Global regularity of a modified Navier-Stokes equation

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Abstract. We introduce a modified equation which shares properties of the incompressible Navier-Stokes equation on the one hand and the Burgers equation on the other hand. For this equation, we demonstrate global well posedness for sufficiently smooth initial conditions in the periodic case and in $\mathbb{R}^3$. The key feature of the modification is the availability of an additional estimate which shows that the L^4-norm remains finite.

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1. Introduction

The problem of whether the three-dimensional incompressible Navier-Stokes equations can develop a finite time singularity from smooth initial conditions or if it has global solutions remains unresolved (see [1, 2, 3] and the references therein). The answer to this important question is recognized as one of the Millennium prize problems [4, 5].

Despite the complexity of the topic, a lot of progress has been made on this field in the past. For the two-dimensional case, global-in-time existence of unique weak and strong solutions is well-known (see [1, 2]). In three dimensions weak solutions are known to exist globally in time. For strong solutions, existence and uniqueness is known for a short interval of time which depends continuously on the initial data [6]. Many results published in the past, starting with [7], provide criteria for the global regularity of solutions via conditions applied to the velocity field [8, 9] or components thereof [10], the vorticity [11], its direction [12] or to the pressure field [13, 14].

The theory for the compressible Navier-Stokes equation is less well developed, and we will not attempt a summary here. The multi-dimensional Burgers equation [15] can be regarded as a crude simplification of this model. Global existence and uniqueness of strong solutions can be established in two and three-dimensions for suitably small initial conditions, much as with the Navier-Stokes system. Irrotational flows do possess global solutions for large data in arbitrary dimension, thanks to the Cole-Hopf transformation

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[16, 17]. However, there is no multidimensional weak theory because of the absence of a mechanism for energy dissipation, unlike Navier-Stokes.

In this paper, we introduce a new equation which is a hybrid of the Navier-Stokes equation and the Burgers equation. For this model equation, we can show existence and regularity for H^1 initial conditions of arbitrary size. This will be carried out in $\mathbb{R}^3$ and periodic domains in $\mathbb{R}^3$. A simple modification of the nonlinearity makes the proof of global solutions possible, insofar as an additional estimate is available showing that the solution remains finite in L^p, $2 < p < \infty$. With $p = 4$, this is then coupled with standard estimates for the H^1-norm to complete the proof.

2. Model equation

We consider a three-dimensional domain Ω which shall be either $\mathbb{R}^3$ or a bounded rectangle in $\mathbb{R}^3$ with periodic boundary conditions. Let $\mathrm{P} = 1 - \Delta^{-1}\nabla \otimes \nabla$ be the Leray-Hopf projection operator (with periodic boundary conditions when Ω is bounded):

$$\mathrm{P}[\mathrm{P}[\mathbf{u}]] = \mathrm{P}[\mathbf{u}] \quad , \quad \nabla \cdot \mathrm{P}[\mathbf{u}] = 0 \; . \tag{1}$$

The usual incompressible Navier-Stokes equation

$$\frac{\partial}{\partial t}\mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p = \nu \Delta \mathbf{v} + \mathbf{f} \quad , \quad \nabla \cdot \mathbf{v} = 0$$

can be written with the projection operator P

$$\frac{\partial}{\partial t}\mathbf{v} + \mathrm{P}[\mathbf{v} \cdot \nabla \mathbf{v}] = \nu \Delta \mathbf{v} + \mathrm{P}[\mathbf{f}] \quad , \quad \nabla \cdot \mathbf{v} = 0$$

such that no explicit pressure term is present in the equation.

We can rewrite the Navier-Stokes equation without the incompressibility constraint in the form

$$\frac{\partial}{\partial t}\mathbf{u} + \mathrm{P}[\mathbf{u}] \cdot \nabla \mathrm{P}[\mathbf{u}] = \nu \Delta \mathbf{u} + \mathbf{f} \; , \tag{2}$$

where the solution of the Navier-Stokes equation can be recovered by taking $\mathbf{v} = P[\mathbf{u}]$.

The equation (2) can be compared with Burgers' equation whose structure is formally similar:

$$\frac{\partial}{\partial t}\mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = \nu \Delta \mathbf{u} + \mathbf{f} \; . \tag{3}$$

For equation (3) the nonlinearity is purely local, whereas for equation (2) the nonlinear interaction involves the nonlocal projection.

A natural hybrid of these two equations leads a new model equation involving a compressible velocity field $\mathbf{u}$ that is convected by its solenoidal part $\mathrm{P}[\mathbf{u}]$:

$$\frac{\partial}{\partial t}\mathbf{u} + \mathrm{P}[\mathbf{u}] \cdot \nabla \mathbf{u} = \nu \Delta \mathbf{u} + \mathbf{f} \; . \tag{4}$$

More accurately this means: The convection of the velocity field $\mathbf{u}$ is local in position space, but the projection operator is local in Fourier space and thus shares this mixture of local and non-local interactions with the original Navier-Stokes equation.

In the next section, we show global regularity for equation (4) for suitable initial data without any size restrictions.

3. Global solutions

In this section, the existence of global solutions is proven for the model equation (4) in a domain Ω which shall either be $\mathbb{R}^3$ or a periodic rectangle in $\mathbb{R}^3$.

Theorem 1. *Let* $\mathbf{u}_0 \in H^1(\Omega)$*. Let* $\mathbf{f} \in L^2_{loc}(\mathbb{R}^+, L^2(\Omega)) \cap L^1_{loc}(\mathbb{R}^+, L^4(\Omega))$*. Then the initial value problem for the model equation (4) has a unique global solution*

$$u \in C(\mathbb{R}^+, H^1(\Omega)) \cap L^2_{loc}(\mathbb{R}^+, H^2(\Omega)) \ .$$

The aim is to show that the solution remains *a priori* bounded in $L^\infty([0,T), H^1(\Omega)) \cap L^2([0,T), H^2(\Omega))$ for any $T > 0$, which implies its existence and uniqueness with standard arguments comparable to e.g. [1, 18]. Throughout the argument, we denote the Euclidean norm of the vector $\mathbf{u} = \sum_i u_i \mathbf{e}_i$ by $u = (\sum_i u_i^2)^{1/2}$. We first prove the following lemma:

Lemma 1. *Let* $\mathbf{u}_0$*,* $\mathbf{f}$*,* Ω *be defined as above. Then the quantity* $\|\mathbf{u}(t)\|_{L^p}$ *remains finite for* $2 \le p \in \mathbb{R}$*.*

Proof. Taking the Euclidean inner product of (4) with $\mathbf{u}$ yields the identity

$$\frac{1}{2}\left(\frac{\partial}{\partial t}u^2 + \mathrm{P}\mathbf{u}\cdot\nabla u^2\right) = \frac{\nu}{2}\Delta u^2 - \nu|\nabla \mathbf{u}|^2 + \mathbf{f}\cdot\mathbf{u} \ . \tag{5}$$

Integrate (5) over Ω, use the fact that $\mathrm{P}\mathbf{u}$ is divergence free, and then apply the Cauchy-Schwarz inequality to obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\|\mathbf{u}\|^2_{L^2} + \nu\|\nabla\mathbf{u}\|^2_{L^2} \le \|\mathbf{f}\|_{L^2}\|\mathbf{u}\|_{L^2} \ . \tag{6}$$

Defining $x(t) = \frac{1}{2}\left(\|\mathbf{u}(t)\|^2_{L^2} + \int_0^t \nu\|\nabla\mathbf{u}(s)\|^2_{L^2}ds\right)$, we have that

$$x'(t) \le \|\mathbf{f}(t)\|_{L^2}(2x(t))^{1/2} \ .$$

Upon integration, this gives the inequality

$$\|\mathbf{u}(t)\|^2_{L^2} + \int_0^t \nu\|\nabla\mathbf{u}(s)\|^2_{L^2}ds \le \left(\|\mathbf{u}_0\|_{L^2} + \int_0^t \|\mathbf{f}(s)\|_{L^2}ds\right)^2 \ . \tag{7}$$

With this estimate the lemma is shown for the case $p = 2$.

Let $2 \le n \in \mathbb{R}$ and multiply the identity (5) by $u^{2(n-1)}$:

$$\begin{aligned}\frac{1}{2n}\left(\frac{\partial}{\partial t}u^{2n} + \mathrm{P}\mathbf{u}\cdot\nabla u^{2n}\right) &= \frac{\nu}{2}\left(\frac{\partial}{\partial x_j}(u^{2(n-1)}\frac{\partial}{\partial x_j}u^2) - \frac{4(n-1)}{n^2}|\nabla u^n|^2\right) \\ &\quad - \nu u^{2(n-1)}|\nabla\mathbf{u}|^2 + u^{2(n-1)}\mathbf{f}\cdot\mathbf{u} \ .\end{aligned}$$

Integrate this over Ω and apply Hölder's inequality:

$$\frac{1}{2n}\frac{\partial}{\partial t}\|\mathbf{u}\|^{2n}_{L^{2n}} + \int_\Omega \left(\tfrac{2\nu(n-1)}{n^2}|\nabla u^n|^2 + \nu u^{2(n-1)}|\nabla\mathbf{u}|^2\right)d\mathbf{x} \le \|\mathbf{f}\|_{L^{2n}}\|\mathbf{u}\|^{2n-1}_{L^{2n}}$$

If we let

$$y(t) = \frac{1}{2n}\|\mathbf{u}(t)\|^{2n}_{L^{2n}} + \int_0^t\int_\Omega \left(\tfrac{2\nu(n-1)}{n^2}|\nabla u^n(s)|^2 + \nu u^{2(n-1)}(s)|\nabla\mathbf{u}(s)|^2\right)d\mathbf{x}ds \ ,$$

then we obtain

$$y'(t) \le \|\mathbf{f}(t)\|_{L^{2n}}(2n\, y(t))^{\frac{2n-1}{2n}} \ .$$

This leads to the estimate

$$\begin{aligned} \|\mathbf{u}(t)\|_{L^{2n}}^{2n} + 2n\int_0^t\int_\Omega \left(\tfrac{2\nu(n-1)}{n^2}\,|\nabla u^n(s)|^2 + \nu u^{2(n-1)}(s)|\nabla\mathbf{u}(s)|^2\right) d\mathbf{x}ds \quad (8)\\ \le \left(\|\mathbf{u}_0\|_{L^{2n}} + \int_0^t \|\mathbf{f}(s)\|_{L^{2n}}ds\right)^{2n} \ . \end{aligned}$$

□

Remark: This key argument fails for the case of the Navier-Stokes equation.

Proof of Theorem 1. Take the L^2-inner product of (4) with $\Delta\mathbf{u}$ and integrate by parts:

$$\frac{1}{2}\frac{\partial}{\partial t}\|\nabla\mathbf{u}\|_{L^2}^2 + \nu\|\Delta\mathbf{u}\|_{L^2}^2 = \underbrace{\int_\Omega(\mathrm{P}\mathbf{u}\cdot\nabla\mathbf{u})\cdot\Delta\mathbf{u}\, d\mathbf{x}}_{(i)} + \underbrace{\int_\Omega \mathbf{f}\cdot\Delta\mathbf{u}\, d\mathbf{x}}_{(ii)} \ .$$

The forcing term (ii) has the bound

$$\int_\Omega \mathbf{f}\cdot\Delta\mathbf{u}d\mathbf{x} \le \|\mathbf{f}\|_{L^2}\|\Delta\mathbf{u}\|_{L^2} \le \frac{\nu}{4}\|\Delta\mathbf{u}\|_{L^2}^2 + \frac{1}{\nu}\|\mathbf{f}\|_{L^2}^2 \ .$$

The nonlinear term (i) is estimated as follows:

$$\begin{aligned} \int_\Omega(\mathrm{P}\mathbf{u}\cdot\nabla\mathbf{u})\cdot\Delta\mathbf{u}\, d\mathbf{x} &= -\int \frac{\partial}{\partial x_k}u_i\frac{\partial}{\partial x_k}\left((\mathrm{P}\mathbf{u})_j\frac{\partial}{\partial x_j}u_i\right)\, d\mathbf{x}\\ &= -\int \frac{\partial}{\partial x_k}u_i\left((\mathrm{P}\mathbf{u})_j\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}u_i + \frac{\partial}{\partial x_k}(\mathrm{P}\mathbf{u})_j\frac{\partial}{\partial x_j}u_i\right)\, d\mathbf{x}\\ &= -\int\left(\frac{1}{2}(\mathrm{P}\mathbf{u})_j\frac{\partial}{\partial x_j}|\nabla\mathbf{u}|^2 + \frac{\partial}{\partial x_k}u_i\frac{\partial}{\partial x_j}(\frac{\partial}{\partial x_k}(\mathrm{P}\mathbf{u})_j u_i)\right)\, d\mathbf{x}\\ &= \int\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}u_i\frac{\partial}{\partial x_k}(\mathrm{P}\mathbf{u})_j u_i\, d\mathbf{x}\\ &\le \|\nabla^2\mathbf{u}\|_{L^2}\|\nabla\mathrm{P}\mathbf{u}\|_{L^4}\|\mathbf{u}\|_{L^4} \ . \end{aligned}$$

The second norm above is handled by interpolation. We first note that

$$\|\nabla\mathrm{P}\mathbf{u}\|_{L^4} \le \|\nabla\mathrm{P}\mathbf{u}\|_{L^6}^{3/4}\|\nabla\mathrm{P}\mathbf{u}\|_{L^2}^{1/4} \ .$$

Now when $\Omega = \mathbb{R}^3$, the Sobolev embedding theorem gives

$$\|\nabla\mathrm{P}\mathbf{u}\|_{L^6} \le C\|\nabla^2\mathrm{P}\mathbf{u}\|_{L^2} \ . \quad (9)$$

When Ω is a periodic domain, the norm on the right must be replaced by $\|\nabla\mathrm{P}\mathbf{u}\|_{H^1}$. However, since $\nabla\mathrm{P}\mathbf{u}$ has zero mean, this is bounded again by $C\|\nabla^2\mathrm{P}\mathbf{u}\|_{L^2}$, by the Poincaré inequality. Therefore, (9) holds in both cases. Using the facts that the operator P commutes with derivatives and that it is a projection in L^2, we have that

$$\|\nabla\mathrm{P}\mathbf{u}\|_{L^2} \le \|\nabla\mathbf{u}\|_{L^2} \quad \text{and} \quad \|\nabla^2\mathrm{P}\mathbf{u}\|_{L^2} \le \|\nabla^2\mathbf{u}\|_{L^2} \ .$$

Next, we use integration by parts to obtain the simple ellipticity identity

$$\begin{aligned} \|\nabla^2\mathbf{u}\|_{L^2}^2 &= \int_\Omega \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}u_i \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}u_i \, d\mathbf{x} \\ &= \int_\Omega \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}u_i \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}u_i \, d\mathbf{x} \\ &= \|\Delta\mathbf{u}\|_{L^2}^2 \, . \end{aligned} \tag{10}$$

Combining these observations with Young's inequality, we conclude that the nonlinear term (i) is bounded by

$$C\|\Delta\mathbf{u}\|_{L^2}^{7/4}\|\nabla\mathbf{u}\|_{L^2}^{1/4}\|\mathbf{u}\|_{L^4} \le \frac{\nu}{4}\|\Delta\mathbf{u}\|_{L^2}^2 + \frac{C}{\nu^7}\|\nabla\mathbf{u}\|_{L^2}^2\|\mathbf{u}\|_{L^4}^8 \, .$$

Altogether, we get the inequality

$$\frac{\partial}{\partial t}\|\nabla\mathbf{u}\|_{L^2}^2 + \nu\|\Delta\mathbf{u}\|_{L^2}^2 \le \frac{C}{\nu^7}\|\nabla\mathbf{u}\|_{L^2}^2\|\mathbf{u}\|_{L^4}^8 + \frac{C}{\nu}\|\mathbf{f}\|_{L^2}^2 \, .$$

Using Gronwall's inequality, we find that

$$\begin{aligned} \|\nabla\mathbf{u}\|_{L^2}^2 + \nu\int_0^t \|\Delta\mathbf{u}(s)\|_{L^2}^2 \, ds &\le \|\nabla\mathbf{u}_0\|_{L^2}^2 \exp\frac{C}{\nu^7}\int_0^t \|\mathbf{u}(s)\|_{L^4}^8 \, ds \\ &\quad + \frac{C}{\nu}\int_0^t \left(\exp\frac{C}{\nu^7}\int_s^t \|\mathbf{u}(\sigma)\|_{L^4}^8 \, d\sigma\right)\|\mathbf{f}(s)\|_{L^2}^2 \, ds \, . \end{aligned} \tag{11}$$

Combining (7) and Lemma 1 with $p = 4$, and (11), we see that the quantity

$$\|\mathbf{u}(t)\|_{H^1}^2 + \int_0^t \nu[\|\nabla\mathbf{u}(s)\|_{L^2}^2 + \|\Delta\mathbf{u}(s)\|_{L^2}^2] \, ds$$

remains finite. However, by (10) and the fact that $L^\infty_{loc}(\mathbb{R}^+, L^2(\Omega)) \subset L^2_{loc}(\mathbb{R}^+, L^2(\Omega))$ we have that

$$\|\mathbf{u}(t)\|_{H^1}^2 + \int_0^t \nu\|\mathbf{u}(s)\|\|_{H^2(\Omega)}^2 \, ds$$

also remains finite. □

4. Final remarks

In this paper, we have focused on the existence of global solutions for an equation similar to the incompressible Navier-Stokes equation. The main feature of this equation is the existence of an infinite number of conserved quantities $\|\mathbf{u}\|_{L^p}$ in the ideal (non-viscous) case without forcing. This property is not only responsible for the existence of global solutions but should show up in the statistics of intermittent turbulent fluctuations. Work in this direction is in progress.

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