a| \leq m} \sum_{b+c=d} \frac{a!}{b!c!} \left[ \langle \tilde{\Gamma}^b u \cdot \Gamma^c \rho + \frac{\gamma-1}{\gamma-1} \Gamma^b \rho \nabla \cdot \tilde{\Gamma}^c u, \Gamma^a \rho \rangle \right. \]

\[\left. + \langle \tilde{\Gamma}^b u \cdot \nabla \Gamma^c \rho + \frac{\gamma-1}{\gamma-1} \Gamma^b \rho \nabla \Gamma^c \rho, \tilde{\Gamma}^a u \rangle \right] \]

with \(\langle \cdot, \cdot \rangle\) the inner product in \(L^2\).

Consider the terms with derivatives of order \(m + 1\):

\[- \sum_{|a| = m} \left[ \langle u \cdot \Gamma^a \rho + \frac{\gamma-1}{\gamma-1} \rho \nabla \cdot \tilde{\Gamma}^a u, \Gamma^a \rho \rangle + \langle u \cdot \nabla \tilde{\Gamma}^a u + \frac{\gamma-1}{\gamma-1} \rho \nabla \Gamma^a \rho, \tilde{\Gamma}^a u \rangle \right]. \]

Because of the symmetry of the system, these can be integrated by parts to give

\[\sum_{|a| = m} \left[ \frac{1}{2} \int \nabla \cdot u |\Gamma^a \rho|^2 + \tilde{\Gamma}^a u |\Gamma^a \rho|^2 \right] dx + \frac{\gamma-1}{\gamma-1} \int \nabla \rho \cdot \tilde{\Gamma}^a u \Gamma^a \rho \, dx, \]
which has the upper bound

\[ C \left[ \| \nabla \cdot u \|_{L^\infty} + \| \nabla \rho \|_{L^\infty} \right] E_m(t). \]

With the aid of (3.10) and (3.2), this, in turn, is dominated by the expression

\[ \frac{C}{(1 + t)^{1/2}} Q(t) E_m(t). \]

All of the other terms in the sum (6.9) are bounded by a like quantity, following the same method as in the earlier estimation of \( Q(t) \), except now we need \( m \geq 4 \) to perform the interpolation.

We have therefore derived the inequality

\[ \frac{d}{dt} E_m(t) \leq \frac{C}{(1 + t)^{1/2}} Q(t) E_m(t), \]

and as a consequence

\[ E_m(t) \leq E_m(0) \exp \left[ C_2 t^{1/2} \sup_{0 \leq s \leq t} Q(s) \right]. \]

Let \( \delta_2 = \ln 2/4C_0C_1C_2 \), and define \( T_3 = \min(T_2, \delta_2^2/\varepsilon^2) \). Then for \( t \leq T_3 \), we have by (6.8)

\[
E_m(t) \leq C_0^2 \varepsilon^2 \exp \left\{ C_2 (\delta_2/\varepsilon) \left[ 2C_1 \sup_{0 \leq s \leq t} E_m(s) \right] \right\}
\leq C_0^2 \varepsilon^2 \exp 4C_0C_1C_2 \delta_2
= 2C_0^2 \varepsilon^2.
\]

This says that \( T_1 \geq T_3 \), which implies that \( E_m(t)^{1/2} \leq 2C_0 \varepsilon \) for \( 0 \leq t \leq A^2/\varepsilon^2 \), with \( A = \min(\delta_1, \delta_2) \). Hence the life span is at least \( A^2/\varepsilon^2 \).

**Corollary 1.** The solution constructed in Theorem 2 has the bound

\[
\left\| \sigma(t - r) \nabla \Gamma^a \rho(t) \right\|_{L^2} + \left\| \sigma(t - r) \nabla \Gamma^a u(t) \right\|_{L^2} \leq CE_m(t)^{1/2} \leq 2C \varepsilon,
\]

for \( 0 \leq t \leq A^2/\varepsilon^2 \).

**7. General case: Theorem 1.** Fix an arbitrary \( T > 0 \) and an integer \( m \geq 4 \). Suppose that \((\rho_0^k, u_0^k)\) satisfies (2.4) with \( k = m + 3 \).

Let \((\rho_1, u_1)\) be the solution of (2.1), (2.2), (2.3) with irrotational initial data \((\rho_0^k, P_1 u_0^k)\). By the result of Theorem 2(i) and its corollary, there is an \( A > 0 \) such
that for all $0 \leq t \leq A^2/\varepsilon^2$

$$\sum_{|\alpha| \leq m+2} \left[ \|\sigma(t-r)\nabla \Gamma^b \rho_t(t)\|_{L^2} + \|\sigma(t-r)\nabla \tilde{\Gamma}^b u_t(t)\|_{L^2} \right] \leq C\varepsilon.$$ 

By (3.10) and (3.4), we then have for all $|\alpha| \leq m + 1$,

$$(7.1) \quad \|\Gamma^\alpha \rho_t\|_{L^\infty} + \|\tilde{\Gamma}^\alpha u_t\|_{L^\infty}$$

$$\leq \frac{C}{(1 + t)^{1/2}} \left[ \|\sigma(r)^{1/2} \sigma(t-r)^{1/2} \Gamma^\alpha \rho_t\|_{L^\infty} + \|\sigma(r)^{1/2} \sigma(t-r)^{1/2} \tilde{\Gamma}^\alpha u_t\|_{L^\infty} \right]$$

$$\leq \frac{C}{(1 + t)^{1/2}} \left[ E_{m+3}(\rho_t(t), u_t(t)) + \sum_{|\alpha| \leq m+2} \left( \|\sigma(t-r)\nabla \Gamma^b \rho_t\|_{L^2} + \|\sigma(t-r)\nabla \tilde{\Gamma}^b u_t\|_{L^2} \right) \right]$$

$$\leq \frac{C\varepsilon}{(1 + t)^{1/2}}.$$  

And for $|\alpha| \leq m + 1$ (trivially),

$$(7.2) \quad \|\Gamma^\alpha \rho_t\|_{L^2} + \|\tilde{\Gamma}^\alpha u_t\|_{L^2} \leq C\varepsilon.$$  

Choose $\varepsilon_0$ small enough so that $A^2/\varepsilon_0 \geq T$. Then for all $0 < \varepsilon \leq \varepsilon_0$, the life span of the solution $(\rho_t, u_t)$ exceeds $T/\varepsilon$.

Also, let $(\tilde{p}, \tilde{v})$ be the global solution of (2.5), (2.6), (2.7) with initial data $(\tilde{v}(0,x) = \frac{1}{\varepsilon} P_2 u_0^\varepsilon(x)$), given by Proposition 1. Then since the data is uniformly bounded in $\tilde{H}^{m+3}_\Lambda$ there is a constant $\tilde{C}$ depending on $T$ such that

$$(7.3) \quad \|\tilde{p}, \tilde{v}\|_{X^{m+3}(T)} \leq \tilde{C}.$$  

Define $(p(t,x), v(t,x)) = (\varepsilon^2 \tilde{p}(\varepsilon t,x), \varepsilon \tilde{v}(\varepsilon t,x))$. Then $(p,v) \in X^{m+3}(T/\varepsilon)$ solves (2.5), (2.6), (2.7) with initial data $v(0,x) = P_2 u_0^\varepsilon(x)$, and the bound (7.3) gives

$$\|\partial^j \gamma v(t)\|_{\tilde{H}^{m+1}_\Lambda} \leq \tilde{C} \varepsilon^{j+1}$$

$$\|\partial^j \gamma p(t)\|_{\tilde{H}^{m+1}_\Lambda} \leq \tilde{C} \varepsilon^{j+2},$$

for $0 \leq t \leq T/\varepsilon$ and $j = 0, \ldots, m + 3$. Consequently, we have, in particular, for all $|\alpha| \leq m + 1$,

$$(7.4) \quad \|\tilde{\Gamma}^\alpha v\|_{L^2} + \|\tilde{\Gamma}^\alpha v\|_{L^\infty} \leq C\varepsilon$$

$$(7.5) \quad \|\Gamma^\alpha p\|_{L^2} + \|\Gamma^\alpha p\|_{L^\infty} \leq C\varepsilon^2.$$
\[ \| \partial \Gamma^a p \|_{L^2} \leq C \varepsilon^3. \]

We point out that the simplest way to get the estimates requires control derivatives of order \( m + 1 \) of the first order terms in \( L^\infty \), which explains the loss of three derivatives in the data.

We seek a solution in the form

\[ \rho = \rho_1 + p + \rho_2 \quad \text{and} \quad u = u_1 + v + u_2. \]

The perturbation will be \( O(\varepsilon^\mu) \) with \( 1 < \mu < 3/2 \). In order that \((\rho, u)\) solve (2.1), (2.2), (2.3), the corrections \((\rho_2, u_2)\) should satisfy

\[ \begin{align*}
\partial_t \rho_2 + \nabla \cdot u_2 &= -[u \cdot \nabla \rho - u_1 \cdot \nabla \rho_1] \\
&\quad - \frac{\gamma - 1}{2} [\rho \nabla \cdot (u_1 + u_2) - \rho_1 \nabla \cdot u_1] - \partial_t p \\
\partial_t u_2 + \nabla \rho_2 &= -[u \cdot \nabla u - u_1 \cdot \nabla u_1 - v \cdot \nabla v] \\
&\quad - \frac{\gamma - 1}{2} [\rho \nabla \rho - \rho_1 \nabla \rho_1] \\
\rho_2(0, x) = -p(0, x), \quad u_2(0, x) = 0.
\end{align*} \]

Note that by (7.5), \( \| \rho_2(0) \|_{L^\infty} = \| p(0) \|_{L^\infty} \leq \hat{C} \varepsilon^2 \).

We are going to construct a solution \((\rho_2, u_2)\) of (7.7), (7.8), (7.9) in the space \( X^m(T/\varepsilon) \). The proof requires only energy estimates, as sufficient decay has already been captured in \((\rho_1, u_1)\).

Now let \( |a| \leq m \) be any multi-index. If \((\rho, u)\) is a solution, then (4.2) and (4.3) hold. Likewise, these equations are verified by \((\rho_1, u_1)\). We also have that \((p, v)\) satisfies (4.4) and (4.5). Subtraction of the first order equations gives the following differentiated versions of (7.7) and (7.8)

\[ \begin{align*}
\partial_t \Gamma^a \rho_2 + \nabla \cdot \tilde{\Gamma}^a u_2 &= - \sum_{b+a=a} \frac{a!}{b! c!} \left\{ \left[ \tilde{\Gamma}^b u \cdot \nabla \Gamma^c \rho - \tilde{\Gamma}^b u_1 \cdot \nabla \Gamma^c \rho_1 \right] \\
&\quad + \frac{\gamma - 1}{2} \left[ \Gamma^b \rho \nabla \cdot \Gamma^c u - \Gamma^b \rho_1 \nabla \cdot \Gamma^c u_1 \right] \right\} + \partial_t \Gamma^a p \\
\partial_t \tilde{\Gamma}^a u_2 + \nabla \Gamma^a \rho_2 &= - \sum_{b+a=a} \frac{a!}{b! c!} \left\{ \left[ \tilde{\Gamma}^b u \cdot \nabla \tilde{\Gamma}^c u_2 - \tilde{\Gamma}^b u_1 \cdot \nabla \tilde{\Gamma}^c u_1 - \tilde{\Gamma}^b v \cdot \nabla \tilde{\Gamma}^c v \right] \\
&\quad + \frac{\gamma - 1}{2} \left[ \Gamma^b \rho \nabla \Gamma^c \rho - \Gamma^b \rho_1 \nabla \Gamma^c \rho_1 \right] \right\}.
\end{align*} \]

Take the \( L^2 \) inner product with \((\Gamma^a \rho_2, \tilde{\Gamma}^a u_2)\) and sum over \( |a| \leq m \). Then, with

\[ E_m(t) = \sum_{|a| \leq m} \left[ \| \Gamma^a \rho_2 \|_{L^2}^2 + \| \Gamma^a u_2 \|_{L^2}^2 \right], \]
we have

\[
\frac{1}{2} \frac{d}{dt} E_m(t) = - \sum_{b+c=m} \sum_{|c|=a} \frac{a!}{b!c!} \left[ \langle \tilde{r}^b v \cdot \nabla \Gamma^c \rho - \tilde{r}^b u_1 \cdot \nabla \Gamma^c \rho_1, \Gamma^a \rho_2 \rangle 
+ \frac{a_1}{2} \langle \tilde{r}^b \rho \nabla \cdot \tilde{r}^c u - \Gamma^b \rho_1 \nabla \cdot \tilde{r}^c u_1, \Gamma^a \rho_2 \rangle 
+ \frac{a_2}{2} \langle \tilde{r}^b \rho \nabla \Gamma^c \rho - \Gamma^b \rho_1 \nabla \Gamma^c \rho_1, \Gamma^a \rho_2 \rangle 
+ \langle \tilde{r}^b \nabla \tilde{r}^c u - \tilde{r}^b u_1 \cdot \nabla \tilde{r}^c u_1 - \tilde{r}^b v \cdot \nabla \tilde{r}^c v, \Gamma^a \rho_2 \rangle \right] 
+ \sum_{|c|_m} \langle \partial_t \Gamma^a p, \Gamma^a \rho_2 \rangle.
\]

\( \equiv S_1 + \cdots + S_5. \)

We will repeatedly make use of the observation that since \( m \geq 4 \), whenever \( b+c = a \) and \( |a| \leq m \), then either \( |b| + 2 \leq m \) or \( |c| + 3 \leq m \).

Consider the sum \( S_1 \). Inserting the terms of the expansion, separating off the terms with derivatives of order \( m+1 \) in \( \rho_2 \) and integrating them by parts, we find

\[
|S_1| \leq \sum_{b+c=a, \ |c| \leq m} \left[ \langle \tilde{r}^b (v + u_2) \cdot \nabla \Gamma^c \rho_1, \Gamma^a \rho_2 \rangle \right] 
+ \langle \tilde{r}^b (u_1 + v + u_2) \cdot \nabla \Gamma^c p, \Gamma^a \rho_2 \rangle 
+ \langle \tilde{r}^b (u_1 + v + u_2) \cdot \nabla \Gamma^c \rho_2, \Gamma^a \rho_2 \rangle
\]

\[
\leq C \sum_{b+c=a, \ |a| \leq m} \left( \| \tilde{r}^b (v + u_2) \|_{L^2} \| \nabla \Gamma^c \rho_1 \|_{L^\infty} 
+ \| \tilde{r}^b (u_1 + v + u_2) \|_{L^2} \| \nabla \Gamma^c p \|_{L^\infty} \right) \| \Gamma^a \rho_2 \|_{L^2}
\]

\[
+ C \sum_{b+c=a, \ |a| \leq m} \left( \| \tilde{r}^b (u_1 + v) \|_{L^\infty} \| \nabla \Gamma^c \rho_2 \|_{L^2} \| \Gamma^a \rho_2 \|_{L^2} \right)
\]

\[
+ C \sum_{b+c=a, \ |a| \leq m} \left( \| \tilde{r}^b u_2 \|_{L^\infty} \| \nabla \Gamma^c \rho_2 \|_{L^2} \| \Gamma^a \rho_2 \|_{L^2} \right)
\]

\[
+ C \sum_{b+c=a, \ |a| \leq m} \left( \| \tilde{r}^b u_2 \|_{L^2} \| \nabla \Gamma^c \rho_2 \|_{L^\infty} \| \Gamma^a \rho_2 \|_{L^2} \right)
\]

\[
+ \frac{1}{2} \sum_{|c| \leq m} \langle \nabla \cdot (u_1 + v + u_2) \Gamma^a \rho_2, \Gamma^a \rho_2 \rangle.
\]
Now use (7.1), (7.2), (7.4), and (7.5) to get

\[ |S_1| \leq C \left[ \frac{\varepsilon^2}{(1 + t)^{1/2}} + \varepsilon^3 + \varepsilon E_m(t)^{1/2} + E_m(t) \right] E_m(t)^{1/2}. \]

The sums \( S_2 \) and \( S_3 \) must be grouped together in order to integrate by parts the top derivatives. Recalling that \( \nabla \cdot v = 0 \), and using the same interpolation as above, we see that

\[
|S_2 + S_3| \leq C \sum_{|b| = a \atop |l| \leq m} \left[ |\langle T^b \rho_2 \nabla \cdot \tilde{\Gamma}^{\alpha} u_1, \Gamma^a \rho_2 \rangle| + |\langle T^b \rho_2 \nabla \cdot \tilde{\Gamma}^{\alpha} u_1, \Gamma^a \rho_2 \rangle| + |\langle T^b \rho_2 \nabla \cdot \tilde{\Gamma}^{\alpha} u_1, \Gamma^a \rho_2 \rangle| + |\langle T^b \rho_2 \nabla \cdot \tilde{\Gamma}^{\alpha} u_1, \Gamma^a \rho_2 \rangle| \right] + \frac{2^{-1}}{2} \sum_{|a| = m} |\langle \nabla \rho_1 + \rho_2, \Gamma^a \rho_2, \tilde{\Gamma}^{\alpha} u_2 \rangle|.
\]

Using (7.2), (7.4), and (7.5) yields

\[ |S_2 + S_3| \leq C \left[ \varepsilon^3 + \varepsilon E_m(t)^{1/2} + E_m(t) \right] E_m(t)^{1/2}. \]

The reader may have noticed that we have not used the full strength of the available inequalities in the preceding. However, the convective term cannot be sharpened in this generality. We have

\[
|S_4| \leq C \sum_{|b| = a \atop |l| \leq m} \left[ |\langle \tilde{\Gamma}^{\alpha} u_1 \cdot \nabla \tilde{\Gamma}^{\alpha} v + \tilde{\Gamma}^{\alpha} v \cdot \nabla \tilde{\Gamma}^{\alpha} u_1, \tilde{\Gamma}^{\alpha} u_2 \rangle| + |\langle \tilde{\Gamma}^{\alpha} u_2 \cdot \nabla \tilde{\Gamma}^{\alpha} (u_1 + v), \tilde{\Gamma}^{\alpha} u_2 \rangle| \right]
\]
\[ + C \sum_{b \leq m} \left| (\tilde{T}^b(u_1 + v + u_2) \cdot \nabla \tilde{\Gamma}^c u_2, \tilde{\Gamma}^d u_2) \right| \\
\leq C \left[ \frac{\varepsilon^2}{(1 + t)^{1/2}} + \varepsilon E_m(t)^{1/2} + E_m(t) \right] E_m(t)^{1/2}, \]

as above.

Finally, (7.6) yields

\[ |S_5| \leq |\partial_t \Gamma^a p|_{L^2} |\Gamma^a p_2|_{L^2} \leq C \varepsilon^3 E_m(t)^{1/2}. \]

We arrive at the bound,

\[ \frac{d}{dt} E_m(t)^{1/2} \leq C_1 \left[ \frac{\varepsilon^2}{(1 + t)^{1/2}} + \varepsilon^3 + \varepsilon E_m(t)^{1/2} + E_m(t) \right], \]

and hence,

\[ E_m(t)^{1/2} \leq \left[ E_m(0)^{1/2} + C_1 \int_0^t \left( \frac{\varepsilon^2}{(1 + s)^{1/2}} + \varepsilon^3 \right) ds \right] \times \exp C_1 \int_0^t (\varepsilon + E_m(s)^{1/2}) ds \]

\[ \leq \left[ E_m(0)^{1/2} + 2 C_1 \varepsilon^2 t^{1/2} + C_1 \varepsilon^3 t \right] \times \exp \sup_{0 \leq s \leq t} E_m(s)^{1/2}. \]

Recall that \( E_m(0)^{1/2} \leq C_0 \varepsilon^2 \). Choose \( 1 < \mu < 3/2 \), and let

\[ T_1 = \sup \left\{ \tau > 0 : E_m(t)^{1/2} < 2 C_0 \varepsilon^\mu, \ 0 \leq t < \tau \right\}. \]

Then for \( t \leq T_2 \equiv \min(T_1, T/\varepsilon) \) and \( 0 < \varepsilon < \varepsilon_0 \), we get from (7.12) that

\[ E_m(t)^{1/2} \leq 2 C_0 \varepsilon^\mu \left\{ \left[ \frac{2 - \mu}{2} C_0 + C_1 T^{1/2} \varepsilon_0^{3/2 - \mu}/C_0 + C_1 T \varepsilon_0^{2 - \mu}/2 C_0 \right] \times \exp C_1 \left( T + 2 C_0 T \varepsilon_0^{\mu-1} \right) \right\}. \]

Simply restrict \( \varepsilon_0 \) further, if necessary, in order that the expression in the brackets is smaller than one. Then, for \( T < T_2 \) and \( 0 < \varepsilon < \varepsilon_0 \), we will have \( E_m(t)^{1/2} \leq \)
2C_0 \varepsilon^h. This shows that \( T/\varepsilon \leq T_1 \), i.e. the corrections \((\rho_2, u_2)\) exist and remain \( O(\varepsilon^h) \) up to time \( T/\varepsilon \).

8. Almost irrotational flow: Theorem 2(ii). We now turn to the proof of part (ii) of Theorem 2. Having gone through the proof of the general case, perturbing from irrotational flow is easy.

Fix \( m \geq 4 \). Suppose that the data \((\rho_0, u_0)\) satisfies (2.4) with \( k = m + 3 \) and that \( \|P_2 u_0\|_{H^m_{\lambda}} \leq C \varepsilon^2 \). Again let \((\rho_1, u_1)\) \( \in X^{m+3}(A^2/\varepsilon^2) \) be the solution of (2.1), (2.2), (2.3) with irrotational initial data \((\rho_0, P_1 u_0)\). Recall the estimates: (7.1) and (7.2). The method is again perturbative. The solution will be constructed as a perturbation of order \( \varepsilon^2 \) from \((\rho_1, u_1)\). Put

\[ \rho = \rho_1 + \rho_2 \quad \text{and} \quad u = u_1 + u_2. \]

The relevant initial value problem for the corrections is

\[
\begin{align*}
\partial_t \rho_2 + \nabla \cdot u_2 &= -[u \cdot \nabla \rho - u_1 \cdot \nabla \rho_1] \\
&\quad - \frac{2}{\varepsilon^2 - 1} [\rho \nabla \cdot u - \rho_1 \nabla \cdot u_1] \\
\partial_t u_2 + \nabla \rho_2 &= -[u \cdot \nabla u - u_1 \cdot \nabla u_1] \\
&\quad - \frac{2}{\varepsilon^2 - 1} [\rho \nabla \rho - \rho_1 \nabla \rho_1] \\
\rho_2(0, x) &= 0, \quad u_2(0, x) = P_2 u_0.
\end{align*}
\]

By assumption, we have \( \|P_2 u_0\|_{H^m_{\lambda}} \leq C_0 \varepsilon^2 \).

Proceed with the energy estimates as in the proof of Theorem 1. Here there will be no inhomogeneous terms since all first order interactions have been subtracted off. The interaction between a first and second order term will produce a term that can be estimated by

\[ \frac{CE}{(1 + t)^{1/2}} E_m(t), \]

the decay factor coming from (7.1). The nonlinear terms give rise in the energy estimates to terms which can be bounded by

\[ C E_m(t)^{3/2}. \]

The result is an inequality of the type

\[ \frac{d}{dt} E_m(t) \leq C \left[ \frac{\varepsilon}{(1 + t)^{1/2}} + E_m(t)^{1/2} \right] E_m(t), \]
for the energy

\[ E_m(t) = \sum_{|\alpha| \leq m} \left[ \| \tilde{\Gamma}^\alpha \rho_2 \|_{L^2}^2 + \| \tilde{\Gamma}^\alpha u_2 \|_{L^2}^2 \right]. \]

We have \( E_m(0)^{1/2} = O(\varepsilon^2) \), so it follows immediately that \( E_m(t)^{1/2} \) remains of order \( \varepsilon^2 \) up to a time of order \( 1/\varepsilon^2 \).

9. The incompressible limit: Theorem 4. In this section only, we restore the superscript \( \varepsilon \).

Let \( m \geq 4 \). Given data \((\rho_0, u_0)\) satisfying (2.4) for \( k = m + 3 \), let \((p, v)\) solve the incompressible Euler equation (2.5), (2.6), (2.7) with data \( v(0) = \frac{1}{\varepsilon} P_2 u_0 \). Define \((p^\varepsilon(t, x), v^\varepsilon(t, x)) = (\varepsilon^2 p(\varepsilon t, x), \varepsilon v(\varepsilon t, x))\). Then \((p^\varepsilon, v^\varepsilon)\) solves (2.5), (2.6) with data \( P_2 u_0 \).

Fix \( T > 0 \), and choose \( \varepsilon_0 \) small enough so that the compressible Euler equation (2.1), (2.2) with data \((\rho_0(x), u_0(x))\) has a solution \((p^\varepsilon, u^\varepsilon) \in X^m(T/\varepsilon)\), by Theorem 1. The estimates of Theorem 1 show that

\[ \| P_2 u^\varepsilon - v^\varepsilon \|_{X^m(T/\varepsilon)} = \| u_0^\varepsilon \|_{X^m(T/\varepsilon)} \leq 2C_0 \varepsilon^{1+\mu}, \]

with \( 0 < \mu < 1/2 \). By (7.1) and (9.1), we have for any \(|a| \leq m - 2\),

\[ \| \tilde{\Gamma}^a (u^\varepsilon - v^\varepsilon)(t, x) \| \leq C \left[ \| \tilde{\Gamma}^a u_1^\varepsilon(t) \|_{L^\infty} + \| \tilde{\Gamma}^a u_2^\varepsilon \|_{X^m(T/\varepsilon)} \right] \]

\[ \leq C \left[ \frac{\varepsilon}{(1 + t)^{1/2}} + \varepsilon^{1+\mu} \right]. \]

Notice however that

\[(\rho, u)(t, x) = \left( p^\varepsilon(t/\varepsilon, x), \frac{1}{\varepsilon} u^\varepsilon(t/\varepsilon, x) \right)\]

is the solution of (2.8), (2.9), (2.10) on \([0, T]\). Moreover, for any time-homogeneous derivative \( \tilde{\Gamma}^a \), \( j + k \leq m \), we have

\[ \left( \tilde{\Gamma}^a u \right)(t, x) = \frac{1}{\varepsilon} \left( \tilde{\Gamma}^a u^\varepsilon \right)(t/\varepsilon, x) \quad \text{and} \quad \left( \tilde{\Gamma}^a v \right)(t, x) = \frac{1}{\varepsilon} \left( \tilde{\Gamma}^a v^\varepsilon \right)(t/\varepsilon, x). \]

So for \( 0 \leq t \leq T \) and any time-homogeneous multi-index \(|a| \leq m\)

\[ \| \tilde{\Gamma}^a (P_2 u - v)(t) \|_{L^2} = \frac{1}{\varepsilon} \| \tilde{\Gamma}^a (P_2 u^\varepsilon - v^\varepsilon)(t/\varepsilon) \|_{L^2} \leq C_0 \varepsilon^{\mu}, \]

\( C_0 \) is a constant.
by (9.1). Likewise, if $0 < \tau \leq t \leq T$ and $|a| \leq m - 2$, then by (9.2)

$$\left| \tilde{F}^a(u - v)(t, x) \right| = \frac{1}{\varepsilon} \left| \tilde{F}^a(u^\varepsilon - v^\varepsilon)(t/\varepsilon, x) \right| \leq C \frac{1}{\varepsilon} \left[ \frac{\varepsilon}{(1 + t/\varepsilon)^{1/2}} + \varepsilon^{1+\mu} \right] \leq C(\tau) \varepsilon^{1+\mu}. $$

Since $p^\varepsilon - p^\varepsilon = \rho^\varepsilon + \rho^\varepsilon$, estimate (9.2) also holds for $\rho - \rho$, and so the last chain of inequalities is also true for $\frac{1}{\varepsilon} \Gamma^a(\rho - \rho)$.

\[\square\]

10. The rotationally symmetric case: Theorem 3. In this section the fluid velocity vector will now be denoted by $w$ instead of $u$. Let $(\rho_0^\varepsilon, w_0^\varepsilon) \in H^{m+3}_\Lambda \times \tilde{H}^{m+3}_\Lambda$, $m \geq 4$, be given rotationally symmetric data, i.e. $\Omega \rho_0^\varepsilon = 0$ and $\tilde{\Omega} w_0^\varepsilon = 0$, such that

$$\|\rho_0^\varepsilon\|_{H^{m+3}_\Lambda} + \|w_0^\varepsilon\|_{\tilde{H}^{m+3}_\Lambda} \leq C_0 \varepsilon. $$

The symmetry possessed by the data is propagated by the flow. This can be readily seen by taking $\Gamma^a = \Omega$ and $\tilde{F}^a = \tilde{\Omega}$ in (4.2) and (4.3) which results in a pair of equations that are linear in $(\Omega \rho, \tilde{\Omega} w)$. Thus, if these derivatives are zero initially, they remain so for positive times, by uniqueness.

Recalling the commutation property (4.1), if $(\rho, w)$ is rotationally symmetric, then for any spatial derivative $\nabla^a$

$$\Omega \nabla^a \rho = \nabla^a \Omega \rho + \left( \nabla^a \right) \rho = \left( \nabla^a \right) \rho, $$

$$\tilde{\Omega} \nabla^a w = \nabla^a \tilde{\Omega} w + \left( \nabla^a \right) w = \left( \nabla^a \right) w. $$

So any derivative containing an angular derivative can be expressed in terms of spatial derivatives of lower order with no angular derivative. Therefore, in estimating the norm in $X^m$ it is unnecessary to include the derivatives $\Omega$ and $\tilde{\Omega}$, and in this section, the norm in $X^m$ is based only on $\partial$ and $S$.

The proof will rely on the same expansion as in Theorem 1. The main difference here is that we will need to establish some decay in the second order corrections in order to sharpen the estimates for the convective terms involving products of the first order (stationary) incompressible flow and the second order corrections. But first, we need to collect a few special facts about rotationally symmetric vectors which will allow us to proceed.

Rotationally symmetric. Suppose that $w \in \tilde{H}^m_\Lambda$ and $\tilde{\Omega} w = 0$, that is, $\Omega w = w_\perp$. Since

$$(10.1) \quad \Omega \left( \frac{x}{r} \right) = \frac{x_\perp}{r} \quad \text{and} \quad \Omega \left( \frac{x_\perp}{r} \right) = -\frac{x}{r}, $$

\[\square\]
it follows that
\[
\Omega \left( \frac{x}{r} \cdot w \right) = \frac{x}{r} \cdot \tilde{\Omega}w = 0 \quad \text{and} \quad \Omega \left( \frac{x \perp}{r} \cdot w \right) = \frac{x \perp}{r} \cdot \tilde{\Omega}w = 0,
\]
and so there are spherically symmetric functions \( f(r) \) and \( g(r) \) such that \( f(r) = \frac{x}{r} \cdot w \) and \( g(r) = \frac{x \perp}{r} \cdot w \). In other words, we have the representation
\[
w(x) = f(r) \frac{x}{r} + g(r) \frac{x \perp}{r}.
\]
A simple calculation using this and (1.1) shows that
\[
\nabla \cdot w = \partial_r f + \frac{1}{r} f \quad \text{and} \quad \nabla \perp \cdot w = \partial_r g + \frac{1}{r} g,
\]
which implies that
\[
(10.2) \quad P_1 w = f(r) \frac{x}{r} \quad \text{and} \quad P_2 w = g(r) \frac{x \perp}{r}.
\]
Thus, the projections are local and they coincide with the radial and angular components. This fact will make it possible to analyze the irrotational and divergence free components of the convective derivative in the following lemma.

**Lemma 4.** Let \( u_j, v_j \in \tilde{H}^1_\Lambda (\mathbb{R}^2) \) be rotationally symmetric, \( j = 1, 2 \). Suppose that \( u_j(x) = f_j(r) \frac{x}{r} \) is irrotational and that \( v_j(x) = g_j(r) \frac{x \perp}{r} \) is divergence free. Then

\[
(10.3) \quad u_1 \cdot \nabla u_2 = P_1(u_1 \cdot \nabla u_2) = (f_1 \partial_r f_2) \frac{x}{r},
\]

\[
(10.4) \quad u_1 \cdot \nabla v_1 = P_2(u_1 \cdot \nabla v_1) = (f_1 \partial_r g_1) \frac{x \perp}{r},
\]

\[
(10.5) \quad v_1 \cdot \nabla u_1 = P_2(v_1 \cdot \nabla u_1) = \left( \frac{1}{r} f_1 g_1 \right) \frac{x \perp}{r},
\]

\[
(10.6) \quad v_1 \cdot \nabla v_2 = P_1(v_1 \cdot \nabla v_2) = - \left( \frac{1}{r} g_1 g_2 \right) \frac{x}{r}.
\]

**Proof:** Note that by (1.1), \( u_1 \cdot \nabla = f_1 \partial_r \) and \( v_1 \cdot \nabla = \frac{1}{r} g_1 \Omega \). The result follows from (10.1) and (10.2). \( \square \)

Rotational symmetry also makes the following localized inequalities possible.

**Lemma 5.** Let \( u \in \tilde{H}^1_\Lambda (\mathbb{R}^2) \) be rotationally symmetric and irrotational. Then

\[
(10.7) \quad \sigma (|x|)^{-1/2} |u(x)| \leq C(1 + t)^{-1} \| \sigma (|t|) \nabla \cdot u \|_{L^2}
\]
\[(10.8) \quad \|\sigma (|x|)^{-1}u(x)\|_{L^2} \leq C(1 + t)^{-1} \|\sigma (t - |\cdot|) \nabla u\|_{L^2}.\]

**Proof.** Write \(u(x) = f(r)\frac{1}{r} \quad \text{and} \quad x = r\omega.\) Then, we get

\[
r|u(x)| = r|f(r)|
\leq \int_0^r |\lambda f'(\lambda) + f(\lambda)|d\lambda
= \int_0^r |\nabla u(\lambda\omega)|\lambda d\lambda
\leq \left( \int_0^r \sigma(t - \lambda)^2|\nabla u(\lambda\omega)|^2\lambda d\lambda \right)^{1/2} \left( \int_0^r \sigma(t - \lambda)^{-2}\lambda d\lambda \right)^{1/2}
\leq C \|\sigma (t - |\cdot|) \nabla u\|_{L^2} I(t, r)^{1/2}.
\]

If \(r < t/2,\) then \(I(t, r)^{1/2} \leq Cr(1 + t)^{-1}.\) If \(r > t/2,\) then

\[
I(t, r)^{1/2} \leq r^{1/2} \left( \int_0^r \sigma(t - \lambda)^{-2}d\lambda \right)^{1/2}
\leq Cr^{1/2}
\leq C r^{3/2} t^{-1}.
\]

Thus, (10.7) follows from these inequalities.

For (10.8), simply note that \(|u| = |\Omega u| \leq r|\nabla u|\). Thus, by (3.10) we find

\[
\|\sigma (|x|)^{-2} u(x)\|_{L^2} \leq \|\sigma(x)^{-1}\nabla u\|_{L^2}
\leq C(1 + t)^{-1} \|\sigma (t - |\cdot|) \nabla u\|_{L^2}.
\]

Without rotational symmetry, an inequality of this type holds, but only with a decay factor of \((1 + t)^{-1/2}.\) The assumption that \(u\) is symmetric with derivative in \(L^2\) restricts \(f(r)\) near \(r = 0.\)

**Lemma 6.** Take any rotationally symmetric and divergence free vector

\[v(x) = g(r)\frac{\chi_{\Lambda}}{r} \in H^4_\Lambda (\mathbb{R}^2).\]

Then for \(\alpha \geq 1/2,\)

\[(10.9) \quad \frac{\alpha}{r} \left| g(r) \right| \leq C \left\|\sigma(\cdot)^{\alpha - 1/2} v\right\|_{H^2_\Lambda},\]

\[(10.10) \quad \sigma(r)^\alpha \left| \partial_r g(r) - \frac{1}{r} g(r) \right| + \sigma(r)^\alpha \left| \nabla \left( \partial_r g(r) - \frac{1}{r} g(r) \right) \right| \leq \left\|\sigma(\cdot)^{\alpha + 1/2} v\right\|_{H^4_\Lambda}.\]
Proof. By symmetry, we have $\Omega v = v^\perp = -\frac{x}{r}g$. So $g = -\frac{x}{r} \cdot \Omega v$. Since $|\Omega v| \leq \frac{r}{r} |\nabla v|$, we have

$$\frac{1}{r} |g| \leq |\nabla v|.$$  

(10.9) follows from this using (3.1) applied to $\sigma(r)\Omega^{-1/2} \nabla v$.

The Laplace operator in two dimensions has the decomposition $\Delta = \partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \Omega^2$. Since $\Omega^2 v = -v$, we find that

$$\partial_r g - \frac{1}{r} g = x^\perp \cdot (\Delta v - \partial_r^2 v),$$

and also,

$$\nabla \left( \partial_r g - \frac{1}{r} g \right) = \frac{x}{r} \left( \partial_r^2 g - \frac{1}{r} \partial_r g + \frac{1}{r^2} g \right) = \frac{x}{r} \left( 2 \frac{x^\perp}{r} \cdot \partial_r v - \frac{x^\perp}{r} \cdot \Delta v \right).$$

Thus,

$$\left| \partial_r g(r) - \frac{1}{r} g(r) \right| \leq C r |\nabla^2 v|,$$

and

$$\left| \nabla \left( \partial_r g(r) - \frac{1}{r} g(r) \right) \right| \leq C |\nabla^2 v|,$$

from which follows (10.10), using (3.1), applied to $\sigma(r)\Omega^{1/2} \nabla^2 v$. \qed

Of course, $v$ need not be divergence free in order that the lemma be true, but that is the context in which we shall apply it.

First order approximation. Let $(\rho_1, u_1)$ be the irrotational solution of (2.1), (2.2), (2.3) with initial data $(\rho_0, P_1 w_0)$, whose life span is $A^2/\varepsilon^2$. We assemble a list of estimates for this solution that we will use later on. We restate the main estimate of Theorem 2(i) and its corollary, from which everything else that we will need about the solution follows. For all $0 \leq t \leq A^2/\varepsilon^2$

\begin{align*}
(10.11) \quad &||\Gamma^a \rho_1(t)||_{L^2} + ||\Gamma^a u_1(t)||_{L^2} \leq C \varepsilon, \quad |a| \leq m + 3 \\
(10.12) \quad &||\sigma(t-r)\nabla \Gamma^a \rho_1(t)||_{L^2} + ||\sigma(t-r)\nabla \Gamma^a u_1(t)||_{L^2} \leq C \varepsilon, \quad |a| \leq m + 2.
\end{align*}

Next, from (3.1) and (10.11), we have

\begin{align*}
(10.13) \quad &\sigma(r)^{1/2} \left[ ||\Gamma^a \rho_1(t,x)|| + ||\Gamma^a u_1(t,x)|| \right] \leq C \varepsilon, \quad |a| \leq m + 1,
\end{align*}
We have shown therefore that
\[
(10.14) \quad \sigma(r)^{1/2} |r - s|^{1/2} \left[ |\Gamma^a \rho_1(t,x)| + |\Gamma^a u_1(t,x)| \right] \leq C \varepsilon, \quad |a| \leq m + 1,
\]
\[
(10.15) \quad \sigma(r)^{1/2} |r - s| \left[ |\nabla \Gamma^a \rho_1(t,x)| + |\nabla \Gamma^a u_1(t,x)| \right] \leq C \varepsilon, \quad |a| \leq m.
\]

By (3.10) and (10.14),
\[
(10.16) \quad |\Gamma^a \rho_1(t,x)| + |\Gamma^a u_1(t,x)| \leq \frac{C \varepsilon}{(1 + t)^{1/2}}, \quad |a| \leq m + 1.
\]

We also have the following localized inequality. Let \(|a| \leq m + 1\). If \(\Gamma^a\) has no spatial derivatives, i.e. \(\Gamma^a = \partial_j \delta^k\) with \(j + k \leq m + 1\), then \(\Gamma^a u_1\) is rotationally symmetric. So by (10.7) and (10.12), we get
\[
\frac{1 + t}{\sigma(r)^{1/2}} |\Gamma^a u_1(t,x)| \leq C \|\sigma - |\cdot|\nabla \cdot \Gamma^a u_1\|_{L^2} \leq C \varepsilon.
\]

If \(\Gamma^a\) contains spatial derivatives, then write \(\Gamma^a = \nabla_j \Gamma^b\) with \(|b| \leq m\). Using (3.10) and (10.15), we find
\[
\frac{1 + t}{\sigma(r)^{1/2}} |\Gamma^a u_1(t,x)| \leq \frac{1 + t}{\sigma(r)^{1/2}} |\nabla \Gamma^b u_1(t,x)|
\leq C \sigma(r)^{1/2} |r - s| |\nabla \Gamma^b u_1(t,x)|
\leq C \varepsilon.
\]

We have shown therefore that
\[
(10.17) \quad \frac{1}{\sigma(r)^{1/2}} |\Gamma^a u_1(t,x)| \leq \frac{C \varepsilon}{(1 + t)}, \quad |a| \leq m + 1.
\]

Now we turn to the first order incompressible flow. Set \(v_1 = P_2 w_0\). Then \(v_1\) is rotationally symmetric and divergence free. Thus we can write \(v_1(x) = g_1(r) \frac{x}{r}\) for some spherically symmetric function \(g_1(r)\). By (10.6),
\[
P_2(v_1 \cdot \nabla v_1) = 0,
\]
and
\[
P_1(v_1 \cdot \nabla v_1) = -\frac{1}{r} \left[ g(r)^2 \right] \frac{x}{r} = \nabla \left( \int_r^\infty \frac{1}{\lambda^2} g(\lambda)^2 d\lambda \right) = -\nabla p(r).
\]

With \(p(r)\) thus defined, we have shown that \((p(r), v_1(x))\) is a stationary solution of the 2d incompressible Euler equation (2.5), (2.6), (2.7) with data \(v_1\). The
assumption \( \| \sigma(r)^2 v_1 \|_{H^m_\Lambda} \leq C \varepsilon \), immediately implies that

\[
(10.18) \quad \| \sigma(r)^2 \Gamma^a v_1 \|_{L^2} \leq C \varepsilon, \quad |a| \leq m + 3
\]

and by Sobolev that,

\[
(10.19) \quad \| \sigma(r)^2 \Gamma^a v_1 \|_{L^\infty} \leq C \varepsilon, \quad |a| \leq m + 1.
\]

Now by (2.5) and (2.6), \( p = \sum_{i,j} R_i R_j (v_i' v_j') \). The Riesz transformations are bounded in \( L^2 \) and commute with \( \Gamma^a \) so by (10.18) and (10.19)

\[
(10.20) \quad \| \Gamma^a p \|_{L^2} \leq C \sum_{i,j} \| \Gamma^a (v_i' v_j') \|_{L^2} \leq C \varepsilon^2, \quad |a| \leq m + 3.
\]

So with (3.1), this implies that

\[
(10.21) \quad \| \sigma(r)^{1/2} \Gamma^a p \|_{L^\infty} \leq C \varepsilon^2, \quad |a| \leq m + 1.
\]

Moreover, from (4.5) and (10.19)

\[
(10.22) \quad \| \sigma(r) \nabla \Gamma^a p \|_{L^\infty} \leq C \sum_{b+c=a} \| \sigma(r) \Gamma^b v_1 \cdot \nabla \Gamma^c v_1 \|_{L^\infty} \leq C \varepsilon^2, \quad |a| \leq m.
\]

Finally, combining (10.9), (10.10) with (10.18), we have

\[
(10.23) \quad \sigma(r) \frac{1}{r} |S^j g_1(r)| \leq C \varepsilon, \quad j \leq m
\]

\[
(10.24) \quad \sigma(r)^{3/2} \left| \partial_r g_1(r) \right| + 4 \sigma(r)^{3/2} \left| \frac{1}{r} g_1(r) \right| \leq C \varepsilon.
\]

**Expansion.** We will construct a solution \((\rho, w)\) of (2.1), (2.2), (2.3) in the form

\[
(10.25) \quad \rho = \rho_1 + p + \rho_2
\]

\[
(10.26) \quad w = w_1 + w_2,
\]

in which the velocity is further decomposed into its irrotational and divergence free components,

\[
(10.27) \quad w_j = P_1 w_j + P_2 w_j = u_j + v_j, \quad j = 1, 2.
\]
The corresponding equations for the corrections \((\rho_2, w_2)\) are

\[
\begin{align*}
\partial_t \rho_2 + \nabla \cdot u_2 &= - [w \cdot \nabla \rho - u_1 \cdot \nabla \rho_1] \\
&\quad - \frac{2\nu}{r} [\rho \nabla w - \rho_1 \nabla \cdot u_1] \\
&\equiv h_0
\end{align*}
\]

\[
\begin{align*}
\partial_t w_2 + \nabla \rho_2 &= - [w \cdot \nabla w - u_1 \cdot \nabla u_1 - v_1 \cdot \nabla v_1] \\
&\quad - \frac{2\nu}{r} [\rho \nabla \rho - \rho_1 \nabla \rho_1] \\
&\equiv h
\end{align*}
\]

\[
\rho_2(0,x) = P(0,x), \quad w_2(0,x) = 0.
\]

There are two differences with the general case in (7.7). Since \(\rho\) is time-independent, \(\partial \rho = 0\), and since \(v_j = P_2 w_j\) is rotationally symmetric and \(\rho\) is radially symmetric, \(v_j \cdot \nabla \rho = \frac{1}{r} \left(\frac{x^1}{r} \cdot v_j\right) \Omega \rho = 0\). Thus,

\[
w \cdot \nabla \rho = (u_1 + u_2) \cdot \nabla \rho.
\]

As before, since \(v_j\) is divergence free

\[
\nabla \cdot w = \nabla \cdot (u_1 + u_2).
\]

By substituting (10.27), we have the following unpleasant but explicit decomposition

\[
\begin{align*}
h &= \{u_1 \cdot \nabla v_1 + v_1 \cdot \nabla u_1 \\
&\quad + u_1 \cdot \nabla u_2 + u_2 \cdot \nabla u_1 \\
&\quad + v_1 \cdot \nabla v_2 + v_2 \cdot \nabla v_1 \\
&\quad + u_1 \cdot \nabla v_2 + v_2 \cdot \nabla u_1 \\
&\quad + v_1 \cdot \nabla u_2 + u_2 \cdot \nabla v_1 \\
&\quad + u_2 \cdot \nabla u_2 + v_2 \cdot \nabla v_2 \\
&\quad + u_2 \cdot \nabla v_2 + v_2 \cdot \nabla u_2 \\
&\quad + \frac{2\nu}{r} [\rho \nabla \rho - \rho_1 \nabla \rho_1]\}
\end{align*}
\]

\[
\equiv D_1 + \ldots + D_8.
\]

Using (10.3)–(10.6), we see that

\[
D_j = P_1 D_j, \ j = 2, 3, 6, 8 \quad \text{and} \quad D_j = P_2 D_j, \ j = 1, 4, 5, 7.
\]
Therefore,

\[(10.30)\quad P_1 h = D_2 + D_3 + D_6 + D_8,\]
\[(10.31)\quad P_2 h = D_1 + D_4 + D_5 + D_7.\]

Take a multi-index \(|a| \leq m\). Then subtraction of the differentiated equations (4.2), (4.3) for \((\rho_1, u_1)\) and (4.4), (4.5) for \((\rho, v_1)\) from those of \((\rho, w)\) yields

\[(10.32)\quad \partial_a \Gamma^a \rho_2 + \nabla \cdot \Gamma^a u_2 = - \sum_{b+c=a} \frac{a^1}{b!c!} \left\{ \left[ \Gamma^b (u_1 + u_2) \cdot \nabla \Gamma^c \rho - \Gamma^b u_1 \cdot \nabla \Gamma^c \rho_1 \right] \\
- \frac{c}{2} \left[ \Gamma^b \rho \nabla \cdot \Gamma^c (u_1 + u_2) - \Gamma^b \rho_1 \nabla \cdot \Gamma^c u_1 \right] \right\} \equiv h_0^a.
\]

\[(10.33)\quad \partial_a \Gamma^a w_2 + \nabla \cdot \Gamma^a \rho_2 = - \sum_{b+c=a} \frac{a^1}{b!c!} \left\{ \left[ \Gamma^b w \cdot \nabla \Gamma^b \rho - \Gamma^b u_1 \cdot \nabla \Gamma^c u_1 - \Gamma^b v_1 \cdot \nabla \Gamma^c v_1 \right] \\
- \frac{c}{2} \left[ \Gamma^b \rho \nabla \Gamma^c \rho - \Gamma^b \rho_1 \nabla \Gamma^c \rho_1 \right] \right\} \equiv h^a.
\]

Substitution of the expansion (10.25), (10.26), (10.27) leads to

\[(10.34)\quad h_0^a = - \sum_{b+c=a} \frac{a^1}{b!c!} \left[ \Gamma^b u_2 \cdot \nabla \Gamma^c \rho_1 \\
+ \Gamma^b (u_1 + u_2) \cdot \nabla p \\
+ \Gamma^b (u_1 + u_2) \cdot \nabla \rho_2 \\
+ \frac{c}{2} \Gamma^b (p + \rho_2) \nabla \cdot \Gamma^c u_1 \\
+ \frac{c}{2} \Gamma^b (\rho_1 + p + \rho_2) \nabla \cdot \Gamma^c u_2 \right] \equiv S_1^0 + \ldots + S_5^0\]

and

\[(10.35)\quad h^a = - \sum_{b+c=a} \frac{a^1}{b!c!} \left\{ \Gamma^b u_1 \cdot \nabla \Gamma^c v_1 + \Gamma^b v_1 \cdot \nabla \Gamma^c u_1 \\
+ \Gamma^b u_1 \cdot \nabla \Gamma^c u_2 + \Gamma^b u_2 \cdot \nabla \Gamma^c u_1 \\
+ \Gamma^b v_1 \cdot \nabla \Gamma^c v_2 + \Gamma^b v_2 \cdot \nabla \Gamma^c v_1 \\
+ \Gamma^b u_1 \cdot \nabla \Gamma^c v_2 + \Gamma^b v_2 \cdot \nabla \Gamma^c u_1 \\
+ \Gamma^b v_1 \cdot \nabla \Gamma^c u_2 + \Gamma^b u_2 \cdot \nabla \Gamma^c v_1 \right\}\]
Thus, by (4.7), (4.6), (10.30), and (10.35), we have from (35) that

\[
\begin{align*}
404 & \equiv S_6^0 + \ldots + S_{15}^0.
\end{align*}
\]

Thus, by (4.7), (4.6), (10.30), and (10.35), we have from (35) that

\[
\begin{align*}
(10.36) & \quad P_1 h^a = S_7^0 + S_8^0 + S_{11}^0 + S_{13}^0 + S_{14}^0 + S_{15}^0 \\
(10.37) & \quad P_2 h^a = S_6^0 + S_9^0 + S_{10}^0 + S_{12}^0.
\end{align*}
\]

**Decay estimates.** Put

\[
J^a(t,x) = \sum_{|\rho| \leq 1} \left[ |\Gamma^b \Gamma^a \rho_2(t,x)| + |\Gamma^b \Gamma^a u_2(t,x)| \right].
\]

Then by (5.11), (5.12), (5.13) it follows that

\[
\begin{align*}
(10.38) & \quad \sigma(t-r)^{1/2} |\nabla \Gamma^a \rho_2(t,x)| \\
& \leq C J^a(t,x) + \frac{tr}{t+r} |h_0^a(t,x)| + \frac{r^2}{(t+r)\sigma(t-r)^{1/2}} |P_1 h^a(t,x)| \\
& \quad \sigma(t-r)^{1/2} |\partial_t \Gamma^a u_2(t,x)| \\
& \leq C J^a(t,x) + \frac{tr}{t+r} |h_0^a(t,x)| + \frac{r^2}{(t+r)\sigma(t-r)^{1/2}} |P_1 h^a(t,x)| \\
& \quad \sigma(t-r) |\nabla \cdot \Gamma^a u_2(t,x)| \\
& \leq C J^a(t,x) + \frac{r^2}{t+r} |h_0^a(t,x)| + \frac{tr}{t+r} |P_1 h^a(t,x)|.
\end{align*}
\]

As a consequence of the weight property (3.10), it is easily verified that

\[
\begin{align*}
\frac{t^2}{(t+r)\sigma(t-r)^{1/2}} & \leq C(1+t)^{1/2}\sigma(r)^{1/2} \\
\frac{r^2}{(t+r)\sigma(t-r)^{1/2}} & \leq C(1+t)^{1/2}\sigma(r)^{1/2}.
\end{align*}
\]
Thus, it follows from (10.38) that

\[
\sum_{|a| \leq m-1} \left[ \| \sigma(t-r)^{1/2} \nabla \Gamma^a \rho_2 \|_{L^2} \right. \\
+ \left. \| \sigma(t-r)^{1/2} \partial_t \Gamma^a u_2 \|_{L^2} + \| \sigma(t-r) \nabla \cdot \Gamma^a u_2 \|_{L^2} \right]
\]

\begin{equation}
(10.39)
\leq E_m(t)^{1/2} + C \sum_{|a| \leq m-1} \left[ (1 + t)^{1/2} \| \sigma(r)^{1/2} \sigma(t-r)^{1/2} h_0^a \|_{L^2} \right. \\
+ \left. (1 + t)^{1/2} \| \sigma(r)^{1/2} P_1 h_a^a \|_{L^2} \right],
\end{equation}

with \( E_m(t) = \sum_{|a| \leq m} \left[ \| \Gamma^a \rho_2 \|_{L^2}^2 + \| \Gamma^a u_2 \|_{L^2}^2 \right] \).

Set

\begin{equation}
Q(t) = \sum_{|a| \leq m-1} \left[ \| \sigma(t-r)^{1/2} \nabla \Gamma^a \rho_2 \|_{L^2} \right. \\
+ \left. \| \sigma(t-r)^{1/2} \partial_t \Gamma^a u_2 \|_{L^2} + \| \sigma(t-r) \nabla \cdot \Gamma^a u_2 \|_{L^2} \right].
\end{equation}

By (6.7), we can gain control of the full gradient of \( \Gamma^a u_2 \) on the left of (10.39). We therefore obtain the bound

\begin{equation}
Q(t) \leq E_m(t)^{1/2} + C \sum_{|a| \leq m-1} \left[ (1 + t)^{1/2} \| \sigma(r)^{1/2} \sigma(t-r)^{1/2} h_0^a \|_{L^2} \right. \\
+ \left. (1 + t)^{1/2} \| \sigma(r)^{1/2} P_1 h_a^a \|_{L^2} \right].
\end{equation}

As in the proof of Theorem 2(i), we will get a bound for the quantity \( Q(t) \) by a bootstrap argument. Although we actually only need to control \( \| \sigma(t-r) \nabla \Gamma^a u_2 \|_{L^2} \) and \( \| \sigma(t-r)^{1/2} \partial_t \Gamma^a u_2 \|_{L^2} \), it is necessary to also include the other term \( \| \sigma(t-r)^{1/2} \nabla \Gamma^a \rho_2 \|_{L^2} \) in order to complete the bootstrap. The loss of decay in \( \partial_t \Gamma^a u_2 \) and \( \nabla \Gamma^a \rho_2 \) is due to the fact that \( S^b_\delta \) has no time decay.

Now from (10.34)

\[
t^{1/2} \| \sigma(r)^{1/2} \sigma(t-r)^{1/2} h_0^a \|_{L^2} \leq \sum_{j=1}^S \sum_{b+c=a} t^{1/2} \| \sigma(r)^{1/2} \sigma(t-r)^{1/2} S^b_j \|_{L^2},
\]

and we proceed to estimate the individual sums.

By (10.14), we have

\[
\sum_{b+c=a} \| \sigma(r)^{1/2} \sigma(t-r)^{1/2} S^b_j \|_{L^2} \leq \sum_{b+c=a} \| \Gamma^b u_2 \|_{L^2} \| \sigma(r)^{1/2} \sigma(t-r)^{1/2} \nabla \Gamma^c \rho_1 \|_{L^2} \\
\leq C \varepsilon E_m(t)^{1/2}.
\]
Using the crude inequality, \( \sigma(t-r) \leq C(1+t)\sigma(r) \), along with (10.14), (10.20), and (10.22), it follows that

\[
\sum_{b+c=n} ||\sigma(r)^{1/2}\sigma(t-r)^{1/2}S_2^n||_{L^2} \leq C \sum_{b+c=n} \left( ||\sigma(r)^{1/2}\sigma(t-r)^{1/2}\Gamma^b u_1||_{L^\infty} ||\nabla \Gamma^c r_2||_{L^2} + ||\Gamma^b u_2||_{L^2}(1+t)^{1/2}||\sigma(r)\nabla \Gamma^c r_2||_{L^\infty}\right)
\leq C \left[ \varepsilon^3 + t^{1/2}\varepsilon E_m(t)^{1/2} \right].
\]

With the aid of (10.14), (3.1), and (3.3), and remembering (10.40), we obtain

\[
\sum_{b+c=n} ||\sigma(r)^{1/2}\sigma(t-r)^{1/2}S_3^n||_{L^2} \leq C \sum_{b+c=n} ||\sigma(r)^{1/2}\sigma(t-r)^{1/2}\Gamma^b u_1||_{L^\infty} ||\nabla \Gamma^c r_2||_{L^2}
+ \sum_{b+c=n} ||\sigma(r)^{1/2}\Gamma^b u_2||_{L^\infty} ||\sigma(t-r)^{1/2}\nabla \Gamma^c r_2||_{L^\infty}
+ \sum_{b+c=n} ||\Gamma^b u_2||_{L^2} ||\sigma(r)^{1/2}\sigma(t-r)^{1/2}\nabla \Gamma^c r_2||_{L^\infty}
\leq C \left[ \varepsilon + Q(t) E_m(t)^{1/2} \right].
\]

The first order estimates (10.14) and (10.20) give

\[
\sum_{b+c=n} ||\sigma(r)^{1/2}\sigma(t-r)^{1/2}S_2^n||_{L^2} \leq \sum_{b+c=n} \left[ ||\Gamma^b p||_{L^2} + ||\Gamma^b r_2||_{L^2} \right] ||\sigma(r)^{1/2}\sigma(t-r)^{1/2}\nabla \cdot \Gamma^c u_1||_{L^\infty}
\leq C \left[ \varepsilon^3 + E_m(t)^{1/2} \varepsilon \right].
\]

The bounds (10.14), (10.20), and (10.21) with the inequalities (3.1) and (3.3) show that

\[
\sum_{b+c=n} ||\sigma(r)^{1/2}\sigma(t-r)^{1/2}S_3^n||_{L^2} \leq C \sum_{b+c=n} ||\sigma(r)^{1/2}\sigma(t-r)^{1/2}\Gamma^b \rho_1||_{L^\infty} ||\nabla \cdot \Gamma^c u_2||_{L^2}
+ C \sum_{b+c=n} ||\sigma(r)^{1/2}\Gamma^b \rho_2||_{L^\infty} ||\sigma(t-r)^{1/2}\nabla \cdot \Gamma^c u_2||_{L^2}
+ \sum_{b+c=n} ||\Gamma^b \rho_2||_{L^2} ||\sigma(r)^{1/2}\sigma(t-r)^{1/2}\nabla \cdot \Gamma^c u_2||_{L^\infty}
\]
The term return to haunt us when we come to the energy estimates.

Next we consider the terms coming from (10.35), (10.37)

To summarize our efforts thus far, we have established

\[
\sum_{\mathcal{P}} \left[ \varepsilon \rho^2 \sigma \frac{1}{2} \right]_{L^2} \lesssim C \left[ \varepsilon E_m(t)^{1/2} + (\varepsilon^2 + E_m(t)^{1/2} Q(t)) \right].
\]

Next we consider the terms coming from (10.35), (10.37)

\[
\left\| \sigma \right\|_{L^2} \lesssim C \sum_{\mathcal{J}} \sum_{b+c=m} \left\| \sigma \right\|_{L^2}, \quad \mathcal{J} = \{7, 8, 11, 13, 14, 15\}. 
\]

Treating the first two sums simultaneously, we estimate using (10.13) and (10.19)

\[
\sum_{b+c=m} \left\| \sigma \right\|_{L^2} \leq C \sum_{b+c=m} \left\| \sigma \right\|_{L^2} \lesssim C \left[ \varepsilon E_m(t)^{1/2} \right].
\]

The term \( S_8 \) is troublesome, and with its cousin \( S_{10} \) in the \( P_2 \) direction, it will return to haunt us when we come to the energy estimates.

An application of (3.1) is enough to see that

\[
\sum_{b+c=m} \left\| \sigma \right\|_{L^2} \leq C \left[ \varepsilon E_m(t)^{1/2} \right].
\]
The bound (10.20) and inequality (3.1) again yield

\[
\sum_{b+c=a} \| \sigma(r)^{1/2} S_{13}^a \|_{L^2} \leq \sum_{b+c=a} \left[ \| \Gamma^b p \|_{L^2} + \| \Gamma^b \rho_2 \|_{L^2} \right] \| \sigma(r)^{1/2} \nabla \Gamma^c \rho_1 \|_{L^\infty}
\leq C \left[ \varepsilon^2 + E_m(t)^{1/2} \right] \varepsilon.
\]

Employing (10.11), (10.20), and (10.21), we see that

\[
\sum_{b+c=a} \| \sigma(r)^{1/2} S_{14}^a \|_{L^2} \leq \sum_{b+c=a} \| \Gamma^b (\rho_1 + p + \rho_2) \|_{L^2} \| \sigma(r)^{1/2} \nabla \Gamma^c p \|_{L^\infty}
\leq C \left[ \varepsilon^3 + \varepsilon E_m(t)^{1/2} \right].
\]

Using (10.13), (10.21), with (3.1), we conclude that

\[
\sum_{b+c=a} \| \sigma(r)^{1/2} S_{15}^a \|_{L^2} \leq \sum_{b+c=a} \| \sigma(r)^{1/2} \Gamma^b (\rho_1 + p) \|_{L^\infty} \| \nabla \Gamma^c \rho_2 \|_{L^2}
+ \sum_{b+c=a} \| \sigma(r)^{1/2} \Gamma^b \rho_2 \|_{L^\infty} \| \nabla \Gamma^c \rho_2 \|_{L^2}
+ \sum_{b+c=a} \| \Gamma^b \rho_2 \|_{L^2} \| \sigma(r)^{1/2} \nabla \Gamma^c \rho_2 \|_{L^\infty}
\leq C \left[ \varepsilon E_m(t)^{1/2} + E_m(t) \right].
\]

If we collect the last group of inequalities, then we deduce

\[
l^{1/2} \| \sigma(r)^{1/2} P_1 h^p \|_{L^2} \leq C l^{1/2} (\varepsilon^3 + \varepsilon E_m(t)^{1/2} + E_m(t)),
\]

and with the earlier bounds (10.41) and (10.42) we have

\[
Q(t) \leq C l^{1/2} \left[ \varepsilon^3 + \varepsilon E_m(t)^{1/2} + t^{1/2} \varepsilon^2 E_m(t)^{1/2}
+ E_m(t) + (\varepsilon^2 + E_m(t)^{1/2} Q(t)) \right].
\]

We are now ready to complete the bootstrap argument which will bound \( Q(t) \) by \( \varepsilon^2 + E_m(t)^{1/2} \).

From (10.28) and (10.20), we have that

\[
E_m(0)^{1/2} \leq C_0 \varepsilon^2.
\]

Define

\[
T_1 = \sup \left\{ T > 0 : E_m(t)^{1/2} \leq 2 C_0 \varepsilon^{3/2}, 0 \leq t \leq T \right\},
\]
and set
\[ T_2 = \min \left( T_1, \frac{A^2}{\varepsilon^2} \right), \]

where \( \frac{A^2}{\varepsilon^2} \) is the life span of \((\rho_1, u_1)\). Then from (10.43), we have for \(0 \leq t \leq T_2\) and \(0 < \varepsilon \leq \varepsilon_0\)
\[
Q(t) \leq C_1 \left( \frac{A}{\varepsilon} \right) \left[ \varepsilon^3 + (1 + A)\varepsilon E_m(t)^{1/2} + 2C_0\varepsilon^{3/2} E_m(t)^{1/2} + (\varepsilon^2 + 2C_0\varepsilon^{3/2}) Q(t) \right] \\
\leq C \left[ \varepsilon^2 + E_m(t)^{1/2} \right] + O \left[ \varepsilon_0^{1/2} \right] Q(t).
\]

If we take \(\varepsilon_0\) sufficiently small, then it follows that
\[
Q(t) \leq C \left[ \varepsilon^2 + E_m(t)^{1/2} \right], \tag{10.45}
\]
for \(0 \leq t \leq T_2\) and \(0 < \varepsilon \leq \varepsilon_0\).

**Energy estimates.** With the decay estimate (10.45) in hand we now turn to the energy estimates. Going back to (10.32), (10.33) take the inner product with \(\Gamma^n \rho_2, \Gamma^n w_2\), and sum over \(|a| \leq m\). This gives
\[
\frac{1}{2} \frac{d}{dt} E_m(t) = \sum_{|a| \leq m} \left[ \langle h_0^a, \Gamma^n \rho_2 \rangle + \langle h^a, \Gamma^n w_2 \rangle \right].
\]

Referring to (10.34), (10.35), we write
\[
\sum_{|a| \leq m} \langle h_0^a, \Gamma^n \rho_2 \rangle = - \sum_{j=1}^{5} \sum_{|a| \leq m} \langle S_j^a, \Gamma^n \rho_2 \rangle \\
\equiv S_1 + \ldots + S_5,
\]
and
\[
\sum_{|a| \leq m} \langle h^a, \Gamma^n w_2 \rangle = - \sum_{j=6}^{15} \sum_{|a| \leq m} \langle S_j^a, \Gamma^n w_2 \rangle \\
\equiv S_6 + \ldots + S_{15}.
\]

We now go down the line estimating each of these sums, all of which can be estimated in a straight-forward manner with the exception of \(S_8 + S_{10}\), the term containing the interaction of the second order corrections with the first order incompressible flow. In spite of the variety of bounds that we derive for the many sums, they all have weight \(\varepsilon^6\) or better, if we count \(E_m(t)\) as \(O(\varepsilon^4)\), its initial size, and \(t\) as \(\varepsilon^{-2}\), its maximum size.
The first term can be interpolated using only (10.16) to get

\[
|S_1| \leq C \sum_{b+c=a} \|\Gamma^b u_2\|_{L^2} \|\nabla \Gamma^c \rho_1\|_{L^\infty} \|\Gamma^a \rho_2\|_{L^2} \\
\leq \frac{C \varepsilon}{(1 + t)^{1/2}} E_m(t).
\]

The next term can be handled with the help of (10.16), (10.20), (10.21) from which we have

\[
|S_2| \leq C \sum_{b+c=a} \|\Gamma^b u_1\|_{L^\infty} \|\nabla \Gamma^c \rho|_{L^2} \|\Gamma^a \rho_2\|_{L^2} \\
+ C \sum_{b+c=a} \|\Gamma^b u_2\|_{L^2} \|\nabla \Gamma^c \rho|_{L^\infty} \|\Gamma^a \rho_2\|_{L^2} \\
\leq C \left[ \frac{\varepsilon^3}{(1 + t)^{1/2}} E_m(t)^{1/2} + \varepsilon^2 E_m(t) \right].
\]

Inequality (10.16) and integration by parts to eliminate the derivatives of order $m + 1$ on $\rho_2$ yields

\[
|S_3| \leq C \sum_{b+c=a} \|\Gamma^b u_1\|_{L^\infty} \|\nabla \Gamma^c \rho_2\|_{L^2} \|\Gamma^a \rho_2\|_{L^2} \\
+ C \sum_{b+c=a} \|\Gamma^b u_2\|_{L^2} \|\nabla \Gamma^c \rho_2\|_{L^\infty} \|\Gamma^a \rho_2\|_{L^2} \\
+ C \sum_{b+c=a} \|\Gamma^b u_2\|_{L^2} \|\nabla \Gamma^c \rho_2\|_{L^\infty} \|\Gamma^a \rho_2\|_{L^2} \\
+ C \|\nabla \cdot (u_1 + u_2)\|_{L^\infty} \|\nabla \cdot \Gamma^a \rho_2\|_{L^2} \\
\leq C \left[ \frac{\varepsilon}{(1 + t)^{1/2}} E_m(t) + E_m(t)^{3/2} \right].
\]

The next bound follows from (10.16) and (10.21):

\[
|S_4| \leq C \sum_{b+c=a} \|\Gamma^b (\rho + \rho_2)\|_{L^2} \|\nabla \cdot \Gamma^c u_1\|_{L^\infty} \|\Gamma^a \rho_2\|_{L^2} \\
\leq C \left[ \frac{\varepsilon^3}{(1 + t)^{1/2}} E_m(t)^{1/2} + \frac{\varepsilon}{(1 + t)^{1/2}} E_m(t) \right].
\]
Employing (10.16) and (10.21) again and using integration by parts, we obtain

\[ |S_3 + S_{13}| \leq C \sum_{b+c=a} \left| \left( \frac{\Gamma^b (\rho_1 + p) \cdot \nabla^c u_2}{\rho_2} \right) \right|_{L^\infty} \left| \left( \frac{\nabla \cdot \Gamma^c u_2}{\rho_2} \right) \right|_{L^2} + \left| \left( \frac{\nabla^c \rho_2}{\rho_2} \right) \right|_{L^2} + \left| \left( \frac{\nabla^c \rho_2}{\rho_2} \right) \right|_{L^2} \]

\[ + C \sum_{b+c=a} \left| \left( \frac{\Gamma^b \rho_2}{\rho_2} \right) \right|_{L^\infty} \left| \left( \frac{\nabla \cdot \Gamma^c u_2}{\rho_2} \right) \right|_{L^2} + \left| \left( \frac{\nabla^c \rho_2}{\rho_2} \right) \right|_{L^2} + \left| \left( \frac{\nabla^c \rho_2}{\rho_2} \right) \right|_{L^2} \]

\[ + \sum_{b+c=a} \left| \left( \frac{\Gamma^b \rho_2}{\rho_2} \right) \right|_{L^2} \left| \left( \frac{\nabla \cdot \Gamma^c u_2}{\rho_2} \right) \right|_{L^\infty} + \left| \left( \frac{\nabla^c \rho_2}{\rho_2} \right) \right|_{L^\infty} + \left| \left( \frac{\nabla^c \rho_2}{\rho_2} \right) \right|_{L^2} \]

\[ + C \left| \left( \frac{\nabla (\rho_1 + p + \rho_2)}{\rho_2} \right) \right|_{L^\infty} \left| \left( \frac{\Gamma^c u_2}{\rho_2} \right) \right|_{L^2} + \left| \left( \frac{\nabla^c \rho_2}{\rho_2} \right) \right|_{L^2} \]

\[ \leq C \left[ \frac{\varepsilon}{(1 + t)^{3/2}} E_m(t) + \varepsilon^2 E_m(t) + E_m(t)^{3/2} \right]. \]

The localized estimate (10.17) and the weighted inequality (10.18) give

\[ |S_6| \leq C \sum_{b+c=a} \left| \left( \frac{\Gamma^b u_1 \cdot \nabla^c v_1}{\rho_2} \right) \right|_{L^2} + \left| \left( \frac{\Gamma^b v_1 \cdot \nabla^c u_1}{\rho_2} \right) \right|_{L^2} \]

\[ \leq C \sum_{b+c=a} \left| \left( \frac{\sigma(r)^{1/2} \Gamma^b u_1}{\rho_2} \right) \right|_{L^\infty} \left| \left( \frac{\sigma(r)^{1/2} \nabla^c v_1}{\rho_2} \right) \right|_{L^2} \]

\[ + \left| \left( \frac{\sigma(r)^{1/2} \nabla^c v_1}{\rho_2} \right) \right|_{L^\infty} \left| \left( \frac{\sigma(r)^{1/2} \nabla^c u_1}{\rho_2} \right) \right|_{L^\infty} \left| \left( \frac{\Gamma^c w_2}{\rho_2} \right) \right|_{L^2} \]

\[ \leq C \frac{\varepsilon^2}{(1 + t)^{1/2}} E_m(t)^{1/2}. \]

The factor \((1 + t)^{-1}\) causes a logarithm to appear later. This might be avoided, if we had a little better decay for \((\rho_1, u_1)\) inside the light cone.

With the help of (10.16) and integration by parts, we have

\[ |S_7 + S_8| \leq C \sum_{b+c=a} \left| \left( \frac{\Gamma^b u_1 \cdot \nabla^c w_2 + \Gamma^b w_2 \cdot \nabla^c u_1}{\Gamma^a w_2} \right) \right| \]

\[ \leq C \sum_{b+c=a} \left| \left( \frac{\Gamma^b u_1}{\rho_2} \right) \right|_{L^\infty} \left| \left( \frac{\nabla^c w_2}{\rho_2} \right) \right|_{L^2} \left| \left( \frac{\Gamma^a w_2}{\rho_2} \right) \right|_{L^2} \]

\[ + C \sum_{b+c=a} \left| \left( \frac{\Gamma^b w_2}{\rho_2} \right) \right|_{L^2} \left| \left( \frac{\nabla^c u_1}{\rho_2} \right) \right|_{L^\infty} \left| \left( \frac{\Gamma^a w_2}{\rho_2} \right) \right|_{L^2} \]

\[ + C \left| \left( \frac{\nabla \cdot w_1}{\rho_2} \right) \right|_{L^\infty} \left| \left( \frac{\Gamma^a w_2}{\rho_2} \right) \right|_{L^2} \]

\[ + C \left( \frac{1}{(1 + t)^{1/2}} \right) \left| \left( \frac{\sigma(r)^{1/2} w_2}{\rho_2} \right) \right|_{L^\infty} \left| \left( \frac{\sigma(t - r)^{1/2} \nabla^a u_1}{\rho_2} \right) \right|_{L^2} \left| \left( \frac{\Gamma^a w_2}{\rho_2} \right) \right|_{L^2} \]

\[ \leq \frac{C \varepsilon}{(1 + t)^{1/2}} E_m(t). \]
An integration by parts for the top derivatives followed by standard interpolation yields

\[
|S_{11} + S_{12}| \leq C \sum_{b+c=m} |\langle \Gamma^b w_2 \cdot \nabla \Gamma^c w_2, \Gamma^d w_2 \rangle|
\]

\[
\leq C \sum_{\substack{b+c=m \\text{ and} \ \Gamma^b \leq m \ \text{ coprime}}} \|\Gamma^b w_2\|_{L^\infty} \|\nabla \Gamma^c w_2\|_{L^2} \|\Gamma^d w_2\|_{L^2}
\]

\[
+ C \sum_{\substack{b+c=m \\text{ and} \ \Gamma^b \leq m \ \text{ coprime}}} \|\Gamma^b w_2\|_{L^2} \|\nabla \Gamma^c w_2\|_{L^\infty} \|\Gamma^d w_2\|_{L^2}
\]

\[
+ \|\nabla \cdot w_2\|_{L^\infty} \|\Gamma^d w_2\|_{L^2}
\]

\[
\leq C E_m(t)^{3/2}.
\]

Inequalities (10.20), (10.21), and (10.16) lead to

\[
|S_{13}| \leq C \sum_{b+c=m} \|\Gamma^b (\rho_1 + p)\|_{L^\infty} \|\nabla \Gamma^c p\|_{L^2} \|\Gamma^d w_2\|_{L^2}
\]

\[
+ C \frac{C}{(1+t)^{1/2}} \sum_{b+c=m} \|\Gamma^b \rho_2\|_{L^2} \|\nabla \Gamma^c p\|_{L^\infty}
\]

\[
\leq C \left[ \frac{\varepsilon^3}{(1+t)^{1/2}} E_m(t)^{1/2} + \varepsilon^2 E_m(t) \right].
\]

By (10.11), (10.16), (10.20), (10.21), we find

\[
|S_{14}| \leq C \sum_{\substack{b+c=m \\text{ and} \ \Gamma^b \leq m \ \text{ coprime}}} \|\Gamma^b (\rho_1 + p + \rho_2)\|_{L^\infty} \|\nabla \Gamma^c p\|_{L^2} \|\Gamma^d w_2\|_{L^2}
\]

\[
+ C \sum_{\substack{b+c=m \\text{ and} \ \Gamma^b \leq m \ \text{ coprime}}} \|\Gamma^b (\rho_1 + p + \rho_2)\|_{L^2} \|\nabla \Gamma^c p\|_{L^\infty} \|\Gamma^d w_2\|_{L^2}
\]

\[
\leq C \left[ \frac{\varepsilon^3}{(1+t)^{1/2}} E_m(t)^{1/2} + \varepsilon^3 E_m(t)^{1/2} + \varepsilon^2 E_m(t) \right].
\]

This completes the energy estimates for the routine terms. We are left with the more challenging sum

\[
S_8 + S_{10} = \sum_{|c| \leq m} [S_8^c + S_{10}^c],
\]

with

\[
S_8^c + S_{10}^c = \sum_{\substack{b+c=d \\text{ and} \ \Gamma^b \leq m \ \text{ coprime}}} \frac{a^1}{b! c! d!} \langle \Gamma^b v_1 \cdot \nabla \Gamma^c u_2, \Gamma^d v_2 \rangle
\]
where the two missing terms with \( c = a \) have been combined and integrated by parts to give zero using \( \nabla \cdot v_1 = 0 \).

This is the point in the proof where we must use the decay estimate (10.45). Since this estimate gives differing results for the various derivatives, we pause for a moment to extract the precise information that we need. Then we will have to juggle derivatives through integration by parts to rewrite certain terms so that the “good derivatives” appear on the irrotational component \( u_2 \).

We single out two multi-indices \( a^* \) and \( a' \) of length \( m \), namely the ones with no spatial derivatives:

\[
\Gamma^{a^*} = S^m \quad \text{and} \quad \Gamma^{a'} = \partial_b S^{m-1}.
\]

If \( a \neq a^*, a' \), then either \( \Gamma^a \) contains spatial derivatives or \( |a| < m \). In the first case, write \( \Gamma^a = \nabla^b \Gamma^b \), with \( |b| \leq m-1 \). Then by (3.10) and (10.40), the definition of \( Q(t) \), we have

\[
||\sigma(t)^{-1} \Gamma^a u_2||_{L^2} \leq \frac{C}{(1+t)} ||\sigma(t-r) \nabla^b \Gamma^b u_2||_{L^2} \leq \frac{C}{(1+t)} Q(t).
\]

On the other hand, if \( \Gamma^a = \partial_b S^k \), with \( |a| = j + k < m \), then \( \Gamma^a u_2 \) is rotationally symmetric and by (10.8)

\[
||\sigma(t)^{-1} \Gamma^a u_2||_{L^2} \leq \frac{C}{(1+t)} ||\sigma(t-r) \nabla^b \Gamma^b u_2||_{L^2} \leq \frac{C}{(1+t)} Q(t).
\]

The previous inequalities show that

\[
(10.46) \quad ||\sigma(t)^{-1} \Gamma^a u_2||_{L^2} \leq \frac{C}{(1+t)} Q(t), \quad a \neq a^*, a'.
\]

Now for \( a = a' \), i.e. \( \Gamma^d = \partial_b S^{m-1} \), (3.10) and (10.40) produce

\[
||\sigma(t)^{-1/2} \Gamma^d u_2||_{L^2} \leq \frac{C}{(1+t)^{1/2}} ||\sigma(t-r)^{1/2} \partial_b S^{m-1} u_2||_{L^2} \leq \frac{C}{(1+t)^{1/2}} Q(t).
\]
which combines with (10.46) in

\[ ||\sigma(r)^{-1}\Gamma^c u_2||_{L^2} \leq \frac{C}{(1 + t)^{1/2}} Q(t), \quad a \neq a^*. \]

Now we are ready to return to the estimation of \( S^i_8 + S^i_{10} \).

First consider the case in which \( a \neq a^* \). Then by (10.47),

\[
|S^i_8 + S^i_{10}| \leq C \sum_{b+c=a \cap \eta} ||\sigma(r)^{b} v_1||_{L^\infty} ||\sigma(r)^{-1}\nabla^c u_2||_{L^2} ||\Gamma^a v_2||_{L^2} \\
+ C \sum_{b+c=a \cap \eta} ||\sigma(r)^{-1}\Gamma^b u_2||_{L^2} ||\sigma(r)\nabla^c v_1||_{L^\infty} ||\Gamma^a v_2||_{L^2} \\
+ C \sum_{b+c=a \cap \eta} ||\sigma(r)^b v_1||_{L^\infty} ||\nabla^c v_2||_{L^2} ||\sigma(r)^{-1}\Gamma^c u_2||_{L^2} \\
+ C \sum_{b+c=a \cap \eta} ||\Gamma^b v_2||_{L^2} ||\sigma(r)\nabla^c v_1||_{L^\infty} ||\sigma(r)^{-1}\Gamma^c u_2||_{L^2} \\
\leq \frac{C \varepsilon}{(1 + t)^{1/2}} E_m(t)^{1/2} Q(t).
\]

We are left with the case \( a = a^* \), when \( \Gamma^{a^*} = S^m \):

\[
S^{a^*}_8 + S^{a^*}_{10} = \sum_{j+k=m \cap \eta} \frac{a!}{b!c!} \langle S^j v_1 \cdot \nabla S^k v_2, S^m u_2 \rangle \\
+ \sum_{j+k=m \cap \eta} \frac{a!}{b!c!} \langle S^j v_2 \cdot \nabla S^k v_1, S^m u_2 \rangle \\
+ \sum_{j+k=m \cap \eta} \frac{a!}{b!c!} \langle S^j v_1 \cdot \nabla S^k u_2, S^m v_2 \rangle \\
+ \sum_{j+k=m \cap \eta} \frac{a!}{b!c!} \langle S^j u_2 \cdot \nabla S^k v_1, S^m v_2 \rangle.
\]

All of the terms above without \( S^m u_2 \) can be estimated by \( \frac{C \varepsilon}{(1 + t)^{1/2}} E_m(t)^{1/2} Q(t) \), as before, using (10.47).

Using (10.4), (10.5) the terms containing \( S^m u_2 \) are bounded by

\[ C \sum_{j+k=m \cap \eta} \left| \left\langle \frac{1}{\rho} S^j g_1, S^k v_2, S^m u_2^\perp \right\rangle \right| + \left| \left\langle (\partial_r g_1 - \frac{1}{\rho} g_1) S^m v_2, S^m u_2^\perp \right\rangle \right|. \]

The last term is the most interesting since it has top order \( \mathcal{S} \)-derivatives of both \( u_2 \) and \( v_2 \). With \( G_1 = \partial_r g_1 - \frac{1}{\rho} g_1 \) and remembering that \( S = t \partial_r + x \cdot \nabla \), we
have

\[
\langle G_1 S^m v_2, S^m u_2 \rangle = \frac{d}{dt} \langle G_1 S^m v_2^\perp, S^m u_2^\perp \rangle
- \langle G_1 S^m v_2, S^m u_2^\perp \rangle
- t \langle G_1 \partial_t S^m v_2, S^m u_2^\perp \rangle
+ \langle G_1 S^m v_2, x \cdot \nabla S^m u_2^\perp \rangle
\equiv \frac{d}{dt} I_1 + I_2 + I_3 + I_4.
\]

For the first, we get by (10.24) and (10.46)

\[
|I_1| \leq C \varepsilon \sigma(t) G_1 \|S^m v_2\| L^\infty \|\sigma(r)^{-1} S^{m-1} u_2\| L^2
\leq C \varepsilon E_m(t)^{1/2} \frac{C}{1 + \varepsilon} Q(t)
\leq C \varepsilon E_m(t)^{1/2} Q(t),
\]

which is sufficient only because of the differentiation in front.

In exactly the same way, we have

\[
|I_2| \leq C \|\sigma(r) G_1\| L^\infty \|S^m v_2\| L^2 \|\sigma(r)^{-1} S^{m-1} u_2\| L^2
\leq \frac{C \varepsilon}{(1 + \varepsilon)} E_m(t)^{1/2} Q(t).
\]

This term gives us no problems.

Jumping now to the last term, we apply (3.10) and (10.24)

\[
|I_4| \leq \frac{C}{(1 + \varepsilon)^{1/2}} \|\sigma(r)^{3/2} G_1\| L^\infty \|S^m v_2\| L^2 \|\sigma(t - r)^{1/2} \nabla S^{m-1} u_2\| L^2
\leq \frac{C \varepsilon}{(1 + \varepsilon)^{1/2}} E_m(t)^{1/2} Q(t).
\]

To handle the third term, we must return to the PDE’s. Applying \( P_2 \) to (10.33) and then using (10.35) and (10.37), we see that

\[
\partial_t \Gamma^a v_2 = \partial_t \Gamma^a P_2 w_2
= - \sum_{b+c=a} \frac{a!}{b!c!} \left\{ \Gamma^b u_1 \cdot \nabla \Gamma^c v_1 + \Gamma^b v_1 \cdot \nabla \Gamma^c u_1 \right\}
+ \left[ \Gamma^b u_1 \cdot \nabla \Gamma^c v_2 + \Gamma^b v_2 \cdot \nabla \Gamma^c u_1 \right]
+ \left[ \Gamma^b v_1 \cdot \nabla \Gamma^c u_2 + \Gamma^b u_2 \cdot \nabla \Gamma^c v_1 \right]
+ \left[ \Gamma^b u_2 \cdot \nabla \Gamma^c v_2 + \Gamma^b v_2 \cdot \nabla \Gamma^c u_2 \right].
\]
Setting $\Gamma^{m} = S^{m}$ and collecting the terms with derivatives of order $m + 1$ in $u_{2}, v_{2}$ and writing them in divergence form, we have that

$$
\partial_{t} S^{m} v_{2} = - \sum_{j \neq k = m} \frac{m!}{j!k!} \left\{ \left[ S^{j} u_{1} \cdot \nabla S^{k} v_{1} + S^{j} v_{1} \cdot \nabla S^{k} u_{1} \right]
+ S^{j} v_{2} \cdot \nabla S^{k} u_{1} + S^{j} u_{2} \cdot \nabla S^{k} v_{1} \right\}
- \sum_{j \neq k = m} \frac{m!}{j!k!} \left\{ S^{j} u_{1} \cdot \nabla S^{k} v_{2} + S^{j} v_{1} \cdot \nabla S^{k} u_{2}
+ \left[ S^{j} u_{2} \cdot \nabla S^{k} v_{2} + S^{j} v_{2} \cdot \nabla S^{k} u_{2} \right] \right\}
+ (\nabla \cdot v_{1}) S^{m} v_{2} + (\nabla \cdot u_{2}) S^{m} v_{2}
- \nabla \cdot \left[ v_{1} \otimes S^{m} v_{2} + u_{1} \otimes S^{m} v_{2}
+ u_{2} \otimes S^{m} v_{2} + v_{2} \otimes S^{m} u_{2} \right]
\equiv \mathcal{R} + \nabla \cdot \mathcal{V}.
$$

It is easily verified that

$$
||\mathcal{R}||_{L^{2}} \leq C \left[ \frac{\varepsilon^{2}}{(1 + t)^{1/2}} + \varepsilon E_{m}(t)^{1/2} + E_{m}(t) \right]
$$

and

$$
||\mathcal{V}||_{L^{2}} \leq C \left[ \varepsilon E_{m}(t)^{1/2} + E_{m}(t) \right].
$$

Therefore, integrating by parts one final time and appealing to (10.24), there results

$$
|J_{3}| = |\langle G_{1} \partial_{t} S^{m} v_{2}, S^{m-1} u_{2}^{1} \rangle|
\leq t |\langle G_{1} \mathcal{R}, S^{m-1} u_{2}^{1} \rangle| + t |\langle \nabla G_{1} \mathcal{V}, S^{m-1} u_{2}^{1} \rangle| + t |\langle G_{1} \mathcal{V}, \nabla S^{m-1} u_{2}^{1} \rangle|
\leq t ||\sigma(r) G_{1}||_{L^{\infty}} ||\mathcal{R}||_{L^{2}} ||\sigma(r)^{-1} S^{m-1} u_{2}||_{L^{2}}
+ t ||\sigma(r) \nabla G_{1}||_{L^{\infty}} ||\mathcal{V}||_{L^{2}} ||\sigma(r)^{-1} S^{m-1} u_{2}||_{L^{\infty}}
+ C ||\sigma(r) G_{1}||_{L^{\infty}} ||\mathcal{V}||_{L^{2}} ||\sigma(t - r) \nabla S^{m-1} u_{2}||_{L^{2}}
\leq C \varepsilon \left[ \frac{\varepsilon^{2}}{(1 + t)^{1/2}} + \varepsilon E_{m}(t)^{1/2} + E_{m}(t) \right] Q(t).
$$

In the same way, the other terms in (10.48) are bounded by the same expression, using (10.23). In fact, the estimates are a bit easier because the last
that the decay estimate (10.45) holds. Using (10.45) together with the bounds for

Thus, if we restrict

Then

that

with

independent of

(10.49)

$$\frac{d}{dt} E_m(t) \leq \frac{d}{dt} I(t)$$

$$+ \hat{C} \left[ \left( \frac{\varepsilon^5}{(1 + t)^{1/2}} + \varepsilon^6 \right) + \left( \frac{\varepsilon^2}{(1 + t)^2} + \varepsilon^4 \right) E_m(t)^{1/2} + E_m(t)^{3/2} \right],$$

with

$$|I(t)| \leq \hat{C} \varepsilon^2 + E_m(t)^{1/2} E_m(t)^{1/2},$$

and repeating (10.44)

$$E_m(0)^{1/2} \leq \hat{C} \varepsilon^2.$$

We shall now conclude by showing that these inequalities imply that $E_m(t)^{1/2}$ remains $O(\varepsilon^2 \ln (1 + 1/\varepsilon^2))$ up to a time of the order $O(1/\varepsilon^2 \ln (1 + 1/\varepsilon^2))$, provided that $0 < \varepsilon < \varepsilon_0$ with $\varepsilon_0$ sufficiently small.

Define

$$T_3 = \sup \{ T > 0 : E_m(t)^{1/2} \leq \varepsilon^2 \ln (1 + 1/\varepsilon^2), 0 \leq t \leq T \}.$$ 

For $\varepsilon_0$ small enough, we have $T_3 < T_1$. Moreover, if $\varepsilon_0$ again is small enough, then

$$T_\varepsilon = \frac{\ln 2}{\hat{C} \varepsilon^2 \ln (1 + 1/\varepsilon^2)} = \frac{A^2}{\varepsilon^2}.$$ 

Thus, if we restrict $t \leq T_4 = \min (T_3, T_\varepsilon)$ we have also that $t < T_2$. We now show that $T_4 = T_\varepsilon$. 

Conclusion of the proof. Assume that $t \leq T_2$. This implies that $E_m(t)^{1/2} \leq 2C_0\varepsilon^2$, that we are within the life span interval of the first order terms, and that the decay estimate (10.45) holds. Using (10.45) together with the bounds for $S_1, \ldots, S_5$ we obtain that for $0 \leq t \leq T_2$ and some large positive constant $\hat{C}$, independent of $\varepsilon$, the energy inequality

$$\frac{d}{dt} E_m(t) \leq \frac{d}{dt} I(t)$$

$$+ \hat{C} \left[ \left( \frac{\varepsilon^5}{(1 + t)^{1/2}} + \varepsilon^6 \right) + \left( \frac{\varepsilon^2}{(1 + t)^2} + \varepsilon^4 \right) E_m(t)^{1/2} + E_m(t)^{3/2} \right],$$

with

$$|I(t)| \leq \hat{C} \varepsilon^2 + E_m(t)^{1/2} E_m(t)^{1/2},$$

and repeating (10.44)

$$E_m(0)^{1/2} \leq \hat{C} \varepsilon^2.$$
Introducing an integrating factor, it follows from (10.49) that

\[
E_m(t) \leq E_m(0) \exp \hat{C} \int_0^t E_m(s)^{1/2} ds \\
+ \int_0^t \frac{d}{d\tau} I(\tau) \left[ \exp \hat{C} \int_\tau^t E_m(s)^{1/2} ds \right] d\tau \\
+ \hat{C} \int_0^t \left[ \frac{\varepsilon^4}{(1 + \varepsilon)^{1/2}} + \varepsilon^6 \right] \left[ \exp \hat{C} \int_\tau^t E_m(s)^{1/2} ds \right] d\tau \\
+ \hat{C} \int_0^t \left[ \frac{\varepsilon^2}{(1 + \varepsilon)^2} + \varepsilon^4 \right] E_m(\tau)^{1/2} \left[ \exp \hat{C} \int_\tau^t E_m(s)^{1/2} ds \right] d\tau \\
\equiv B_1 + \cdots + B_4.
\]

Thanks to our restriction on \( t \), we have

\[
(10.52) \quad \exp \hat{C} \int_\tau^t E_m(s)^{1/2} ds \leq \exp \hat{C} T \varepsilon^2 \ln (1 + 1/\varepsilon^2) = 2.
\]

Thus for \( \varepsilon_0 \) sufficiently small, we immediately get by (10.51) and (10.52) for all \( 0 \leq \varepsilon \leq \varepsilon_0 \)

\[
|B_1| \leq 2\hat{C}^2 \varepsilon^4 \leq (1/4)\varepsilon^4 \ln^2 (1 + 1/\varepsilon^2),
\]

\[
|B_3| \leq 2\hat{C} \left[ 5T \varepsilon^{1/2} + \varepsilon^6 T \varepsilon \right]
\leq (1/4)\varepsilon^4 \ln^2 (1 + 1/\varepsilon^2)[o(\varepsilon_0)]
\leq (1/4)\varepsilon^4 \ln^2 (1 + 1/\varepsilon^2),
\]

and

\[
|B_4| \leq 2\hat{C} \varepsilon^2 \ln (1 + 1/\varepsilon^2) \left[ \varepsilon^2 \ln (1 + T \varepsilon) + \varepsilon^4 T \varepsilon \right]
\leq (1/4)\varepsilon^4 \ln^2 (1 + 1/\varepsilon^2)[o(\varepsilon_0)]
\leq (1/4)\varepsilon^4 \ln^2 (1 + 1/\varepsilon^2).
\]

An integration by part gives

\[
B_2 = \int_0^t I(\tau) \hat{C} E_m(\tau)^{1/2} \left[ \exp \hat{C} \int_\tau^t E_m(s)^{1/2} ds \right] d\tau \\
+ I(t) - I(0) \exp \hat{C} \int_0^t E_m(s)^{1/2} ds \\
\equiv B_{21} + B_{22} + B_{23}.
\]
And now we easily get using (10.50) and (10.52) for small enough $\varepsilon_0$
\[
|I(t)| \leq \tilde{C}\varepsilon \left[\varepsilon^2 + \varepsilon^2 \ln(1 + 1/\varepsilon^2)\right] \varepsilon^2 \ln(1 + 1/\varepsilon^2) \\
\leq 2\tilde{C}\varepsilon_0 \varepsilon^4 \ln^2(1 + 1/\varepsilon^2),
\]
so that
\[
|B_{21}| \leq \varepsilon^4 \ln^2(1 + 1/\varepsilon^2)[o(\varepsilon_0)] \leq (1/12)\varepsilon^4 \ln^2(1 + 1/\varepsilon^2),
\]
\[
|B_{22}| = |I(t)| \leq (1/12)\varepsilon^4 \ln^2(1 + 1/\varepsilon^2),
\]
and
\[
|B_{32}| \leq 2|I(0)| \leq (1/12)\varepsilon^4 \ln^2(1 + 1/\varepsilon^2).
\]
Hence,
\[
|B_2| \leq (1/4)\varepsilon^4 \ln^2(1 + 1/\varepsilon^2).
\]

Looking at the inequalities for $B_1, \ldots, B_4$, we have shown that
\[
E_m(t) \leq \varepsilon^4 \ln^2(1 + 1/\varepsilon^2)
\]
for $t \leq T_4$. Thus, the life span of the solution is at least $T_\varepsilon$ for $\varepsilon < \varepsilon_0$. \qed

11. Remarks on local existence. We give a sketch of a proof of Proposition 1.

Let $\phi \in S(\mathbb{R}^2)$ be radially symmetric with $\int \phi(x)dx = 1$. Then $x_j \phi(x)$ is odd in $x_j$, $j = 1, 2$, and the first moments of $\phi$ vanish: $\int x_j \phi(x)dx = 0$, $j = 1, 2$. Define the mollifier $J_\varepsilon$ by convolution with the function $\varepsilon^{-2}\phi(x/\varepsilon)$. $J_\varepsilon$ commutes with $\partial$ and $P_2$, and also $\Omega$ since $\phi$ is radially symmetric. The commutator $[S_0, J_\varepsilon]$ is given by convolution with $\varepsilon^{-2}\psi(x/\varepsilon)$ with $\psi(x) = \nabla \cdot x \phi(x)$. Note that $\psi$ has zero mean and first moments. Thus, $[S_0, J_\varepsilon]$ is a smoothing operator which tends to zero as $\varepsilon \to 0$. Higher order commutators behave similarly.

Now for any $v \in \tilde{H}_A^1$, we have
\begin{align}
\|J_\varepsilon v\|_{\tilde{H}_A^j} &\leq C\varepsilon^{-j}\|v\|_{\tilde{H}_A^j}, &j = 0, 1, 2, \ldots, \\
\|J_{\varepsilon_1} v - J_{\varepsilon_2} v\|_{\tilde{H}_A^j} &= o(\varepsilon_1 - \varepsilon_2), &0 \leq \varepsilon_2 \leq \varepsilon_1, \\
\|J_{\varepsilon_1} v - J_{\varepsilon_2} v\|_{\tilde{H}_A^{j-1}} &\leq C(\varepsilon_1 - \varepsilon_2)\|v\|_{\tilde{H}_A^j}, &0 \leq \varepsilon_2 \leq \varepsilon_1.
\end{align}

These inequalities can be demonstrated using the Fourier transform. They are

To prove Proposition 1, it is enough to construct a solution

\[ v \in Y_k \equiv C \left( [0, T); \tilde{H}^k_\Lambda \right) \cap C^1 \left( [0, T); \tilde{H}^{k-1}_\Lambda \right), \quad k \geq 4, \]

of the initial value problem

\[ \partial_t v + P_2(v \cdot \nabla v) = 0, \]
\[ v(0) = v_0 = P_2 v_0. \]

The remaining regularity in \( t \) would then follow from the PDE, and the regularity statements for the pressure would be a consequence of the formula

\[ p + \sum_{i,j} R_i R_j (v_i v_j) = 0. \]

Following [20], we regularize the initial value problem as follows

\[ \partial_t v^{\varepsilon, \delta} + J_\varepsilon P_2(\varepsilon v^{\varepsilon, \delta}) = 0, \]
\[ v^{\varepsilon, \delta}(0) = J_\varepsilon v_0. \]

Treating the regularized problem as an ODE, we immediately get a solution

\[ v^{\varepsilon, \delta} \in C \left( [0, T_{\varepsilon, \delta}); \tilde{H}^k_\Lambda \right). \]

By (11.1) and (11.4), it follows that

\[ v^{\varepsilon, \delta} \in C^{\infty} \left( [0, T_{\varepsilon, \delta}); \tilde{H}^{k+j}_\Lambda \right), \quad j = 0, 1, 2, \ldots \]

Next, adopting a PDE point of view, energy estimates give a uniform bound for the lifespan \( T_{\varepsilon, \delta} \) and the size of the solution in \( Y_k, \quad k \geq 4 \). Moreover, again using energy estimates we have

\[ v^{\varepsilon, \delta} \in C \left( [0, T); \tilde{H}^{k+j}_\Lambda \right), \]

with the bound

\[ \| v^{\varepsilon, \delta}(t) \|_{\tilde{H}^{k+j}_\Lambda} \leq C \delta^{-j} \| v_0 \|_{\tilde{H}^k_\Lambda}, \]

thanks to (11.1) and (11.5).

The key point is to show that \( v^{\varepsilon, \delta} \) with \( \varepsilon = \varepsilon^{1/3} \) is a Cauchy sequence in \( Y_k \). This can be done using the energy method applied to the equation satisfied by
w = v_{\varepsilon_1} - v_{\varepsilon_2}, \varepsilon_1 > \varepsilon_2 > 0. To this end, it is useful to note that by (11.3) and (11.6),
\[
\|\nabla (J_{\varepsilon_1} - J_{\varepsilon_2})\|_{H^k} \leq C(\varepsilon_1 - \varepsilon_2)\|\nabla v^{\varepsilon_1 \delta}\|_{H^{k+1}}
\leq C(\varepsilon_1 - \varepsilon_2)\delta^{-2}\|v_0\|_{L^k},
\]
which is \(o(\varepsilon_1 - \varepsilon_2)\) for \(\delta = \varepsilon^{1/3}\). It is then immediate to pass to the limit in the equation using (11.2).

A similar argument can be applied to the compressible case.

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