Nonreflexive Banach SSD spaces

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Abstract

In this paper, we unify the theory of SSD spaces, part of the theory of strongly representable multifunctions, and the theory of the equivalence of various classes of maximally monotone multifunctions.

0 Introduction

In this paper, we unify three different lines of investigation: the theory of SSD spaces as expounded in [11] and [13], part of the theory of strongly representable multifunctions as expounded in [15] and [4], and the equivalence of various classes of maximally monotone multifunctions, as expounded in [5].

The purely algebraic concepts of *SSD space* and *q–positive set* are introduced in Definition 1.2. These were originally defined in [11], and the development of the theory was continued in [13]. Apart from the fact that we write "$\mathcal{P}$" instead of "pos", we use the notation of the latter of these references. We show in Lemma 1.9 how certain proper convex functions f on an SSD space lead to a *q–positive* set, $\mathcal{P}(f)$. In Definition 1.10, we define the *intrinsic conjugate*, $f^{@}$, of a proper convex function on an SSD space, and we end Section 1 by proving in Lemma 1.11 a simple, but useful, property of intrinsic conjugates.

In Definition 2.1, we introduce the concept of a *Banach SSD space*, which is an SSD space with a Banach space structure satisfying the compatibility conditions (2.1.1) and (2.1.2). A proper convex function on a Banach SSD space may be a *VZ function*, which is introduced in Definition 2.5. Our main result on VZ functions, established in Theorem 2.9(c,d), is that *if f is a lower semicontinuous VZ function then $\mathcal{P}(f)$ is* maximally *q–positive, $f^{@}$ is also a VZ function, and $\mathcal{P}\big(f^{@}\big) = \mathcal{P}(f)$.* Lemma 2.7(b) is an important stepping–stone to Theorem 2.9. In Definition 2.12 and Lemma 2.13, we introduce and discuss the properties of various convex functions on a Banach SSD space and its dual, and show in Theorem 2.15(c) that if f is a lower semicontinuous VZ function on a Banach SSD then there is a whole family of VZ functions h associated with f such that $\mathcal{P}(h) = \mathcal{P}(f)$.

If E is a nonzero Banach space then it is shown in Examples 1.4, 2.3, and 2.4 that $E \times E^*$ is a Banach SSD space under various different norms. We show in Section 3 how the definitions and results of Section 2 specialize to this case. Theorem 3.1 extends some concepts and results from [1] and [5]. The definition of VZ function involves the norm of B in an essential way. Looking ahead, we will see in Theorem 5.3 that there is a large class of norms on $E \times E^*$ for which the classes of VZ functions coincide. This follows from the analysis in Section 4, which we will now discuss.

In Definition 4.1, we introduce the concept of a *Banach SSD dual space*, which is the dual of a Banach SSD space which has an SSD structure in its own right, satisfying the compatibility conditions (4.1.1) and (4.1.2). In this situation, a proper convex function on the (original) Banach SSD space may be an *MAS function*, which is introduced in Definition 4.8. The main result here is Theorem 4.9(c), in which we prove that, under the *$\widetilde{p}$–density* condition (4.2.1), a function is an MAS function if, and only if, it is a VZ function. The main stepping stone to Theorem 4.9 is Lemma 4.7, which relies on Rockafellar's formula for the conjugate of the sum of two convex functions.

The subtlety of the analysis outlined in the previous paragraph is that definition of MAS function does not use the norm of B explicitly — it only uses the knowledge of B^*. In a certain sense, the analysis of Section 2 is *isometric*, while the analysis of Section 4 is *isomorphic*, though it would be a mistake to push this analogy too far because, despite the fact that the definition of MAS function does not use the norm of B, the conditions (4.1.2) and (4.2.1) referred to above do use the norm very strongly.

In Section 5, we show how the results of Section 4 specialize to the $E \times E^*$ case. In Theorem 5.5, we show how the *negative alignment* analysis introduced in [10, Section 8, pp. 274–280] and [13, Section 42, pp. 161–167] can be used to obtain, and in some cases strengthen, results from [4] and [15]. In Theorem 5.8, we generalize some equivalencies from [5, Theorem 1.2]. In particular, we give a proof of the very nice result from [5] that a maximally monotone multifunction is strongly representable if, and only if, it is of type (NI).

At one point in this paper, we will use the Fenchel–Moreau theorem for a not necessarily Hausdorff locally convex space. For the convenience of the reader, we give a proof of this result in the Appendix, Section 6.

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1 SSD spaces

We first introduce the concepts of an SSD space and q–positive set. As pointed out in the introduction, these were introduced in [11] and [13]. The first of these references has a detailed discussion of the finite dimensional case.

Definition 1.1. If X is a nonzero vector space and $f\colon\ X \to]{-\infty},\infty]$, we write $\operatorname{dom} f$ for the set $\big\{x \in X\colon\ f(x) \in \mathbb{R}\big\}$. $\operatorname{dom} f$ is the *effective domain* of f. We say that f is *proper* if $\operatorname{dom} f \neq \emptyset$. We write $\mathcal{PC}(X)$ for the set of all proper convex functions from X into $]{-\infty},\infty]$. If X is a nonzero Banach space, we write $\mathcal{PCLSC}(X)$ for the set

$$\{f \in \mathcal{PC}(X)\colon\ f \text{ is lower semicontinuous on } X\},$$

and $\mathcal{PCLSC}^*(X^*)$ for the set

$$\{f \in \mathcal{PC}(X^*)\colon\ f \text{ is } w(X^*,X)\text{–lower semicontinuous on } X^*\}.$$

Definition 1.2. We will say that $\big(B, \lfloor\cdot,\cdot\rfloor\big)$ is a *symmetrically self–dual space (SSD space)* (if there is no risk of confusion, we will say simply "B is an SSD space") if B is a nonzero real vector space and $\lfloor\cdot,\cdot\rfloor\colon B \times B \to \mathbb{R}$ is a symmetric bilinear form. We define the quadratic form q on B by $q(b) := \frac{1}{2}\lfloor b, b\rfloor$. Let $A \subset B$. We say that A is *q–positive* if $A \neq \emptyset$ and

$$b, c \in A \Longrightarrow q(b-c) \geq 0.$$

We say that A is *maximally q–positive* if A is q–positive and A is not properly contained in any other q–positive set. We make the elementary observation that if $b \in B$ and $q(b) \geq 0$ then the linear span $\mathbb{R}b$ of $\{b\}$ is q–positive.

We now give some examples of SSD spaces and their associated q–positive sets.

Example 1.3. Let B be a Hilbert space with inner product $(b,c)\mapsto\langle b,c\rangle$ and $T\colon B\to B$ be a self–adjoint linear operator. Then B is an SSD space with $\lfloor b,c\rfloor := \langle Tb,c\rangle$, and then $q(b)=\frac{1}{2}\langle Tb,b\rangle$. Here are three special cases of this example:

(a) If, for all $b\in B$, $Tb=b$ then $\lfloor b,c\rfloor := \langle b,c\rangle$, $q(b)=\frac{1}{2}\|b\|^2$ and every subset of B is q–positive

(b) If, for all $b\in B$, $Tb=-b$ then $\lfloor b,c\rfloor := -\langle b,c\rangle$, $q(b)=-\frac{1}{2}\|b\|^2$ and the q–positive sets are the singletons.

(c) If $B=\mathbb{R}^3$ and $T(b_1,b_2,b_3)=(b_2,b_1,b_3)$ then

$$\big\lfloor (b_1,b_2,b_3),(c_1,c_2,c_3)\big\rfloor := b_1c_2+b_2c_1+b_3c_3,$$

and $q(b_1,b_2,b_3)=b_1b_2+\frac{1}{2}b_3^2$. Here, If M is any nonempty monotone subset of $\mathbb{R}\times\mathbb{R}$ (in the obvious sense) then $M\times\mathbb{R}$ is a q–positive subset of B. The set $\mathbb{R}(1,-1,2)$ is a q–positive subset of B which is not contained in a set $M\times\mathbb{R}$ for any monotone subset of $\mathbb{R}\times\mathbb{R}$. The helix $\big\{(\cos\theta,\sin\theta,\theta)\colon\theta\in\mathbb{R}\big\}$ is a q–positive subset of B, but if $0<\lambda<1$ then the helix $\big\{(\cos\theta,\sin\theta,\lambda\theta)\colon\theta\in\mathbb{R}\big\}$ is not.

Example 1.4. Let E be a nonzero Banach space and $B:=E\times E^*$. For all $b=(x,x^*)$ and $c=(y,y^*)\in B$, we set $\lfloor b,c\rfloor := \langle x,y^*\rangle+\langle y,x^*\rangle$. Then B is an SSD space and

$$q(b)=\tfrac{1}{2}\big[\langle x,x^*\rangle+\langle x,x^*\rangle\big]=\langle x,x^*\rangle.$$

Consequently, if $b=(x,x^*)$ and $c=(y,y^*)\in B$ then

$$\langle x-y,x^*-y^*\rangle=q(x-y,x^*-y^*)=q\big((x,x^*)-(y,y^*)\big)=q(b-c).$$

Thus if $A\subset B$ then A is q–positive exactly when A is a nonempty monotone subset of B in the usual sense, and A is maximally q–positive exactly when A is a maximally monotone subset of B in the usual sense. We point out that any finite dimensional SSD space of the form described here must have *even* dimension. Thus cases of Example 1.3 with finite odd dimension cannot be of this form.

Example 1.5. $\mathbb{R}^3$ is *not* an SSD space with

$$\big\lfloor (b_1,b_2,b_3),(c_1,c_2,c_3)\big\rfloor := b_1c_2+b_2c_3+b_3c_1.$$

(The bilinear form $\lfloor\cdot,\cdot\rfloor$ is not symmetric.)

Lemma 1.6. *Let B be an SSD space, $f\in\mathcal{PC}(B)$, $f\ge q$ on B and $b,c\in B$. Then*

$$-q(b-c)\le\Big[\sqrt{(f-q)(b)}+\sqrt{(f-q)(c)}\Big]^2.$$

Proof. We can and will suppose that $0\le(f-q)(b)<\infty$ and $0\le(f-q)(c)<\infty$. Let $\sqrt{(f-q)(b)}<\beta<\infty$ and $\sqrt{(f-q)(c)}<\gamma<\infty$, so that $\beta^2+q(b)>f(b)$ and

$\gamma^2 + q(c) > f(c)$. Then

$$\begin{aligned}\beta\gamma + \frac{\gamma q(b) + \beta q(c)}{\beta+\gamma} &= \frac{\gamma}{\beta+\gamma}\big(\beta^2 + q(b)\big) + \frac{\beta}{\beta+\gamma}\big(\gamma^2 + q(c)\big)\\ &> \frac{\gamma}{\beta+\gamma} f(b) + \frac{\beta}{\beta+\gamma} f(c) \ge f\left(\frac{\gamma b + \beta c}{\beta+\gamma}\right)\\ &\ge q\left(\frac{\gamma b + \beta c}{\beta+\gamma}\right) = \frac{\gamma^2 q(b) + \gamma\beta\lfloor b,c\rfloor + \beta^2 q(c)}{(\beta+\gamma)^2}.\end{aligned}$$

Clearing of fractions, we obtain

$$(\beta+\gamma)^2\beta\gamma + (\beta+\gamma)\big(\gamma q(b) + \beta q(c)\big) > \gamma^2 q(b) + \gamma\beta\lfloor b,c\rfloor + \beta^2 q(c),$$

from which $(\beta+\gamma)^2\beta\gamma > -\beta\gamma q(b) + \beta\gamma\lfloor b,c\rfloor - \beta\gamma q(c) = -\beta\gamma q(b-c)$. If we now divide by $\beta\gamma$, we obtain $(\beta+\gamma)^2 > -q(b-c)$, and the result follows by letting $\beta \to \sqrt{(f-q)(b)}$ and $\gamma \to \sqrt{(f-q)(c)}$. □

Remark 1.7. It follows from Lemma 1.6 and the Cauchy–Schwarz inequality that

$$-q(b-c) \le 2(f-q)(b) + 2(f-q)(c).$$

In the situation of Example 1.4, we recover [15, Proposition 1].

Definition 1.8. If B be an SSD space, $f \in \mathcal{PC}(B)$ and $f \ge q$ on B, we write

$$\mathcal{P}(f) := \big\{b \in B\colon\ f(b) = q(b)\big\}.$$

The following result is suggested by Burachik–Svaiter, [1, Theorem 3.1, pp. 2381–2382] and Penot, [7, Proposition 4(h)$\Longrightarrow$(a), pp. 860–861].

Lemma 1.9. *Let B be an SSD space, $f \in \mathcal{PC}(B)$, $f \ge q$ on B and $\mathcal{P}(f) \ne \emptyset$. Then $\mathcal{P}(f)$ is a q–positive subset of B.*

Proof. This is immediate from Lemma 1.6. □

We now introduce a concept of conjugate that is intrinsic to an SSD space without any topological conditions.

Definition 1.10. If B is an SSD space and $f \in \mathcal{PC}(B)$, we write $f^{@}$ for the Fenchel conjugate of f with respect to the pairing $\lfloor\cdot,\cdot\rfloor$, that is to say,

$$\text{for all } c \in B, \qquad f^{@}(c) := \sup\nolimits_{b\in B}\big[\lfloor b,c\rfloor - f(b)\big].$$

Our next result represents an improvement of the result proved in [13, Lemma 19.12, p. 82], and uses a disguised differentiability argument. See Remark 1.12 below for another proof of Lemma 1.11, due to Constantin Zălinescu.

Lemma 1.11. *Let B be an SSD space, $f \in \mathcal{PC}(B)$ and $f \ge q$ on B. Then:*

$$(a) \qquad a \in \mathcal{P}(f) \text{ and } b \in B \quad \Longrightarrow \quad \lfloor b, a \rfloor \le q(a) + f(b).$$

$$(b) \qquad a \in \mathcal{P}(f) \quad \Longrightarrow \quad f^{@}(a) = q(a).$$

Proof. Let $a \in \mathcal{P}(f)$ and $b \in B$. Let $\lambda \in]0,1[$. For simplicity in writing, let $\mu := 1 - \lambda \in]0,1[$. Then

$$\lambda^2 q(b) + \lambda\mu\lfloor b, a \rfloor + \mu^2 q(a) = q\big(\lambda b + \mu a\big) \le f(\lambda b + \mu a)$$
$$\le \lambda f(b) + \mu f(a) = \lambda f(b) + \mu q(a).$$

Thus $\lambda^2 q(b) + \lambda\mu\lfloor b, a \rfloor \le \lambda f(b) + \lambda\mu q(a)$. We now obtain (a) by dividing by λ and letting $\lambda \to 0$. Now let $a \in \mathcal{P}(f)$. From (a), $b \in B \Longrightarrow \lfloor a, b \rfloor - f(b) \le q(a)$, and it follows by taking the supremum over $b \in B$ that $f^{@}(a) \le q(a)$. On the other hand, $f^{@}(a) \ge \lfloor a, a \rfloor - f(a) = 2q(a) - q(a) = q(a)$, completing the proof of (b). $\square$

Remark 1.12. The author is grateful to Constantin Zălinescu for pointing out to him the following alternative proof of Lemma 1.11(a). From Lemma 1.6, with c replaced by a, $-q(b) + \lfloor b, a \rfloor - q(a) = -q(b - a) \le (f - q)(b)$. Thus $\lfloor b, a \rfloor - q(a) \le f(b)$, as required.

2 Banach SSD spaces

Definition 2.1. We say that B is a *Banach SSD space* if B is an SSD space and $\|\cdot\|$ is a norm on B with respect to which B is a Banach space with norm–dual B^*,

$$\tfrac{1}{2}\|\cdot\|^2 + q \ge 0 \text{ on } B \tag{2.1.1}$$

and there exists $\iota \in L(B, B^*)$ such that

$$\text{for all } b, c \in B, \quad \big\langle b, \iota(c)\big\rangle = \lfloor b, c \rfloor, \quad \big(\text{from which } \big|\lfloor b, c \rfloor\big| \le \|\iota\|\|b\|\|c\|\big). \tag{2.1.2}$$

Then, for all $d, e \in B$,

$$|q(d) - q(e)| = \tfrac{1}{2}\big|\lfloor d, d \rfloor - \lfloor e, e \rfloor\big| = \tfrac{1}{2}\big|\lfloor d - e, d + e \rfloor\big| \le \tfrac{1}{2}\|\iota\|\|d - e\|\|d + e\|. \tag{2.1.3}$$

We define the continuous convex functions g and p on B by $g := \frac{1}{2}\|\cdot\|^2$ and $p := g + q$, so that $p \ge 0$ on B. Since $p(0) = 0$, in fact

$$\inf\nolimits_B p = 0. \tag{2.1.4}$$

Also, for all $d, e \in B$, $|g(d) - g(e)| = \frac{1}{2}\big|\|d\| - \|e\|\big|\big(\|d\| + \|e\|\big) \le \frac{1}{2}\|d - e\|\big(\|d\| + \|e\|\big)$. Combining this with (2.1.3), for all $d, e \in B$,

$$|p(d) - p(e)| \le \tfrac{1}{2}\big(1 + \|\iota\|\big)\|d - e\|\big(\|d\| + \|e\|\big). \tag{2.1.5}$$

(2.1.2) implies that, for all $f \in \mathcal{PC}(B)$ and $c \in B$, $f^{@}(c) = \sup_{b \in B}\big[\big\langle b, \iota(c)\big\rangle - f(b)\big] = f^*\big(\iota(c)\big)$, that is to say,

$$f^{@} = f^* \circ \iota \text{ on } B. \tag{2.1.6}$$

Remark 2.2. Example 1.3 is a Banach SSD space provided that $\|T\| \le 1$. This is the case with (a), (b) and (c) of Example 1.3.

Example 2.3. We now continue our discussion of Example 1.4. We suppose that $B = E \times E^*$ and $\big(B, \|\cdot\|\big)$ is a Banach SSD space such that $B^* = E^{**} \times E^*$, under the pairing

$$\langle b, c^*\rangle := \langle x, y^*\rangle + \langle x^*, y^{**}\rangle \quad \big(b = (x, x^*) \in B,\ c^* = (y^{**}, y^*) \in B^*\big). \tag{2.3.1}$$

We recall that, for all $(x, x^*) \in B$, $q(x, x^*) = \langle x, x^*\rangle$. It is clear that, for all $(x, x^*) \in B$, $\iota(x, x^*) := (\widehat{x}, x^*)$ where $\widehat{x}$ is the canonical image of x in E^{**}. We note that if $\big(B, \|\cdot\|\big)$ is a Banach SSD space and $\|\cdot\|'$ is a larger norm on B such that $\big(B, \|\cdot\|'\big)^* = E^{**} \times E^*$ then $\big(B, \|\cdot\|'\big)$ is also a Banach SSD space.

Example 2.4. We now discuss some specific examples of the above concepts. Here it is convenient to introduce a parameter $\tau > 0$. (τ stands for "torsion".) Then $E \times E^*$ is a Banach SSD space if we use the norm $\|(x, x^*)\|_{1,\tau} := \frac{1}{\sqrt{2}}\big(\tau\|x\| + \|x^*\|/\tau\big)$ or $\|(x, x^*)\|_{2,\tau} := \sqrt{\tau^2\|x\|^2 + \|x^*\|^2/\tau^2}$ or $\|(x, x^*)\|_{\infty,\tau} := \sqrt{2}\big(\tau\|x\| \vee \|x^*\|/\tau\big)$. (These are arranged in order of increasing size.) Then the dual norm of $\big(B, \|\cdot\|_{1,\tau}\big)$ is given by $\|(y^{**}, y^*)\|_{\infty,\tau} := \sqrt{2}\big(\tau\|y^{**}\| \vee \|y^*\|/\tau\big)$, the dual norm of $\big(B, \|\cdot\|_{2,\tau}\big)$ is given by $\|(y^{**}, y^*)\|_{2,\tau} := \sqrt{\tau^2\|y^{**}\|^2 + \|y^*\|^2/\tau^2}$, and the dual norm of $\big(B, \|\cdot\|_{\infty,\tau}\big)$ is given by $\|(y^{**}, y^*)\|_{1,\tau} := \frac{1}{\sqrt{2}}\big(\tau\|y^{**}\| + \|y^*\|/\tau\big)$.

Definition 2.5. Let X be a vector space and $h, k\colon X \to\,]-\infty, \infty]$. The inf–convolution of h and k is defined by $(h \mathbin{\nabla} k)(x) := \inf_{y\in X}\big[h(y) + k(x - y)\big]$ $(x \in X)$. It is clear that

$$\inf\nolimits_X k = 0 \quad\Longrightarrow\quad \inf\nolimits_X\big[h \mathbin{\nabla} k\big] = \inf\nolimits_X h. \tag{2.5.1}$$

Now let $\big(B, \|\cdot\|\big)$ be a Banach SSD space and $f \in \mathcal{PC}(B)$. We say that f is a *VZ function* $\big($*with respect to* $\|\cdot\|\big)$ if

$$(f - q) \mathbin{\nabla} p = 0 \text{ on } B. \tag{2.5.2}$$

It follows from (2.1.4) and (2.5.1) that

$$\text{if } f \text{ is a VZ function with respect to } \|\cdot\| \text{ then } \inf\nolimits_B[f - q] = 0. \tag{2.5.3}$$

"VZ" stands for "Voisei–Zălinescu", since (2.5.2) is an extension to Banach SSD spaces of a condition introduced in [15, Proposition 3].

Definition 2.6. Let A be a subset of a Banach SSD space B. We say that A is *p–dense* if, for all $c \in B$, $\inf p(c - A) = 0$.

We now come to our main results on Banach SSD spaces. Lemma 2.7(b) is interesting since it tells us that we can determine whether f is a VZ function by inspecting $\mathcal{P}(f)$.

Lemma 2.7. *Let B be a Banach SSD space and $f \in \mathcal{PCLSC}(B)$.*
(a) *Let f be a VZ function. Then $\mathcal{P}(f)$ is a q–positive subset of B and*

$$c \in B \quad\Longrightarrow\quad \operatorname{dist}(c, \mathcal{P}(f)) \le \sqrt{2}\sqrt{(f - q)(c)}. \tag{2.7.1}$$

(b) *The following three conditions are equivalent:*
(i) *f is a VZ function.*
(ii) *$f \ge q$ on B and, for all $c \in B$ there exists a bounded sequence $\{a_n\}_{n\ge1}$ of elements of $\mathcal{P}(f)$ such that* $\lim_{n\to\infty} p(c - a_n) = 0$.
(iii) *$f \ge q$ on B and $\mathcal{P}(f)$ is p–dense.*

Proof. (a) (2.5.3) implies that $f \ge q$ on B, and so $\mathcal{P}(f)$ is defined. Since (2.7.1) is trivial if $c \in B \setminus \operatorname{dom} f$, we can and will suppose that $c \in \operatorname{dom} f$. Let $\varepsilon \in \,]0,1[$. We first prove that there exists a Cauchy sequence $\{b_n\}_{n\ge 1}$ such that, for all $n \ge 1$,

$$(f-q)(b_n) \le (f-q)(c)/16^n \quad \text{and} \quad \|c - b_n\| \le (1+\varepsilon)\sqrt{2}\sqrt{(f-q)(c)}. \tag{2.7.2}$$

Since we can take $b_n = c$ if $(f-q)(c) = 0$, we can and will suppose that

$$\alpha := \sqrt{(f-q)(c)} > 0. \tag{2.7.3}$$

Let $\lambda := \varepsilon/(3+\varepsilon) \in \,]0,1/4[$ and write $b_0 := c$. Then we can choose inductively $b_1, b_2, \ldots \in B$ such that, for all $n \ge 1$, $(f-q)(b_n) + p(b_{n-1} - b_n) \le \lambda^{2n}\alpha^2$. It follows from this and (2.5.3) that,

$$\text{for all } n \ge 1, \qquad p(b_{n-1} - b_n) \le \lambda^{2n}\alpha^2, \tag{2.7.4}$$

and, combining with (2.1.4),

$$\text{for all } n \ge 0, \qquad (f-q)(b_n) \le \lambda^{2n}\alpha^2 \le \alpha^2/16^n. \tag{2.7.5}$$

Substituting the first inequality of (2.7.5) into Lemma 1.6, for all $n \ge 1$,

$$-q(b_{n-1} - b_n) \le \left[\sqrt{(f-q)(b_{n-1})} + \sqrt{(f-q)(b_n)}\right]^2 \le (1+\lambda)^2\lambda^{2n-2}\alpha^2.$$

Consequently, since $g(b_{n-1} - b_n) = p(b_{n-1} - b_n) - q(b_{n-1} - b_n)$, (2.7.4) gives,

$$\text{for all } n \ge 1, \qquad g(b_{n-1} - b_n) \le (1+\lambda)^2\lambda^{2n-2}\alpha^2 + \lambda^{2n}\alpha^2 \le (1+2\lambda)^2\lambda^{2n-2}\alpha^2,$$

and so, for all $n \ge 1$, $\|b_{n-1} - b_n\| \le \sqrt{2}(1+2\lambda)\lambda^{n-1}\alpha$. Adding up this inequality for $n = 1, \ldots, m$, we derive that, for all $m \ge 1$, $\|c - b_m\| \le \sqrt{2}(1+2\lambda)\alpha/(1-\lambda)$. Since $(1+2\lambda)/(1-\lambda) = 1+\varepsilon$, this and (2.7.5) give (2.7.2). Now set $a = \lim_n b_n$, so that $\|c - a\| \le (1+\varepsilon)\sqrt{2}\sqrt{(f-q)(c)}$. (2.7.5) and the lower semicontinuity of $f - q$ now imply that $(f-q)(a) \le 0$, that is to say, $a \in \mathcal{P}(f)$. Since $\operatorname{dom} f \ne \emptyset$, it follows that $\mathcal{P}(f) \ne \emptyset$ and so, from Lemma 1.9, $\mathcal{P}(f)$ is a q–positive subset of B. We also have

$$\operatorname{dist}(c, \mathcal{P}(f)) \le (1+\varepsilon)\sqrt{2}\sqrt{(f-q)(c)},$$

and so if we now let $\varepsilon \to 0$, we obtain (2.7.1). This completes the proof of (a).

(b) Suppose first that (i) is satisfied. (2.5.3) implies that $f \ge q$ on B. Let $c \in B$. We choose inductively $b_1, b_2, \ldots \in B$ such that, for all $n \ge 1$,

$$f(b_n) + g(c - b_n) + q(c) - \lfloor c, b_n \rfloor = (f-q)(b_n) + p(c - b_n) < 1/n^2.$$

Consequently, using (2.1.4) and (2.1.2), for all $n \ge 1$,

$$(f-q)(b_n) < 1/n^2, \; p(c - b_n) < 1/n^2 \tag{2.7.6}$$

and

$$f(b_n) + g(c - b_n) + q(c) - \|\iota\|\|c\|\|b_n\| < 1/n^2. \tag{2.7.7}$$

Since $f \in \mathcal{PCLSC}(B)$, f dominates a continuous affine function, and so (2.7.7) and the usual coercivity argument imply that $K := \sup_{n\ge1}\|b_n\| < \infty$. From (a) and (2.7.6), there exists $a_n \in \mathcal{P}(f)$ such that $\|a_n - b_n\| \le \sqrt{2}/n$. Now, from (2.1.5), for all $n \ge 1$,

$$\begin{aligned}|p(c-a_n) - p(c-b_n)| &\le \tfrac{1}{2}(1+\|\iota\|)\|a_n - b_n\|(2\|c\| + \|a_n\| + \|b_n\|)\\ &\le \tfrac{1}{2}(1+\|\iota\|)\big(2\|c\| + \big(K+\sqrt{2}\big) + K\big)\sqrt{2}/n.\end{aligned}$$

Thus $\lim_{n\to\infty}\big[p(c-a_n) - p(c-b_n)\big] = 0$, and (ii) follows by combining this with (2.7.6). It is trivial that (ii)$\Longrightarrow$(iii). Suppose, finally, that (iii) is satisfied. Then, for all $c \in B$,

$$\big((f-q)\mathbin{\nabla} p\big)(c) \le \inf_{a\in\mathcal{P}(f)}\big[(f-q)(a) + p(c-a)\big] = \inf p(c - \mathcal{P}(f)) = 0,$$

from which $(f-q)\mathbin{\nabla} p \le 0$ on B. On the other hand, since $f - q \ge 0$ on B and, from (2.1.4), $p \ge 0$ on B, we have $(f-q)\mathbin{\nabla} p \ge 0$ on B. Thus f is a VZ function, giving (i). $\square$

Lemma 2.8. *Let A be a closed, p–dense and q–positive subset of a Banach SSD space B.*

(a) *For all $c \in B$, $\inf q(c-A) \le 0$ and $\operatorname{dist}(c,A) \le \sqrt{2}\sqrt{-\inf q(c-A)}$.*

(b) *Let $h \in \mathcal{PC}(B)$, $h \ge q$ on B, and $\mathcal{P}(h) \supset A$. Then h is a VZ function.*

(c) *A is a maximally q–positive subset of B.*

Proof. (a) Let $c \in B$. Then $\inf g(c-A) + \inf q(c-A) \le \inf p(c-A) = 0$. Thus $\frac{1}{2}\operatorname{dist}(c,A)^2 = \inf g(c-A) \le -\inf q(c-A)$, from which (a) is an immediate consequence.

(b) Clearly, $\mathcal{P}(h)$ is also p–dense, and it follows as in Lemma 2.7(b)$\big($(iii)$\Longrightarrow$(i)$\big)$ (which does not use any semicontinuity) that h is a VZ function, which gives (b).

(c) We suppose that $c \in B$ and $\inf q(c-A) \ge 0$, and we must prove that $c \in A$. From (a), in fact $\inf q(c-A) = 0$ and $\operatorname{dist}(c,A) = 0$. Since A is closed, $c \in A$. This completes the proof of (c). $\square$

Theorem 2.9. *Let B be a Banach SSD space and $f \in \mathcal{PCLSC}(B)$ be a VZ function. Then:*

(a) *For all $c \in B$, $\inf q(c-\mathcal{P}(f)) \le 0$ and $\operatorname{dist}(c,\mathcal{P}(f)) \le \sqrt{2}\sqrt{-\inf q(c-\mathcal{P}(f))}$.*

(b) *Let $h \in \mathcal{PC}(B)$, $h \ge q$ on B, and $\mathcal{P}(h) \supset \mathcal{P}(f)$. Then h is a VZ function.*

(c) *$\mathcal{P}(f)$ is a maximally q–positive subset of B.*

(d) *$f^{@} \in \mathcal{PCLSC}(B)$, $f^{@}$ is a VZ function and $\mathcal{P}\big(f^{@}\big) = \mathcal{P}(f)$.*

Proof. (a), (b) and (c) are immediate from Lemma 2.7(b)$\big($(i)$\Longrightarrow$(iii)$\big)$ and the corresponding parts of Lemma 2.8.

(d) Let $c \in B$. Then, since $q \le p$ on B, Definition 1.10 gives

$$q(c) - f^{@}(c) = \inf_{b\in B}\big[f(b) - \lfloor b,c\rfloor + q(c)\big] = \big((f-q)\mathbin{\nabla} q\big)(c) \le \big((f-q)\mathbin{\nabla} p\big)(c) = 0,$$

and so $f^{@} \ge q$ on B. It now follows from Lemma 1.11(b) that $\mathcal{P}\big(f^{@}\big) \supset \mathcal{P}(f)$, and so (b) and (c) imply that $f^{@}$ is a VZ function and $\mathcal{P}\big(f^{@}\big) = \mathcal{P}(f)$. Since $\mathcal{P}(f) \ne \emptyset$, it is evident that $f^{@} \in \mathcal{PCLSC}(B)$. ($\mathcal{P}(f)$ is closed because f is lower semicontinuous.) $\square$

Remark 2.10. In general, Theorem 2.9(a) is strictly stronger than Lemma 2.7(a). While this can be proved directly, we will see in Remark 2.17 that it follows easily from the properties of the Φ–functions. We will also see in Remark 2.17 that the constant $\sqrt{2}$ in (2.7.1) is sharp.

The proof of Theorem 2.9 relies heavily on the lower semicontinuity of f. We will show in Corollary 2.11 below that part of Theorem 2.9(d) can be recovered even if f is not assumed to be lower semicontinuous.

Corollary 2.11. *Let B be a Banach SSD space and $f \in \mathcal{PC}(B)$ be a VZ function. Then $f^{@} \in \mathcal{PCLSC}(B)$, $f^{@}$ is a VZ function and $\mathcal{P}\big(f^{@}\big)$ is a maximally q–positive subset of B.*

Proof. Let $\overline{f}$ be the lower semicontinuous envelope of f. Since q is continuous and $f \ge q$ on B, it follows that $f \ge \overline{f} \ge q$ on B. Thus, from (2.1.4),

$$0 = (f - q)\,\nabla\, p \ge (\overline{f} - q)\,\nabla\, p \ge 0\,\nabla\, p = 0 \text{ on } B,$$

and so $\overline{f}$ is a VZ function. Since $\overline{f} \in \mathcal{PCLSC}(B)$, Theorem 2.9(d) implies that $\overline{f}^{@}$ is a VZ function also. It is well known that $\overline{f}^{*} = f^{*}$ on B^{*} thus, composing with ι and using (2.1.6), $\overline{f}^{@} = f^{@}$ on B. The result now follows from Theorem 2.9(d,c), with f replaced by $f^{@}$. □

Definition 2.12. Let B be a Banach SSD space and A be a nonempty q–positive subset of B. We define the function $\Theta_A\colon\ B^{*} \to]{-\infty}, \infty]$ by: for all $b^{*} \in B^{*}$,

$$\Theta_A(b^{*}) := \sup\nolimits_{a \in A} \big[\langle a, b^{*}\rangle - q(a)\big].$$

We define the function $\Phi_A\colon\ B \to]{-\infty}, \infty]$ by $\Phi_A := \Theta_A \circ \iota$.
We define the function ${}^{*}\Theta_A\colon\ B \to]{-\infty}, \infty]$ by: for all $c \in B$,

$${}^{*}\Theta_A(c) := \sup\nolimits_{b^{*} \in B^{*}} \big[\langle c, b^{*}\rangle - \Theta_A(b^{*})\big].$$

We collect together in Lemma 2.13 some elementary properties of Θ_A, Φ_A, ${}^{*}\Theta_A$, and $\Phi_A^{@}$. The properties of Φ_A and $\Phi_A^{@}$ have already appeared in [13].

Lemma 2.13. *Let B be a Banach SSD space and A be a nonempty q–positive subset of B.*

(a) *For all $b \in B$, $\Phi_A(b) = \sup_{a \in A} \big[\lfloor a, b\rfloor - q(a)\big] = q(b) - \inf q(b - A)$.*

(b) *$\Phi_A \in \mathcal{PCLSC}(B)$ and $\Phi_A = q$ on A.*

(c) *$\Theta_A \in \mathcal{PCLSC}^{*}(B^{*})$.*

(d) *$({}^{*}\Theta_A)^{*} = \Theta_A$ and $({}^{*}\Theta_A)^{@} = \Phi_A$.*

(e) *${}^{*}\Theta_A \le q$ on A. Consequently, ${}^{*}\Theta_A \in \mathcal{PCLSC}(B)$.*

(f) *${}^{*}\Theta_A \ge {\Phi_A}^{@} \ge \Phi_A \vee q$ on B.*

(g) *${}^{*}\Theta_A = {\Phi_A}^{@} = q$ on A.*

(h) *Let A be maximally q–positive. Then ${}^{*}\Theta_A \ge {\Phi_A}^{@} \ge \Phi_A \ge q$ on B and $A \subset \mathcal{P}\big({}^{*}\Theta_A\big)$.*

(i) *Let A be maximally q–positive. Then $\mathcal{P}\big({}^{*}\Theta_A\big) = \mathcal{P}\big({\Phi_A}^{@}\big) = \mathcal{P}\big(\Phi_A\big) = A$.*

Proof. (a) is immediate from (2.1.2), (b) from (a), and (c) from (b) and the definition of Θ_A.

The first assertion in (d) follows from (c) and the Fenchel–Moreau theorem for the locally convex space $\big(B^*, w(B^*,B)\big)$, while the second assertion follows from the first by composing with ι, and using (2.1.6) and the definition of Φ_A.

(e) Let $a \in A$. The definition of Θ_A implies that, for all $b^* \in B^*$, $\langle a, b^*\rangle - \Theta_A(b^*) \le q(a)$. Taking the supremum over $b^* \in B^*$, ${}^*\Theta_A(a) \le q(a)$, as required.

(f) Let $c \in B$. Then, from (2.1.2), the definition of Φ_A and (b),

$$\begin{aligned}{}^*\Theta_A(c) &\ge \sup\nolimits_{b\in B}\big[\langle c,\iota(b)\rangle - \Theta_A(\iota(b))\big]\\ &= \sup\nolimits_{b\in B}\big[\lfloor c,b\rfloor - \Phi_A(b)\big] \quad \big(= {\Phi_A}^{@}(c)\big)\\ &\ge \big[\lfloor c,c\rfloor - \Phi_A(c)\big] \vee \sup\nolimits_{a\in A}\big[\lfloor c,a\rfloor - \Phi_A(a)\big]\\ &= \big[2q(c) - \Phi_A(c)\big] \vee \sup\nolimits_{a\in A}\big[\lfloor c,a\rfloor - q(a)\big]\\ &= \big[2q(c) - \Phi_A(c)\big] \vee \Phi_A(c).\end{aligned}$$

Now if $\Phi_A(c) = \infty$ then obviously $\big[2q(c) - \Phi_A(c)\big] \vee \Phi_A(c) \ge q(c)$, while if $\Phi_A(c) \in \mathbb{R}$ then $\big[2q(c) - \Phi_A(c)\big] \vee \Phi_A(c) \ge \frac{1}{2}\big[2q(c) - \Phi_A(c)\big] + \frac{1}{2}\Phi_A(c) = q(c)$. Thus ${\Phi_A}^{@}(c) \ge \Phi(c) \vee q(c)$. This completes the proof of (f).

(g) is immediate from (e) and (f).

(h) In this case, for all $b \in B \setminus A$, there exists $a \in A$ such that $q(b-a) < 0$, and so $\inf q(b - A) < 0$. Thus, from (a), $\Phi_A > q$ on $B \setminus A$. Combining this with (b), $\Phi_A \ge q$ on B and $\mathcal{P}\big(\Phi_A\big) = A$. Thus (h) follows from (f) and (g).

It is clear from (h) that $A \subset \mathcal{P}\big({}^*\Theta_A\big) \subset \mathcal{P}\big({\Phi_A}^{@}\big) \subset \mathcal{P}\big(\Phi_A\big)$, and so (i) follows from the maximality of A. $\square$

Remark 2.14. We will see in (2.15.2) and (2.15.3) that ${}^*\Theta$ and ${\Phi_A}^{@}$ are both "upper limiting" functions in various situations, so the question arises whether these two functions are identical. If ${}^*\Theta_A = {\Phi_A}^{@}$ then ${}^*\Theta_A$ is obviously $w(B,B)$–lower semicontinuous. If, conversely, ${}^*\Theta_A$ is $w(B,B)$–lower semicontinuous then the Fenchel–Moreau theorem for the (possibly nonhausdorff) locally convex space $\big(B, w(B,B)\big)$ and Lemma 2.13(d) imply that ${}^*\Theta_A = \big({}^*\Theta_A\big)^{@@} = {\Phi_A}^{@}$. The author is grateful to Constantin Zălinescu for the following example showing that, in general, the functions ${}^*\Theta$ and ${\Phi_A}^{@}$ are not identical. Let B be a Banach space, $\lfloor\cdot,\cdot\rfloor = 0$ on $B \times B$ and A be a nonempty proper closed convex subset of B. Then ${}^*\Theta_A$ is the indicator function of A and ${\Phi_A}^{@} = 0$ on B. We do not know what the situation is if A is maximally q–positive, or in the special situation of Example 2.4. For the convenience of the reader, we will give a proof of the Fenchel–Moreau theorem for nonhausdorff locally convex spaces in Theorem 6.1.

Theorem 2.15. *Let B be a Banach SSD space.*
(a) *Let $f \in \mathcal{PCLSC}(B)$, $f \ge q$ on B and $A := \mathcal{P}(f) \ne \emptyset$. Then ${}^*\Theta_A \ge f \ge \Phi_A$ on B and ${\Phi_A}^* \ge f^* \ge \Theta_A$ on B^*.*

(b) *Let A be a maximally q–positive subset of B, $h \in \mathcal{PC}(B)$ and ${}^*\Theta_A \ge h \ge \Phi_A$ on B. Then $h \ge q$ on B, $h^{@} \ge q$ on B and $\mathcal{P}(h) = \mathcal{P}\big(h^{@}\big) = A$.*

(c) *Let $f \in \mathcal{PCLSC}(B)$ be a VZ function and $A := \mathcal{P}(f)$. Then*

$$^*\Theta_A \ge f \ge \Phi_A \ge q \text{ on } B \quad\text{and}\quad {\Phi_A}^* \ge f^* \ge \Theta_A \text{ on } B^*. \tag{2.15.1}$$

Now let $h \in \mathcal{PC}(B)$ and $^*\Theta_A \ge h \ge \Phi_A$ on B. Then h and $h^@$ are VZ functions. In particular, $\mathcal{P}\big({}^*\Theta_A\big) = \mathcal{P}\big({\Phi_A}^@\big) = \mathcal{P}\big(\Phi_A\big) = \mathcal{P}(f)$ and Φ_A, ${\Phi_A}^@$ and $^*\Theta_A$ are all VZ functions.

Proof. (a) Let $b \in B$ and $a \in \mathcal{P}(f)$. Then, from Lemma 1.11(a), $f(b) \ge \lfloor b, a\rfloor - q(a)$. Taking the supremum over $a \in \mathcal{P}(f)$ and using Lemma 2.13(a), $f(b) \ge \Phi_A(b)$. Thus $f \ge \Phi_A$ on B and, taking conjugates, ${\Phi_A}^* \ge f^*$ on B^*. Now, for all $b^* \in B^*$,

$$\begin{aligned} f^*(b^*) &= \sup\nolimits_{b\in B}\big[\langle b, b^*\rangle - f(b)\big] \ge \sup\nolimits_{a\in\mathcal{P}(f)}\big[\langle a, b^*\rangle - f(a)\big] \\ &= \sup\nolimits_{a\in\mathcal{P}(f)}\big[\langle a, b^*\rangle - q(a)\big] = \Theta_A(b^*). \end{aligned}$$

Thus $f^* \ge \Theta_A$ on B^*. Taking conjugates and using the Fenchel–Moreau theorem for the normed space B, $^*\Theta_A \ge f$ on B. This completes the proof of (a).

(b) From Lemma 2.13(h),

$$^*\Theta_A \ge h \ge \Phi_A \ge q \text{ on } B, \quad\text{from which}\quad \mathcal{P}\big({}^*\Theta_A\big) \subset \mathcal{P}(h) \subset \mathcal{P}\big(\Phi_A\big). \tag{2.15.2}$$

It is clear from our assumptions that ${\Phi_A}^@ \ge h^@ \ge ({}^*\Theta_A)^@$ on B. If we now combine this with Lemma 2.13(d,h), we derive that

$${\Phi_A}^@ \ge h^@ \ge \Phi_A \ge q \text{ on } B, \quad\text{from which}\quad \mathcal{P}\big({\Phi_A}^@\big) \subset \mathcal{P}\big(h^@\big) \subset \mathcal{P}\big(\Phi_A\big). \tag{2.15.3}$$

(b) now follows from (2.15.2), (2.15.3) and Lemma 2.13(i).

(c) The assertions about f follow from (2.5.3), Theorem 2.9(c), (a) and Lemma 2.13(h), the assertions about h and $h^@$ follow from Theorem 2.9(c,b) and (b), and then the assertions about Φ_A, ${\Phi_A}^@$ and $^*\Theta_A$ follow from Theorem 2.9(c) and Lemma 2.13(h,i). □

In Theorem 2.16 below, we show that $^*\Theta_A$ has a certain maximal property. This result was motivated by results originally proved by Burachik and Svaiter in [1] for maximally monotone multifunctions.

Theorem 2.16. *Let A be a nonempty q–positive subset of a Banach SSD space B and*

$$\sigma_A := \sup\big\{h\colon\ h \in \mathcal{PCLSC}(B),\ h \le q \text{ on } A\big\}.$$

Then $^*\Theta_A = \sigma_A$ on B.

Proof. Let $h \in \mathcal{PCLSC}(B)$ and $h \le q$ on A. The Fenchel Young inequality implies that, for all $b^* \in B^*$ and $a \in A$, $h^*(b^*) \ge \langle a, b^*\rangle - h(a) \ge \langle a, b^*\rangle - q(a)$. Thus, taking the supremum over $a \in A$, $h^*(b^*) \ge \Theta_A(b^*)$. In other words, $h^* \ge \Theta_A$ on B^*. Taking conjugates and using the Fenchel Moreau theorem for the normed space B, $^*\Theta_A \ge h$ on B. It follows by taking the supremum over h that $^*\Theta_A \ge \sigma_A$ on B. On the other hand, it is clear from Lemma 2.13(e) that $\sigma_A \ge {}^*\Theta_A$ on B. □

Remark 2.17. Let B be a Banach SSD space and $f \in \mathcal{PCLSC}(B)$ be a VZ function. We know from Theorem 2.15(c) that $\mathcal{P}\big(\Phi_{\mathcal{P}(f)}\big) = \mathcal{P}(f)$, $\Phi_{\mathcal{P}(f)}$ is a VZ function and $\Phi_{\mathcal{P}(f)} \le f$ on B. Thus Lemma 2.7(a) implies that, for all $c \in B$,

$$\operatorname{dist}(c, \mathcal{P}(f)) = \operatorname{dist}\big(c, \mathcal{P}\big(\Phi_{\mathcal{P}(f)}\big)\big) \le \sqrt{2}\sqrt{\big(\Phi_{\mathcal{P}(f)} - q\big)(c)} \le \sqrt{2}\sqrt{(f-q)(c)}.$$

From Lemma 2.13(a), $\big(\Phi_{\mathcal{P}(f)} - q\big)(c) = \Phi_{\mathcal{P}(f)}(c) - q(c) = -\inf q(c - \mathcal{P}(f))$, thus we have

$$\operatorname{dist}(c, \mathcal{P}(f)) \le \sqrt{2}\sqrt{-\inf q(c - \mathcal{P}(f))} \le \sqrt{2}\sqrt{(f-q)(c)}.$$

This shows that Theorem 2.9(a) is as least as strong as Lemma 2.7(a). Now let $E := \mathbb{R}$ and B be the Banach SSD space $\mathbb{R}^2$ as in Example 2.4, using the norm $\|\cdot\|_{2,1}$. Define $f \in \mathcal{PCLSC}(B)$ by $f(x_1, x_2) := \frac{1}{2}(x_1^2 + x_2^2)$. Then $(f-q)(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2) - x_1 x_2 = \frac{1}{2}(x_1 - x_2)^2$ and $p(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2) + x_1 x_2 = \frac{1}{2}(x_1 + x_2)^2$. Let $c := (z_1, z_2) \in B$ and $b := \big(\frac{1}{2}(z_1 + z_2), \frac{1}{2}(z_1 + z_2)\big) \in B$. Then $(f-q)(b) = 0$ and $p(c - b) = 0$. Consequently, f is a VZ function. Now $\mathcal{P}(f)$ is the diagonal of $\mathbb{R}^2$ and so, by direct computation, for all $c = (x_1, x_2) \in \mathbb{R}^2$, $-\inf q(c - \mathcal{P}(f)) = \frac{1}{4}(x_1 - x_2)^2$. Since $\frac{1}{4}(x_1 - x_2)^2 < \frac{1}{2}(x_1 - x_2)^2$ when $x_1 \ne x_2$, Theorem 2.9(a) is strictly stronger than Lemma 2.7(a) in this case.

Now let $h := \Phi_{\mathcal{P}(f)}$. Lemma 2.13(a) gives us that, for all $(x_1, x_2) \in B$,

$$\sqrt{(h-q)(x_1, x_2)} = \sqrt{\tfrac{1}{4}(x_1 - x_2)^2} = \tfrac{1}{2}|x_1 - x_2|.$$

On the other hand, by direct computation, $\operatorname{dist}\big((x_1, x_2), \mathcal{P}(h)\big) = \frac{1}{\sqrt{2}}|x_1 - x_2|$. Thus the constant $\sqrt{2}$ in (2.7.1) is sharp. The genesis of this argument and example can be found in the results of Martínez-Legaz and Théra in [6].

Remark 2.18. We note that the inequalities for B in (2.15.1) have four functions, while the inequality for B^* has only three. The reason for this is that we do not have a function on B^* that plays the role that the function q plays on B. We will introduce such a function in Definition 4.1.

3 Applications of Section 2 to $E \times E^*$

In this section, we suppose that E is a nonzero Banach space, and follow the notation of Example 2.3. Let A be a nonempty monotone subset of $E \times E^*$. In this case, the definitions and results obtained in Definition 2.12 and Lemma 2.13 specialize as follows. The function $\Theta_A \in \mathcal{PCLSC}^*(E^{**} \times E^*)$ is defined by:

$$\Theta_A(x^{**}, x^*) := \sup\nolimits_{(s,s^*) \in A} \big[\langle s, x^* \rangle + \langle s^*, x^{**} \rangle - \langle s, s^* \rangle\big].$$

The function $\Phi_A \in \mathcal{PCLSC}(E \times E^*)$ is defined by:

$$\Phi_A(x, x^*) = \sup\nolimits_{(s,s^*) \in A} \big[\langle x, s^* \rangle + \langle s, x^* \rangle - \langle s, s^* \rangle\big].$$

Φ_A is the *Fitzpatrick function* of A, first introduced in [2], which has been discussed by many authors in recent years. The function ${}^*\Theta_A \in \mathcal{PCLSC}(E \times E^*)$ is defined by:

$$^*\Theta_A(y, y^*) := \sup_{(x^{**}, x^*) \in E^{**} \times E^*} \big[\langle y, x^* \rangle + \langle y^*, x^{**} \rangle - \Theta_A(x^{**}, x^*)\big].$$

Then $({}^*\Theta_A)^* = \Theta_A$ and $({}^*\Theta_A)^@ = \Phi_A$. Furthermore,

$$^*\Theta_A \ge {\Phi_A}^@ \ge \Phi_A \vee q \text{ on } E \times E^* \quad \text{and} \quad {}^*\Theta_A = {\Phi_A}^@ = \Phi_A = q \text{ on } A.$$

If $f \in \mathcal{PC}(E \times E^*)$ and $f \ge q$ on $E \times E^*$ then we define $\mathcal{M}f$ to be the monotone set $\{(x, x^*) \in E \times E^*\colon f(x, x^*) = \langle x, x^* \rangle\}$. $\mathcal{M}f$ is identical with $\mathcal{P}(f)$ as in Definition 1.8, but the "$\mathcal{M}$" notation seems more appropriate in this case. Continuing with the consequences of Lemma 2.13, we have:

$$^*\Theta_A \ge {\Phi_A}^@ \ge \Phi_A \ge q \text{ on } E \times E^* \quad \text{and} \quad \mathcal{M}\big({}^*\Theta_A\big) = \mathcal{M}\big({\Phi_A}^@\big) = \mathcal{M}\big(\Phi_A\big) = A.$$

The following results are then immediate from Theorems 2.15 and 2.16. The expression $\sup\big\{h\colon\ h \in \mathcal{PCLSC}(E \times E^*),\ h \le q \text{ on } A\big\}$ that appears in Theorem 3.1(b) was first introduced by Burachik and Svaiter in [1] (for A maximally monotone) and further studied by Marques Alves and Svaiter in [5]. The analysis of Lemma 2.13 and Theorem 2.15 suggests that the natural framework in which to consider these results is that of Banach SSD spaces.

Theorem 3.1. *Let E be a nonzero Banach space, $E \times E^*$ be normed as in Example 2.3, and A be a nonempty monotone subset of $E \times E^*$.*
(a) *Let $f \in \mathcal{PCLSC}(E \times E^*)$, $f \ge q$ on $E \times E^*$ and $A := \mathcal{M}f \ne \emptyset$. Then*

$$^*\Theta_A \ge f \ge \Phi_A \text{ on } E \times E^*, \quad \text{and} \quad {\Phi_A}^* \ge f^* \ge \Theta_A \text{ on } E^{**} \times E^*.$$

(b) *Let A be maximally monotone, $h \in \mathcal{PC}(E \times E^*)$ and ${}^*\Theta_A \ge h \ge \Phi_A$ on $E \times E^*$. Then $h \ge q$ on $E \times E^*$, $h^@ \ge q$ on $E \times E^*$ and $\mathcal{M}h = \mathcal{M}\big(h^@\big) = A$.*
(c) *Let $f \in \mathcal{PCLSC}(E \times E^*)$ be a VZ function and $A := \mathcal{M}f$. Then*

$$^*\Theta_A \ge f \ge \Phi_A \ge q \text{ on } E \times E^* \quad \text{and} \quad {\Phi_A}^* \ge f^* \ge \Theta_A \text{ on } E^{**} \times E^*$$

Now let $h \in \mathcal{PC}(E \times E^*)$ and ${}^*\Theta_A \ge h \ge \Phi_A$ on $E \times E^*$. Then h and $h^@$ are VZ functions. In particular, $\mathcal{M}\big({}^*\Theta_A\big) = \mathcal{M}\big({\Phi_A}^@\big) = \mathcal{M}\big(\Phi_A\big) = A$, and Φ_A, ${\Phi_A}^@$ and ${}^*\Theta_A$ are all VZ functions.

(d) $$^*\Theta_A = \sup\big\{h\colon\ h \in \mathcal{PCLSC}(E \times E^*),\ h \le q \text{ on } A\big\}.$$

4 Banach SSD dual spaces

Definition 4.1. Let $(B, \|\cdot\|)$ be a Banach SSD space and $(B^*, \|\cdot\|)$ be the norm–dual of B. We say that $(B^*, \lceil\cdot,\cdot\rceil)$ is a *Banach SSD dual of* B if $\lceil\cdot,\cdot\rceil\colon\ B^*\times B^*\to\mathbb{R}$ is a symmetric bilinear form,

$$\text{for all } b\in B \text{ and } c^*\in B^*,\qquad \lceil\iota(b),c^*\rceil = \langle b,c^*\rangle. \tag{4.1.1}$$

Writing $\widetilde{q}(c^*) := \frac{1}{2}\lceil c^*,c^*\rceil$ and $\widetilde{p}(c^*) := \frac{1}{2}\|c^*\|^2 + \widetilde{q}(c^*)$, we suppose also that

$$\widetilde{p}\ge 0 \text{ on } B^*. \tag{4.1.2}$$

Now if we take $c^* = \iota(c)$ in (4.1.1) and use (2.1.2), we obtain

$$\text{for all } b,c\in B,\qquad \lceil\iota(b),\iota(c)\rceil = \big\langle b,\iota(c)\big\rangle = \lfloor b,c\rfloor, \tag{4.1.3}$$

from which

$$\widetilde{q}\circ\iota = q. \tag{4.1.4}$$

It is easy to see from these definitions that,

$$\text{for all } b^*\in B^*,\qquad \Theta_A(b^*) = \widetilde{q}(b^*) - \inf\widetilde{q}(b^*-\iota(A)). \tag{4.1.5}$$

This should be compared with Lemma 2.13(a).

Definition 4.2. Let $(B, \|\cdot\|)$ be a Banach SSD space and $(B^*, \lceil\cdot,\cdot\rceil)$ be a Banach SSD dual of B. We say that $\iota(B)$ is $\widetilde{p}$*–dense in* B^* if

$$\text{for all } b^*\in B^*,\quad \inf\widetilde{p}\big(b^*-\iota(B)\big) = 0. \tag{4.2.1}$$

Remark 4.3. In Example 1.3 with $\|T\|\le 1$ (see also Remark 2.2), for all $c\in B$, $\iota(c) = Tc$. Suppose now that T^2 is the identity on B. Since $B^* = B$,

$$\text{for all } b\in B \text{ and } c^*\in B^* = B,\qquad \lfloor\iota(b),c^*\rfloor = \lfloor Tb,c^*\rfloor = \langle T^2b,c^*\rangle = \langle b,c^*\rangle.$$

Thus (4.1.1) is satisfied with $\lceil\cdot,\cdot\rceil := \lfloor\cdot,\cdot\rfloor$, and so $\big(B,\lfloor\cdot,\cdot\rfloor\big)$ is its own Banach SSD dual. We note that T^2 is the identity on B in (a), (b) and (c) of Example 1.3.

Example 4.4. We now continue our discussion of Examples 1.4, 2.3 and 2.4. We recall that $B = E\times E^*$, $B^* = E^{**}\times E^*$ and, for all $(x,x^*)\in B$, $\iota(x,x^*) = (\widehat{x},x^*)$. We define the symmetric bilinear form $\lceil\cdot,\cdot\rceil\colon\ B^*\times B^*\to\mathbb{R}$ by

$$\lceil b^*,c^*\rceil := \langle y^*,x^{**}\rangle + \langle x^*,y^{**}\rangle\quad \big(b^* = (x^{**},x^*)\in B^*,\ c^* = (y^{**},y^*)\in B^*\big).$$

It is then easily checked from (2.3.1) that (4.1.1) is satisfied and, for all $c^* = (y^{**},y^*)\in B^*$, $\widetilde{q}(c^*) = \frac{1}{2}\big[\langle y^*,y^{**}\rangle + \langle y^*,y^{**}\rangle\big] = \langle y^*,y^{**}\rangle$. We now discuss briefly the limitations of this definition. Let $E := \mathbb{R}$ and B be the SSD space $\mathbb{R}^2$ as in Example 1.4, using the norm $2\|\cdot\|_{2,1}$. As we observed in Example 2.4, $\mathbb{R}^2$ is a Banach SSD space under $\|\cdot\|_{2,1}$, and consequently also a Banach SSD space under the larger norm $2\|\cdot\|_{2,1}$. Since ι is the identity on $\mathbb{R}^2$, (4.1.3) implies that $\lceil\cdot,\cdot\rceil := \lfloor\cdot,\cdot\rfloor$. Now the norm on $B^* = B$ dual to $2\|\cdot\|_{2,1}$ is

$\frac{1}{2}\|\cdot\|_{2,1}$. Since $\widetilde{p}(1,-1)=\frac{1}{8}(2)+(1)(-1)=-\frac{3}{4}<0$, B does not admit a Banach SSD dual. We now return to the general case. If $c^*=(y^{**},y^*)\in B^*$ then

$$\tfrac{1}{2}\|c^*\|_{1,\tau}^2+\widetilde{q}(c^*)\ge\tfrac{1}{4}\big(\tau\|y^{**}\|+\|y^*\|/\tau\big)^2-\|y^{**}\|\|y^*\|=\tfrac{1}{4}\big(\tau\|y^{**}\|-\|y^*\|/\tau\big)^2\ge 0.$$

Consequently, $\big(B^*,\|\cdot\|_{1,\tau}\big)$ is a Banach SSD dual of $\big(B,\|\cdot\|_{\infty,\tau}\big)$. Since $\|\cdot\|_{\infty,\tau}\ge\|\cdot\|_{2,\tau}\ge\|\cdot\|_{1,\tau}$ on B^*, $\big(B^*,\|\cdot\|_{2,\tau}\big)$ is a Banach SSD dual of $\big(B,\|\cdot\|_{2,\tau}\big)$ and $\big(B^*,\|\cdot\|_{\infty,\tau}\big)$ is a Banach SSD dual of $\big(B,\|\cdot\|_{1,\tau}\big)$. Next, if $b^*=(y^{**},y^*)\in B^*$ and $\varepsilon>0$ then there exists $z^*\in E^*$ such that $\|z^*\|\le\|\tau y^{**}\|$ and $\langle z^*,\tau y^{**}\rangle\ge\|\tau y^{**}\|^2-\varepsilon$. Let $c:=(0,y^*+\tau z^*)\in B$, so that $b^*-\iota(c)=(y^{**},-\tau z^*)\in B^*$. Thus

$$\begin{aligned}\tfrac{1}{2}\|b^*-\iota(c)\|_{\infty,\tau}^2+\widetilde{q}(b^*-\iota(c))&=\big(\tau\|y^{**}\|\vee\|z^*\|\big)^2-\langle\tau z^*,y^{**}\rangle\\&=\big(\|\tau y^{**}\|\vee\|z^*\|\big)^2-\langle z^*,\tau y^{**}\rangle=\|\tau y^{**}\|^2-\langle z^*,\tau y^{**}\rangle\le\varepsilon.\end{aligned}$$

Consequently, if B is normed by $\|\cdot\|_{1,\tau}$ then $\iota(B)$ is $\widetilde{p}$–dense in B^*. Since $\|\cdot\|_{1,\tau}\le\|\cdot\|_{2,\tau}\le\|\cdot\|_{\infty,\tau}$ on B^*, the same is true if B is normed by $\|\cdot\|_{2,\tau}$ or $\|\cdot\|_{\infty,\tau}$.

We now recall Rockafellar's formula for the conjugate of a sum:

Lemma 4.5. *Let X be a nonzero real Banach space and $f\in\mathcal{PC}(X)$, and let $h\in\mathcal{PC}(X)$ be real–valued and continuous. Then, for all $x^*\in X^*$,*

$$(f+h)^*(x^*)=\min\nolimits_{y^*\in X^*}\big[f^*(y^*)+h^*(x^*-y^*)\big].$$

Proof. See Rockafellar, [8, Theorem 3(a), p. 85], Zălinescu, [16, Theorem 2.8.7(iii), p. 127], or [13, Corollary 10.3, p. 52]. □

Remark 4.6. [13, Theorem 7.4, p. 43] contains a version of the Fenchel duality theorem with a sharp lower bound on the functional obtained.

Our next result exhibits a certain pleasing symmetry between B and B^*.

Lemma 4.7. *Let B be a Banach SSD space with a Banach SSD dual B^* and $f\in\mathcal{PC}(B)$. Then $\big((f-q)\,\nabla\,p\big)+\big((f^*-\widetilde{q})\,\nabla\,\widetilde{p}\big)\circ\iota=0$ on B.*

Proof. Let $c\in B$. Define h: $B\to\mathbb{R}$ by $h(b):=g(c-b)$. Then, by direct computation using the fact that g is an even function,

$$\text{for all } c^*\in B^*,\qquad h^*(c^*)=g^*(c^*)+\langle c,c^*\rangle. \tag{4.7.1}$$

Then, using (2.1.2), the continuity of h, Lemma 4.5, (4.7.1), (4.1.4) and the fact that, for all $c^*\in B^*$, $g^*(c^*)=\frac{1}{2}\|c^*\|^2$,

$$\begin{aligned}-\big((f-q)\,\nabla\,p\big)(c)&=\sup\nolimits_{b\in B}\big[-(f-q)(b)-p(c-b)\big]\\&=\sup\nolimits_{b\in B}\big[\langle b,\iota(c)\rangle-f(b)-h(b)\big]-q(c)=(f+h)^*\big(\iota(c)\big)-q(c)\\&=\min\nolimits_{b^*\in B^*}\big[f^*(b^*)+h^*\big(\iota(c)-b^*\big)\big]-q(c)\\&=\min\nolimits_{b^*\in B^*}\big[f^*(b^*)+g^*\big(\iota(c)-b^*\big)+\big\langle c,\iota(c)-b^*\big\rangle\big]-q(c)\\&=\min\nolimits_{b^*\in B^*}\big[f^*(b^*)+g^*\big(\iota(c)-b^*\big)-\lceil\iota(c),b^*\rceil+\widetilde{q}\big(\iota(c)\big)\big]\\&=\min\nolimits_{b^*\in B^*}\big[(f^*-\widetilde{q})(b^*)+\widetilde{p}\big(\iota(c)-b^*\big)\big]\\&=\big((f^*-\widetilde{q})\,\nabla\,\widetilde{p}\big)\big(\iota(c)\big).\end{aligned}$$

This completes the proof of Lemma 4.7. □

Definition 4.8. Let B be a Banach SSD space with Banach SSD dual B^* and $f \in \mathcal{PC}(B)$. We say that f is an *MAS function* if $f \ge q$ on B and $f^* \ge \widetilde{q}$ on B^*. This is an extension to Banach SSD spaces of the concept introduced by Marques Alves and Svaiter in [4] for the situation described in Example 4.4.

Theorem 4.9. *Let B be a Banach SSD space with Banach SSD dual B^* and $f \in \mathcal{PC}(B)$.*
(a) *Let f be an MAS function. Then f is a VZ function.*
(b) *Let $\iota(B)$ be $\widetilde{p}$–dense in B^* and f be a VZ function. Then f is an MAS function.*
(c) *Let $\iota(B)$ be $\widetilde{p}$–dense in B^*. Then f is a VZ function if, and only if, f is an MAS function.*

Proof. (a) We have $\big($using (2.1.4)$\big)$ $f - q \ge 0$ and $p \ge 0$ on B, and $\big($using (4.1.2)$\big)$, $f^* - \widetilde{q} \ge 0$ and $\widetilde{p} \ge 0$ on B^*. Thus $(f - q) \nabla p \ge 0$ and $\big((f^* - \widetilde{q}) \nabla \widetilde{p}\big) \circ \iota \ge 0$ on B. It now follows from Lemma 4.7 that f is a VZ function.

(b) Let $b^* \in B^*$ and $c \in B$. Then, from Lemma 4.7 again,

$$(f^* - \widetilde{q})(b^*) + \widetilde{p}\big(\iota(c) - b^*\big) \ge \big((f^* - \widetilde{q}) \nabla \widetilde{p}\big)\big(\iota(c)\big) = -\big((f - q) \nabla p\big)(c) = 0.$$

Taking the infimum over $c \in B$ and using (4.2.1), $(f^* - \widetilde{q})(b^*) \ge 0$ on B^*. Since this holds for all $b^* \in B^*$, f is an MAS function.

(c) is immediate from (a) and (b). $\square$

In Theorem 4.10, we shift the emphasis from the properties of a given function $f \in \mathcal{PC}(B)$ to the properties of a given maximally q–positive subset A of B. We note that (a), (b), (c), (f) and (g) of Theorem 4.10 do not involve any functions on B other than those introduced in Definition 2.12.

Theorem 4.10. *Let $(B, \|\cdot\|)$ be a Banach SSD space with Banach SSD dual B^* and $\iota(B)$ be $\widetilde{p}$–dense in B^*. Let A be a maximally q–positive subset of B. Then the following conditions are equivalent:*
(a) *For all $b^* \in B^*$, $\inf \widetilde{q}\big(b^* - \iota(A)\big) \le 0$.*
(b) *$\Theta_A \ge \widetilde{q}$ on B^*.*
(c) *${\Phi_A}^* \ge \widetilde{q}$ on B^*.*
(d) *There exists an MAS function $f \in \mathcal{PCLSC}(B)$ such that $\mathcal{P}(f) = A$.*
(e) *There exists a VZ function $f \in \mathcal{PCLSC}(B)$ such that $\mathcal{P}(f) = A$.*
(f) *Φ_A is a VZ function.*
(g) *${}^*\Theta_A$ is a VZ function.*
(b_1) *If $h \in \mathcal{PC}(B)$ and ${}^*\Theta_A \ge h$ on B then $h^* \ge \widetilde{q}$ on B^*.*
(b_2) *If $h \in \mathcal{PCLSC}(B)$ and ${}^*\Theta_A \ge h \ge \Phi_A$ on B then $h^* \ge \widetilde{q}$ on B^*.*
(c_1) *There exists $h \in \mathcal{PCLSC}(B)$ such that ${}^*\Theta_A \ge h \ge \Phi_A$ on B and $h^* \ge \widetilde{q}$ on B^*.*
(c_2) *There exists $h \in \mathcal{PC}(B)$ such that $h \ge \Phi_A$ on B and $h^* \ge \widetilde{q}$ on B^*.*

Proof. The equivalence of (a) and (b) is immediate from (4.1.5). Taking the conjugate of the inequality in Lemma 2.13(f) and using Lemma 2.13(d) implies that $\Phi_A{}^* \ge \Theta_A$ on B^*. Thus (b)$\Longrightarrow$(c). If (c) is satisfied then Lemma 2.13(b,h,i) give (d) with $f := \Phi_A$. It is immediate from Theorem 4.9(c) that (d)$\Longrightarrow$(e). If (e) is satisfied then Theorem 2.15(c) gives (f) and (g). If (f) or (g) is satisfied then, from Theorem 4.9(c) again, Φ_A or ${}^*\Theta_A$ (respectively) are MAS functions. The first of these possibilities implies (c), and the second of these possibilities together with Lemma 2.13(d) implies (b). Thus (a), (b), (c), (d), (e), (f) and (g) are equivalent.

If $h \in \mathcal{PC}(B)$ and ${}^*\Theta_A \ge h$ on B then, from Lemma 2.13(d), $h^* \ge ({}^*\Theta_A)^* = \Theta_A$ on B^*, thus (b) implies (b_1). It is trivial that (b_1) implies (b_2), and it follows by taking $h := {}^*\Theta_A$ and using Lemma 2.13(f,d) that (b) is true. Thus (b), (b_1) and (b_2) are equivalent.

If (c) is true then (c_1) follows by taking $h := \Phi_A$ and using Lemma 2.13(b,f). It is trivial that (c_1) implies (c_2). If $h \in \mathcal{PC}(B)$ and $h \ge \Phi_A$ on B then $\Phi_A{}^* \ge h^*$ on B^*, and (c_2) implies (c). Thus (c), (c_1) and (c_2) are equivalent. $\square$

5 Applications of Section 4 to $E \times E^*$

In this section, we suppose that E is a nonzero Banach space, and show how the results of Section 4 can be applied to Example 4.4. We refer the reader to Section 3 for the definitions of Θ_A and Φ_A in this case.

Remark 5.1. Before proceeding with our analysis, we make some remarks about the essential difference between the concepts of MAS function introduced in Definition 4.8 and VZ function introduced in Definition 2.5. As observed in Example 4.4, we have $(E \times E^*)^* = E^{**} \times E^*$ and $\widetilde{q}(x^{**}, x^*) = \langle x^*, x^{**}\rangle$, so we have all the information needed to decide whether a function $f \in \mathcal{PC}(E \times E^*)$ is an MAS function. The situation with VZ functions is different since that involves the function g in an essential way, and this is determined by the precise norm we are using on $E \times E^*$. In order to clarify the situation, we make the following definition.

Definition 5.2. We say that the norm $\|\cdot\|$ on $E \times E^*$ is *special* if, for some $\tau > 0$, $\|\cdot\|$ is identical with one of the norms $\|\cdot\|_{1,\tau}$, $\|\cdot\|_{2,\tau}$ or $\|\cdot\|_{\infty,\tau}$ introduced in Example 2.4. As we pointed out in the comments in Example 4.4, if $E \times E^*$ is normed by a special norm then $\iota(E \times E^*)$ is $\widetilde{p}$–dense in $(E \times E^*)^*$.

Theorem 5.3. *Let E be a nonzero Banach space and $f \in \mathcal{PC}(E \times E^*)$ be a VZ function with respect to a given special norm on $E \times E^*$. Then f is a VZ function with respect to all special norms on $E \times E^*$.*

Proof. This is clear from the comments above and Theorem 4.9(c). $\square$

Definition 5.4. Let Let E be a nonzero Banach space and $f \in \mathcal{PC}(E \times E^*)$. We say that f is a *VZ function on $E \times E^*$* if f is a VZ function with respect to any one special norm on $E \times E^*$ or, equivalently, with respect to all special norms on $E \times E^*$. This is also equivalent to the statement that f is an MAS function.

Theorem 5.5(a) was obtained in [15, Theorem 8] under the VZ hypothesis and, in [4, Theorem 4.2(2)] under the MAS hypothesis.

Theorem 5.5(c) extends the result proved in [15, Corollary 25] that $\mathcal{M}f$ is of type (ANA).

Theorem 5.5(d) extends the result proved in [4, Theorem 4.2(2)].

Theorem 5.5(f) was obtained in [15, Corollary 7]. This is a very significant result, because maximally monotone sets A of $E \times E^*$ are known such that $\overline{\pi_{E^*}(A)}$ is not convex. $\big($The first such example was given by Gossez in [3, Proposition, p. 360]$\big)$. Thus $\big($as was first observed in [15]$\big)$ Theorem 5.5(f) implies that there exist maximally monotone sets A that are not of the form $\mathcal{M}f$ for any lower semicontinuous VZ function on $E \times E^*$ or, equivalently, not of the form $\mathcal{M}f$ for any lower semicontinuous MAS function on $E \times E^*$. Theorem 5.5(f) can also be proved directly from Lemma 2.7(a) rather than from the more circuitous argument given here.

The techniques used in Theorem 5.5 originated in the *negative alignment* analysis of [10, Section 8, pp. 274–280] and [13, Section 42, pp. 161–167].

Theorem 5.5. *Let E be a nonzero Banach space and $f \in \mathcal{PCLSC}(E \times E^*)$. Assume either that f is a VZ function on $E \times E^*$ or, equivalently, that f is an MAS function. Then:*

(a) *$\mathcal{M}f$ is a maximally monotone subset of $E \times E^*$.*

(b) *Let $(x, x^*) \in E \times E^*$ and $\alpha, \beta > 0$. Then there exists a unique value of $\omega \geq 0$ for which there exists a bounded sequence $\big\{(y_n, y_n^*)\big\}_{n\geq 1}$ of elements of $\mathcal{M}f$ such that,*

$$\lim_{n\to\infty} \|y_n - x\| = \alpha\omega, \quad \lim_{n\to\infty} \|y_n^* - x^*\| = \beta\omega \quad \textit{and} \quad \lim_{n\to\infty} \langle y_n - x, y_n^* - x^*\rangle = -\alpha\beta\omega^2.$$

(c) *Let $(x, x^*) \in E \times E^* \setminus \mathcal{M}f$ and $\alpha, \beta > 0$. Then there exists a bounded sequence $\big\{(y_n, y_n^*)\big\}_{n\geq 1}$ of elements of $\mathcal{M}f \cap \big[(E \setminus \{x\}) \times (E^* \setminus \{x^*\})\big]$ such that,*

$$\lim_{n\to\infty} \frac{\|y_n - x\|}{\|y_n^* - x^*\|} = \frac{\alpha}{\beta} \quad \textit{and} \quad \lim_{n\to\infty} \frac{\langle y_n - x, y_n^* - x^*\rangle}{\|y_n - x\|\|y_n^* - x^*\|} = -1. \tag{5.5.1}$$

In particular, $\mathcal{M}f$ is of type (ANA) $\big($see [13, Definition 36.11, p. 152]$\big)$.

(d) *Let $(x, x^*) \in E \times E^* \setminus \mathcal{M}f$, $\alpha, \beta > 0$ and $\inf_{(y,y^*)\in\mathcal{M}f}\langle y - x, y^* - x^*\rangle > -\alpha\beta$. Then there exists a bounded sequence $\big\{(y_n, y_n^*)\big\}_{n\geq 1}$ in $\mathcal{M}f \cap \big[(E \setminus \{x\}) \times (E^* \setminus \{x^*\})\big]$ such that* (5.5.1) *is satisfied, $\lim_{n\to\infty} \|y_n - x\| < \alpha$ and $\lim_{n\to\infty} \|y_n^* - x^*\| < \beta$. In particular, $\mathcal{M}f$ is of type (BR)* $\big($see [13, Definition 36.13, p. 153]$\big)$.

(e) *Let $(x, x^*) \in E \times E^* \setminus \mathcal{M}f$, $\alpha, \beta > 0$ and $f(x, x^*) < \langle x, x^*\rangle + \alpha\beta$. Then there exists a bounded sequence $\big\{(y_n, y_n^*)\big\}_{n\geq 1}$ of elements of $\mathcal{M}f \cap \big[(E \setminus \{x\}) \times (E^* \setminus \{x^*\})\big]$ such that* (5.5.1) *is satisfied, $\lim_{n\to\infty} \|y_n - x\| < \alpha$ and $\lim_{n\to\infty} \|y_n^* - x^*\| < \beta$.*

(f) *We define the* projection maps *$\pi_E\colon E \times E^* \to E$ and $\pi_{E^*}\colon E \times E^* \to E^*$ by $\pi_E(x, x^*) := x$ and $\pi_{E^*}(x, x^*) := x^*$. Then $\overline{\pi_E(\mathcal{M}f)} = \overline{\pi_E(\operatorname{dom} f)}$ and $\overline{\pi_{E^*}(\mathcal{M}f)} = \overline{\pi_{E^*}(\operatorname{dom} f)}$. Consequently, the sets $\overline{\pi_E(\mathcal{M}f)}$ and $\overline{\pi_{E^*}(\mathcal{M}f)}$ are convex.*

Proof. (a) is immediate from Theorem 2.9(c).

(b) Let $\tau := \sqrt{\beta/\alpha}$ and use the norm $\|\cdot\|_{\infty,\tau}$ on $E \times E^*$. Lemma 2.7(b) provides us with a bounded sequence $\big\{(y_n, y_n^*)\big\}_{n\ge 1}$ of elements of $\mathcal{M}f$ such that

$$\lim_{n\to\infty} \big[\beta\|y_n - x\|^2/\alpha \vee \alpha\|y_n^* - x^*\|^2/\beta + \langle y_n - x, y_n^* - x^*\rangle\big] = 0.$$

By passing to an appropriate subsequence, we can and will suppose that the three limits $\rho := \lim_{n\to\infty}\|y_n - x\|$, $\sigma := \lim_{n\to\infty}\|y_n^* - x^*\|$ and $\lim_{n\to\infty}\langle y_n - x, y_n^* - x^*\rangle$ all exist. Consequently, $\beta\rho^2/\alpha \vee \alpha\sigma^2/\beta + \lim_{n\to\infty}\langle y_n - x, y_n^* - x^*\rangle = 0$, from which

$$\beta\rho^2/\alpha \vee \alpha\sigma^2/\beta = -\lim\nolimits_{n\to\infty}\langle y_n - x, y_n^* - x^*\rangle \le \rho\sigma = \sqrt{\beta\rho^2/\alpha}\sqrt{\alpha\sigma^2/\beta}.$$

It follows easily from this that $\beta\rho^2/\alpha = \alpha\sigma^2/\beta$ and $\lim_{n\to\infty}\langle y_n - x, y_n^* - x^*\rangle = -\rho\sigma$. The first of these equalities implies that $\rho/\alpha = \sigma/\beta$. We take $\omega := \rho/\alpha = \sigma/\beta$, and it is immediate that ω has the required properties. The uniqueness of ω was established in [10, Theorem 8.4(b), p. 276] and [13, Theorem 42.2(b), pp. 163–164].

(c) Following on from (b), if $\omega = 0$ then $(\rho,\sigma) = (0,0)$, that is to say $\lim_{n\to\infty} y_n = x$ in E and $\lim_{n\to\infty} y_n^* = x^*$ in E^*. Since $\mathcal{M}f$ is closed, this would contradict the hypothesis that $(x, x^*) \notin \mathcal{M}f$. Thus $\omega > 0$, from which $\rho > 0$ and $\sigma > 0$. (c) now follows by truncating the sequences so that, for all n, $\|y_n - x\| > 0$ and $\|y_n^* - x^*\| > 0$.

(d) Continuing with the notation of (c), we have

$$-\alpha\beta < \inf\nolimits_{(y,y^*)\in\mathcal{M}f}\langle y - x, y^* - x^*\rangle \le \lim\nolimits_{n\to\infty}\langle y_n - x, y_n^* - x^*\rangle = -\rho\sigma,$$

from which $(\rho/\alpha)(\sigma/\beta) < 1$. Since $\rho/\alpha = \sigma/\beta$, in fact $\rho/\alpha < 1.$ and $\sigma/\beta < 1$, that is to say $\rho = \lim_{n\to\infty}\|y_n - x\| < \alpha$ and $\sigma = \lim_{n\to\infty}\|y_n^* - x^*\| < \beta$. This gives (d).

(e) is immediate from (d) and the comment in Remark 2.17 that, for all $(x, x^*) \in E \times E^*$, $-\inf_{(y,y^*)\in\mathcal{M}f}\langle y - x, y^* - x^*\rangle \le f(x, x^*) - \langle x, x^*\rangle$. .

(f) If $x \in \pi_E(\operatorname{dom} f)$ then there exists $x^* \in E^*$ such that $f(x, x^*) < \infty$, and so it follows from (e) that there exists $(y, y^*) \in \mathcal{M}f$ such that $\|y - x\| < 1/n$. Consequently, $x \in \overline{\pi_E(\mathcal{M}f)}$. Thus we have proved that $\pi_E(\operatorname{dom} f) \subset \overline{\pi_E(\mathcal{M}f)}$. On the other hand, $\mathcal{M}f \subset \operatorname{dom} f$, and so $\overline{\pi_E(\mathcal{M}f)} = \overline{\pi_E(\operatorname{dom} f)}$. We can prove in an exactly similar way that $\overline{\pi_{E^*}(\mathcal{M}f)} = \overline{\pi_{E^*}(\operatorname{dom} f)}$. The convexity of the sets $\overline{\pi_E(\mathcal{M}f)}$ and $\overline{\pi_{E^*}(\mathcal{M}f)}$ now follows immediately. $\square$

Remark 5.6. If we combine Theorem 2.9(a) (using the norm $\|\cdot\|_{2,1}$ on $E \times E^*$) with the comments made in the proof of Theorem 5.5(e) we obtain the following result: *Let E be a nonzero Banach space, $f \in \mathcal{PCLSC}(E \times E^*)$, and f be a VZ function on $E \times E^*$. Then, for all $(x, x^*) \in E \times E^*$,*

$$\begin{aligned}\inf\nolimits_{(y,y^*)\in\mathcal{M}f}\sqrt{\|y - x\|^2 + \|y^* - x^*\|^2} &\le \sqrt{2}\sqrt{-\inf\nolimits_{(y,y^*)\in\mathcal{M}f}\langle y - x, y^* - x^*\rangle}\\ &\le \sqrt{2}\sqrt{f(x, x^*) - \langle x, x^*\rangle}.\end{aligned}$$

This strengthens the result proved in [15, Theorem 4], namely that

$$\inf\nolimits_{(y,y^*)\in\mathcal{M}f}\sqrt{\|y - x\|^2 + \|y^* - x^*\|^2} \le 2\sqrt{f(x, x^*) - \langle x, x^*\rangle}.$$

As we observed in Remark 2.17, the constant $\sqrt{2}$ is sharp.

Definition 5.7. Let E be a nonzero Banach space and A be a nonempty monotone subset of $E \times E^*$. We say that A is *of type (NI)* if, for all $(x^{**}, x^*) \in E^{**} \times E^*$,

$$\inf_{(s,s^*)\in A} \langle x^* - s^*, x^{**} - \widehat{s}\rangle \le 0.$$

This concept was introduced in [9, Definition 10, p. 183]. We say that A is *strongly representable* if there exists $f \in \mathcal{PCLSC}(E \times E^*)$ such that $f \ge q$ on $E \times E^*$, $f^* \ge \widetilde{q}$ on $E^{**} \times E^*$ (i.e., f is a lower semicontinuous MAS function) and $\mathcal{M}f = A$. This concept was introduced and studied in [4], [5] and [15].

Theorem 5.8 was motivated by and extends that proved in [5, Theorem 1.2]. The most significant part of it is the fact that (a) implies (d) and (a) implies (e). In particular, if A is maximally monotone of type (NI), then the conclusions of Theorem 5.5(b–f) hold (with $\mathcal{M}f$ replaced by A). This leads to a substantial generalization of [10, Theorem 8.6, pp. 277–278] and [13, Theorem 42.6, pp. 163–164]. The fact that $\overline{\pi_E A}$ and $\overline{\pi_{E^*} A}$ are convex whenever A is of type (NI) was first proved by Zagrodny in [14].

Theorem 5.8. *Let E be a nonzero Banach space and A be a maximally monotone subset of $E \times E^*$. Then the following conditions are equivalent:*

(a) *A is of type (NI).*

(b) *For all $(x^{**}, x^*) \in E^{**} \times E^*$, $\sup_{(s,s^*)\in A} \big[\langle s, x^*\rangle + \langle s^*, x^{**}\rangle - \langle s, s^*\rangle\big] \ge \langle x^*, x^{**}\rangle$.*

(c) *For all $(x^{**}, x^*) \in E^{**} \times E^*$, $\sup_{(y,y^*)\in E\times E^*} \big[\langle y, x^*\rangle + \langle y^*, x^{**}\rangle - \Phi_A(y, y^*)\big] \ge \langle x^*, x^{**}\rangle$.*

(d) *A is strongly representable.*

(e) *There exists a lower semicontinuous VZ function on $E \times E^*$ such that $\mathcal{M}f = A$.*

(f) *Φ_A is a VZ function on $E \times E^*$.*

(g) *${}^*\Theta_A$ is a VZ function on $E \times E^*$.*

(b_1) *If $h \in \mathcal{PC}(E \times E^*)$ and ${}^*\Theta_A \ge h$ on $E \times E^*$ then, for all $(x^{**}, x^*) \in E^{**} \times E^*$,*

$$h^*(x^{**}, x^*) = \sup_{(y,y^*)\in E\times E^*} \big[\langle y, x^*\rangle + \langle y^*, x^{**}\rangle - h(y, y^*)\big] \ge \langle x^*, x^{**}\rangle. \tag{5.8.1}$$

(b_2) *If $h \in \mathcal{PCLSC}(E \times E^*)$ and ${}^*\Theta_A \ge h \ge \Phi_A$ on $E \times E^*$ then, for all $(x^{**}, x^*) \in E^{**} \times E^*$, (5.8.1) is satisfied.*

(c_1) *There exists $h \in \mathcal{PCLSC}(E \times E^*)$ such that ${}^*\Theta_A \ge h \ge \Phi_A$ on $E \times E^*$ and, for all $(x^{**}, x^*) \in E^{**} \times E^*$, (5.8.1) is satisfied.*

(c_2) *There exists $h \in \mathcal{PC}(E \times E^*)$ such that $h \ge \Phi_A$ on $E \times E^*$ and, for all $(x^{**}, x^*) \in E^{**} \times E^*$, (5.8.1) is satisfied.*

Proof. These results are all immediate from the corresponding parts of Theorem 4.10. □

6 Appendix: a nonhausdorff Fenchel–Moreau theorem

In Remark 2.14, we referred to the Fenchel–Moreau theorem for (possibly nonhausdorff) locally convex spaces. We shall give a proof of this result in Theorem 6.1. When we say that X is a *locally convex space*, we mean that X is a nonzero real vector space endowed with a topology compatible with its vector structure with a base of neighborhoods of 0 of the form $\big\{x \in X\colon\ S(x) \le 1\big\}_{S\in\mathcal{S}(X)}$, where $\mathcal{S}(X)$ is a family of seminorms on X such that if $S_1 \in \mathcal{S}(X)$ and $S_2 \in \mathcal{S}(X)$ then $S_1 \vee S_2 \in \mathcal{S}(X)$; and if $S \in \mathcal{S}(X)$ and $\lambda \ge 0$ then $\lambda S \in \mathcal{S}(X)$. If L is a linear functional on X then L is continuous if, and only if, there exists $S \in \mathcal{S}(X)$ such that $L \le S$ on X.

As an example of the construction above, we can suppose that X and Y are vector spaces paired by a bilinear form $\langle\cdot,\cdot\rangle$. Then $\big(X, w(X,Y)\big)$ is a locally convex space with determining family of seminorms $\big\{|\langle\cdot,y_1\rangle| \vee \cdots \vee |\langle\cdot,y_n\rangle|\big\}_{n\ge 1,\ y_1,\ldots,y_n\in Y}$.

The author is grateful to Constantin Zălinescu for showing him a proof of Theorem 6.1 based on the standard (Hausdorff) result and a quotient construction. The proof we give here is a simplification of the result on "Fenchel–Moreau points" of [12, Theorem 5.3, pp. 157–158] or [13, Theorem 12.2, pp. 59–60], which is also valid in the nonhausdorff setting.

Theorem 6.1. *Let X be a locally convex space and $f \in \mathcal{PC}(X)$ be lower semicontinuous. Write X^* for the set of continuous linear functionals on X. If $L \in X^*$, define $f^*(L) := \sup_X \big[L - f\big]$. Let $y \in X$. Then*

$$f(y) = \sup\nolimits_{L\in X^*}\big[L(y) - f^*(L)\big]. \tag{6.1.1}$$

Proof. Since, for all $L \in X^*$, $L(y) - f^*(L) = \inf_{x\in X}\big[L(y) - L(x) + f(x)\big] = (f \mathbin{\nabla} L)(y)$ and the inequality "$\ge$" in (6.1.1) is obvious from the definition of $f^*(L)$, we only have to prove that

$$f(y) \le \sup\nolimits_{L\in X^*}(f \mathbin{\nabla} L)(y)\big]. \tag{6.1.2}$$

Let $\lambda \in \mathbb{R}$ and $\lambda < f(y)$. Since f is proper, there exists $z \in \operatorname{dom} f$. Choose $Q \in \mathcal{S}(X)$ such that

$$Q(z - x) \le 1 \quad\Longrightarrow\quad f(x) > f(z) - 1 \tag{6.1.3}$$

and

$$Q(y - x) \le 1 \quad\Longrightarrow\quad f(x) > \lambda. \tag{6.1.4}$$

We first prove that

$$(f \mathbin{\nabla} Q)(z) \ge f(z) - 1. \tag{6.1.5}$$

To this end, let x be an arbitrary element of X. If $Q(z - x) \le 1$ then (6.1.3) implies that $f(x) + Q(z - x) \ge f(x) > f(z) - 1$. If, on the other hand, $Q(z - x) > 1$, let $\gamma := 1/Q(z - x) \in \,]0,1[$ and put $u := \gamma x + (1-\gamma)z$. Then $Q(z - u) = \gamma Q(z - x) = 1$ and so, from the convexity of f, and (6.1.3) with x replaced by u,

$$\gamma f(x) + (1-\gamma)f(z) \ge f\big(\gamma x + (1-\gamma)z\big) = f(u) > f(z) - 1.$$

Substituting in the formula for γ and clearing of fractions yields $f(x) + Q(z - x) \ge f(z)$. This completes the proof of (6.1.5).

Now let $M \ge 1$ and $M \ge \lambda + 2 + Q(z - y) - f(z)$. We will prove that

$$(f \mathbin{\nabla} MQ)(y) \ge \lambda. \tag{6.1.6}$$

To this end, let x be an arbitrary element of X. If $Q(y-x) \le 1$ then (6.1.4) implies that $f(x) + MQ(y-x) \ge f(x) > \lambda$. If, on the other hand, $Q(y-x) > 1$ then, from (6.1.5),

$$\begin{aligned} f(x) + MQ(y-x) &= f(x) + Q(y-x) + (M-1)Q(y-x) \\ &\ge f(x) + Q(z-x) - Q(z-y) + (M-1) \\ &\ge f(z) - 1 - Q(z-y) + M - 1 \ge \lambda, \end{aligned}$$

which completes the proof of (6.1.6). The "Hahn–Banach–Lagrange theorem" of [12, Theorem 2.9, p. 153] or [13, Theorem 1.11, p. 21] now provides us with a linear functional L on X such that $L \le MQ$ on X and $(f \mathbin{\nabla} L)(y) \ge \lambda$. (6.1.2) now follows by letting $\lambda \to f(y)$. □

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