Quadrivariate versions of the Attouch–Brezis theorem and strong representability

0. Introduction

In a recent paper, Marques Alves and Svaiter introduced the class of *strongly representable* multifunctions on a Banach space, and proved the remarkable result that any such multifunction is maximally monotone. This analysis was continued by Voisei and Zălinescu, who proved that, under certain circumstances, the sum of strongly representable multifunctions is strongly representable, and that strongly representable multifunctions have a number of other very desirable properties. A number of these proofs involve some very intricate calculations, and it is the purpose of this paper how some of these results can be established in a much more transparent fashion using an enhancement of the Attouch–Brezis theorem. (The original Attouch–Brezis theorem is stated in Theorem 1).

We now give a more detailed discussion of this enhancement of the Attouch–Brezis theorem, which we discuss in Section 1. Given two proper, convex lower semicontinuous functions f and g on the product of two (possible unrelated) (real) Banach spaces, let h be the inf–convolution of f and g with respect to the second variable. In Simons–Zălinescu [9, Theorem 4.2, pp. 9–10], there was a generalization of the Attouch–Brezis theorem giving sufficient conditions for h^* to be an exact inf–convolution of f^* and g^*. On the other hand, given Banach spaces X and Y, $A \in L(X,Y)$, and proper, convex lower semicontinuous functions f on $X \times X^*$ and g on $Y \times Y^*$, Voisei and Zălinescu provide conditions in [10, Theorem 16] under which they can give a formula for the conjugate of the funtion defined on $X \times X^*$ by the formula $(x,x^*) \mapsto \inf_{y^* \in Y^*} \left[f(x, x^* - A^{\mathbf{T}}y^*) + g(Ax, y^*)\right]$. In Theorem 3, we give a direct proof of a result that unifies [9, Theorem 4.2] and [10, Theorem 16]. Of course, such a result must use four variables. We show in Corollary 4 how to deduce [9, Theorem 4.2], and we give in Theorem 5 two results involving two Banach spaces and their duals, which will be applied later to strongly representable multifunctions.

In Section 2, we define representative and strongly representative functions on the product of a Banach space and its dual. After discussing a couple of examples which we will use in our analysis of strongly maximal multifunctions, we apply Theorem 5 in Theorem 8 and obtain sufficient condition for the generalized sum and generalized parallel sum of strongly representative functions to "lead to" strongly representative functions. Theorem 8(a) appears in [10, Theorem 16], but Theorem 8(b) seems to be new.

In Section 3, we define strongly maximally monotone multifunctions. Our main result here is Theorem 11, in which we give new sufficient conditions (in terms of the examples introduced in Section 2) for a maximally monotone multifunction to be strongly maximally monotone.

In Section 4, we define strongly representable multifunctions. In Theorem 15 and Corollary 16, we give sufficient conditions for the generalized sum and generalized parallel sum of strongly representable multifunctions to be strongly representable. The results on sums appear in [10], but the results on parallel sums seem to be new. We use these results in Theorem 17 to give short proofs that strongly representable multifunctions are strongly maximally monotone. In Section 5, we show how Theorem 8 can be generalized to allow a greater variety of qualification conditions.

In Section 6, we give another quadrivariate version of the Attouch–Brezis theorem,

Theorem 21, which might well find applications elsewhere. The proof of this involves simpler computations than Theorem 3, and it seems to be a more basic result in the sense that Theorem 3 can easily be deduced from it. (The details of this deduction can be found in Remark 22.) We chose to give a direct proof of Theorem 3 (which is what we need for the applications to strong representability) in order to spare the reader the task of working through all these substitutions.

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1. The first quadrivariate version of the Attouch–Brezis theorem

If E is a nonzero Banach space, we write $\mathcal{PCLSC}(E)$ for the set of all proper, convex lower semicontinuous functions from E into $]-\infty,\infty]$. Theorem 1 below was first proved in Attouch–Brezis, [2, Corollary 2.3, pp. 131–132], – there is a somewhat different proof in Simons, [8, Theorem 15.1, p. 66], and a much more general result was established in Zălinescu, [11, Theorem 2.8.6, p. 125–126]:

Theorem 1. *Let E be a nonzero Banach space, $\sigma,\tau \in \mathcal{PCLSC}(E)$ and $\sigma+\tau \ge 0$ on E. Suppose that $\bigcup_{\lambda>0}\lambda\big[\operatorname{dom}\sigma - \operatorname{dom}\tau\big]$ is a closed subspace of E. Then there exists $z^* \in E^*$ such that $\sigma^*(-z^*)+\tau^*(z^*) \le 0$.*

We will use the following simple algebraic lemma in both Theorems 3 and 21.

Lemma 2. *Let X and Z be vector spaces and $G \subset X \times Z$. Let $R\colon X \to Z$ be linear, and define $Q\colon X\times Z \to Z$ by $Q(x,y) := y - Rx$. Then*

$$\big\{(x-x_1, y-Rx_1)\colon\ (x,y)\in G,\ x_1\in X\big\} = Q^{-1}\big[Q(G)\big].$$

Proof. We suppose first that there exist $(x,y)\in G$ and $x_1 \in X$ such that $(\xi,\eta) = (x-x_1, y-Rx_1)$. Then $Q(\xi,\eta) = y - Rx_1 - R(x-x_1) = y - Rx \in Q(G)$, and so $(\xi,\eta)\in Q^{-1}\big[Q(G)\big]$. If, conversely, $(\xi,\eta)\in Q^{-1}\big[Q(G)\big]$ then there exists $(x,y)\in G$ such that $\eta - R\xi = y - Rx$. Defining $x_1 = x-\xi$, we have $(\xi,\eta) = (\xi, \eta - R\xi + R\xi) = (\xi, y - Rx + R\xi) = (x-x_1, y-Rx_1)$. $\square$

We now come to our first quadrivariate version of the Attouch–Brezis theorem. In what follows, we write $\mathbb{I}_\cdot$ for the indicator function of the set $\cdot$ in the appropriate space, and $\cdot^{\mathbf{T}}$ for the adjoint of the map $\cdot$. The following chart should help the reader keep track of the various spaces and maps.

$$\begin{array}{ccccccc} & & X & \xrightarrow{A} & Y & & \\ ]-\infty,\infty] & \xleftarrow{f,h} & \times & & \times & \xrightarrow{g} &]-\infty,\infty]. \\ & & U & \xleftarrow{B} & V & & \end{array}$$

Theorem 3. *Let X, Y, U and V be nonzero Banach spaces, $A\in L(X,Y)$, $B \in L(V,U)$, π_X map (x,u) to x and π_Y map (y,v) to y. Let $f \in \mathcal{PCLSC}(X\times U)$, $g\in\mathcal{PCLSC}(Y\times V)$ and*

$$L := \textstyle\bigcup_{\lambda>0}\lambda\big[\pi_Y\operatorname{dom} g - A(\pi_X \operatorname{dom} f)\big] \text{ be a closed subspace of } Y. \tag{3.1}$$

For all $(x,u) \in X \times U$, *let*

$$h(x,u) := \inf\big\{f(x,u-Bv) + g(Ax,v)\colon\ v \in V\big\} > -\infty.$$

Finally, let $(x_0^*, u_0^*) \in X^* \times U^*$. *Then*

$$h^*(x_0^*, u_0^*) = \min\big\{f^*(x_0^* - A^{\mathbf{T}}y^*, u_0^*) + g^*(y^*, B^{\mathbf{T}}u_0^*)\colon\ y^* \in Y^*\big\}.$$

Proof. We first note that it is easy to see that h is convex. Furthermore, (3.1) implies that $\pi_Y \operatorname{dom} g \cap A(\pi_X \operatorname{dom} f) \neq \emptyset$, from which it follows immediately that h is proper. Let $y^* \in Y^*$, $(x,u) \in X \times U$ and $v \in V$. Then, from the Fenchel–Young inequality,

$$\begin{aligned} f(x,u-Bv) &+ g(Ax,v) + f^*(x_0^* - A^{\mathbf{T}}y^*, u_0^*) + g^*(y^*, B^{\mathbf{T}}u_0^*) \\ &\geq \langle x, x_0^* - A^{\mathbf{T}}y^*\rangle + \big\langle u - Bv, u_0^*\big\rangle + \big\langle Ax, y^*\big\rangle + \big\langle v, B^{\mathbf{T}}u_0^*\big\rangle \\ &= \langle x, x_0^*\rangle + \langle u, u_0^*\rangle = \big\langle (x,u), (x_0^*, u_0^*)\big\rangle. \end{aligned}$$

Taking the infimum over v, $h(x,u) + f^*(x_0^* - A^{\mathbf{T}}y^*, u_0^*) + g^*(y^*, B^{\mathbf{T}}u_0^*) \geq \big\langle (x,u), (x_0^*, u_0^*)\big\rangle$. It follows from this that $h^*(x_0^*, u_0^*) \leq f^*(x_0^* - A^{\mathbf{T}}y^*, u_0^*) + g^*(y^*, B^{\mathbf{T}}u_0^*)$. So what we must prove is that there exists $y^* \in Y^*$ such that

$$f^*(x_0^* - A^{\mathbf{T}}y^*, u_0^*) + g^*(y^*, B^{\mathbf{T}}u_0^*) \leq h^*(x_0^*, u_0^*). \tag{3.2}$$

Since h is proper, $h^*(x_0^*, u_0^*) > -\infty$, so we can suppose that $h^*(x_0^*, u_0^*) \in \mathbb{R}$. Let P stand for the product space $X \times Y \times U \times V$, and define $\sigma \in \mathcal{PCLSC}(P)$ by

$$\sigma(x,y,u,v) := h^*(x_0^*, u_0^*) - \langle x, x_0^*\rangle - \langle u + Bv, u_0^*\rangle + f(x,u) + g(y,v),$$

and $\tau \in \mathcal{PCLSC}(P)$ by $\tau(x,y,u,v) := \mathbb{I}_{\{0\}}(y - Ax)$. Now let $(x,y,u,v) \in P$. Then, from the Fenchel–Young inequality,

$$\begin{aligned} (\sigma + \tau)(x,y,u,v) &= h^*(x_0^*, u_0^*) - \langle x, x_0^*\rangle - \langle u + Bv, u_0^*\rangle + f(x,u) + g(Ax,v) \\ &\geq h^*(x_0^*, u_0^*) - \langle x, x_0^*\rangle - \big\langle u + Bv, u_0^*\big\rangle + h(x, u + Bv) \\ &= h(x, u+Bv) + h^*(x_0^*, u_0^*) - \big\langle (x, u+Bv), (x_0^*, u_0^*)\big\rangle \geq 0. \end{aligned}$$

We now define $Q \in L(X \times Y, Y)$ by $Q(x,y) := y - Ax$. Then, from the definitions of σ and τ, and Lemma 2 with $Z = Y$, $R = A$ and $G = \pi_X \operatorname{dom} f \times \pi_Y \operatorname{dom} g$,

$$\begin{aligned} \operatorname{dom}\sigma - \operatorname{dom}\tau &= \big\{(x - x_1, y - Ax_1, u - u_1, v - v_1)\colon \\ &\qquad (x,u) \in \operatorname{dom} f,\ (y,v) \in \operatorname{dom} g,\ x_1 \in X,\ u_1 \in U,\ v_1 \in V\big\} \\ &= \big\{(x - x_1, y - Ax_1, u_2, v_2)\colon \\ &\qquad (x,y) \in \pi_X \operatorname{dom} f \times \pi_Y \operatorname{dom} g,\ x_1 \in X,\ u_2 \in U,\ v_2 \in V\big\} \\ &= \big\{(x - x_1, y - Ax_1)\colon\ (x,y) \in \pi_X \operatorname{dom} f \times \pi_Y \operatorname{dom} g,\ x_1 \in X\big\} \times U \times V \\ &= Q^{-1}\big[Q(\pi_X \operatorname{dom} f \times \pi_Y \operatorname{dom} g)\big] \times U \times V \\ &= Q^{-1}\big[\pi_Y \operatorname{dom} g - A(\pi_X \operatorname{dom} f)\big] \times U \times V. \end{aligned}$$

It follows easily from this that $\bigcup_{\lambda>0}\lambda\big[\operatorname{dom}\sigma-\operatorname{dom}\tau\big]=Q^{-1}(L)\times U\times V$, which is a closed subspace of P. Theorem 1 now gives $(x^*,y^*,u^*,v^*)\in P^*$ such that

$$\sigma^*(x^*,y^*,u^*,v^*)+\tau^*(-x^*,-y^*,-u^*,-v^*)\le 0, \tag{3.3}$$

which clearly implies that $\tau^*(-x^*,-y^*,-u^*,-v^*)<\infty$. However, by direct computation, $\tau^*(-x^*,-y^*,-u^*,-v^*)=\mathbb{I}_{\{0\}}(x^*+A^{\mathbf{T}}y^*)+\mathbb{I}_{\{0\}}(u^*)+\mathbb{I}_{\{0\}}(v^*)$, and so it follows from (3.3) that $\sigma^*(-A^{\mathbf{T}}y^*,y^*,0,0)\le 0$. But

$$\begin{aligned}
&\sigma^*(-A^{\mathbf{T}}y^*,y^*,0,0)\\
&=\sup_{x,y,u,v}\big[\langle x,-A^{\mathbf{T}}y^*\rangle+\langle y,y^*\rangle-h^*(x_0^*,u_0^*)+\langle x,x_0^*\rangle+\langle u+Bv,u_0^*\rangle-f(x,u)-g(y,v)\big]\\
&=\sup_{x,y,u,v}\big[\langle x,x_0^*-A^{\mathbf{T}}y^*\rangle+\langle y,y^*\rangle+\langle u,u_0^*\rangle+\langle v,B^{\mathbf{T}}u_0^*\rangle-f(x,u)-g(y,v)\big]-h^*(x_0^*,u_0^*)\\
&=f^*\big(x_0^*-A^{\mathbf{T}}y^*,u_0^*\big)+g^*\big(y^*,B^{\mathbf{T}}u_0^*\big)-h^*(x_0^*,u_0^*).
\end{aligned}$$

Thus (3.3) reduces to (3.2). This completes the proof of Theorem 3. $\square$

We now show how to deduce [9, Theorem 4.2]:

Corollary 4. *Let E and F be nonzero Banach spaces and π_E map (x,x^*) to x. Let $f,g\in\mathcal{PCLSC}(E\times F)$ and $\bigcup_{\lambda>0}\lambda\big[\pi_E\operatorname{dom}g-\pi_E\operatorname{dom}f\big]$ be a closed subspace of E. For all $(x,y)\in E\times F$, let*

$$h(x,u):=\inf\big\{f(x,y-v)+g(x,v)\colon\ v\in F\big\}>-\infty.$$

Then, for all $(x^*,u^*)\in X^*\times U^*$,

$$h^*(x^*,u^*)=\min\big\{f^*(x^*-y^*,u^*)+g^*(y^*,u^*)\colon\ y^*\in E^*\big\}.$$

Proof. This is immediate from Theorem 3 with $X=Y=E$, $U=V=F$, and A and B identity maps. $\square$

The next theorem contains two results involving two Banach spaces and their duals. These results will be applied in Theorem 8.

Theorem 5. *Let E and F be nonzero Banach spaces, $f\in\mathcal{PCLSC}(E\times E^*)$ and $g\in\mathcal{PCLSC}(F\times F^*)$. Let π_E map (x,x^*) to x, π_F map (y,y^*) to y, π_{E^*} map (x,x^*) to x^* and π_{F^*} map (y,y^*) to y^*.*
(a) *Let $A\in L(E,F)$, $B\in L(F^*,E^*)$, $\bigcup_{\lambda>0}\lambda\big[\pi_F\operatorname{dom}g-A(\pi_E\operatorname{dom}f)\big]$ be a closed subspace of F and, for all $(x,x^*)\in E\times E^*$,*

$$h(x,x^*):=\inf\big\{f(x,x^*-By^*)+g(Ax,y^*)\colon\ y^*\in F^*\big\}>-\infty.$$

Then, for all $(x^*,x^{**})\in E^*\times E^{**}$,

$$h^*(x^*,x^{**})=\min\big\{f^*(x^*-A^{\mathbf{T}}y^*,x^{**})+g^*(y^*,B^{\mathbf{T}}x^{**})\colon\ y^*\in F^*\big\}.$$

(b) *Let* $A \in L(E^*, F^*)$, $B \in L(F, E)$, $\bigcup_{\lambda>0} \lambda\big[\pi_{F^*} \operatorname{dom} g - A(\pi_{E^*} \operatorname{dom} f)\big]$ *be a closed subspace of* F^* *and, for all* $(x, x^*) \in E \times E^*$,

$$h(x, x^*) := \inf\big\{f(x - By, x^*) + g(y, Ax^*)\colon\ y \in F\big\} > -\infty.$$

Then, for all $(x^*, x^{**}) \in E^* \times E^{**}$,

$$h^*(x^*, x^{**}) = \min\big\{f^*(x^*, x^{**} - A^{\mathbf{T}}y^{**}) + g^*(B^{\mathbf{T}}x^*, y^{**})\colon\ y^{**} \in F^{**}\big\}.$$

Proof. (a) is immediate from Theorem 3 with $X = E$, $Y = F$, $U = E^*$ and $V = F^*$. (b) is immediate from Theorem 3 with $X = E^*$, $Y = F^*$, $U = E$ and $V = F$, and changing the order of the arguments of f, g and h. □

2. Representative and strongly representative functions

We start of by recalling some facts from convex analysis. Let X be a nonzero Banach space and $h\colon X \to]-\infty, \infty]$ be proper and convex. We write $\overline{h}$ for the (convex) function on X such that the epigraph of $\overline{h}$ is the closure of the epigraph of h in $X \times \mathbb{R}$. Clearly, if $q\colon X \to \mathbb{R}$ is continuous and $h \ge q$ on X then $\overline{h} \ge q$ on X. As a particular case of this, let $x^* \in \operatorname{dom} h^*$ Then the Fenchel–Young inequality implies that $x^* - h^*(x^*) \le h$ on X, and so $x^* - h^*(x^*) \le \overline{h}$ on X, from which $x^* - \overline{h} \le h^*(x^*)$ on X. Taking the supremum over X, $\overline{h}^* \le h^*$ on X^*. On the other hand, since $\overline{h} \le h$ on X, we also have $h^* \le \overline{h}$ on X^*. Thus $\overline{h}^* = h^*$.

Let E be a nonzero Banach space. We say that f is a *representative function* on $E \times E^*$ if $f \in \mathcal{PCLSC}(E \times E^*)$ and, for all $(x, x^*) \in E \times E^*$, $f(x, x^*) \ge \langle x, x^*\rangle$. We say that f is a *strongly representative function* on $E \times E^*$ if f is a representative function on $E \times E^*$ and, further, for all $(x^*, x^{**}) \in E^* \times E^{**}$, $f^*(x^*, x^{**}) \ge \langle x^*, x^{**}\rangle$.

If $h\colon E \times E^* \to]-\infty, \infty]$ is proper and convex and, for all $(x, x^*) \in E \times E^*$, $h(x, x^*) \ge \langle x, x^*\rangle$ then, since the function $(x, x^*) \mapsto \langle x, x^*\rangle$ is continuous, $\overline{h}$ is a representative function. If, in addition, for all $(x^*, x^{**}) \in E^* \times E^{**}$, $h^*(x^*, x^{**}) \ge \langle x^*, x^{**}\rangle$ then, since $\overline{h}^* = h^*$, $\overline{h}$ is a strongly representative function.

The main result of this section is Theorem 8, in which we show how two strongly representative functions give rise to a third.

We now give two examples of strongly representative functions. (These results can be deduced from [8, Lemma 35.1, p. 140], but we give direct proofs.) In what follows, for all $x \in E$, we write $\widehat{x}$ for the canonical image of x in the bidual, E^{**}.

Example 6. If K is a nonempty $w(E, E^*)$–compact convex subset of E and $y^* \in E^*$, we define the function $h_{K,y^*} \in \mathcal{PCLSC}(E \times E^*)$ by

$$h_{K,y^*}(x, x^*) := \mathbb{I}_K(x) + \langle x, y^*\rangle + \sup\langle K, x^* - y^*\rangle.$$

Since $h^*_{K,y^*}(x^*, x^{**}) = \sup\big\langle K, x^* - y^*\big\rangle + \mathbb{I}_{\widehat{K}}(x^{**}) + \langle y^*, x^{**}\rangle$, h_{K,y^*} is a strongly representative function. We define the multifunction $N_{K,y^*}\colon E \rightrightarrows E^*$ by $x^* \in N_{K,y^*}x$ exactly when $h_{K,y^*}(x, x^*) = \langle x, x^*\rangle$. Clearly,

$$v \in E,\ s^* \in N_{K,y^*}v \text{ and } u \in K \quad\Longrightarrow\quad v \in K \text{ and } \langle v - u, s^* - y^*\rangle \ge 0. \tag{6.1}$$

We note that $N_{K,0}$ is the “normal cone” multifunction in the usual sense.

Example 7. If $y \in E$ and K is a nonempty $w(E^*, E)$–compact convex subset of E^*, we define the function $g_{y,K} \in \mathcal{PCLSC}(E \times E^*)$ by

$$g_{y,K}(x, x^*) := \sup\langle x - y, K\rangle + \mathbb{I}_K(x^*) + \langle y, x^*\rangle.$$

Since $g^*_{y,K}(x^*, x^{**}) = \mathbb{I}_K(x^*) + \langle y, x^*\rangle + \sup\big\langle K, x^{**} - \widehat{y}\big\rangle$, $g_{y,K}$ is a strongly representative function. We define the multifunction $M_{y,K}\colon E \rightrightarrows E^*$ by $x^* \in M_{y,K}x$ exactly when $g_{y,K}(x, x^*) = \langle x, x^*\rangle$. Clearly,

$$s \in E,\ w^* \in M_{y,K}s \text{ and } v^* \in K \quad\Longrightarrow\quad w^* \in K \text{ and } \langle s - y, w^* - v^*\rangle \ge 0. \tag{7.1}$$

(a) of Theorem 8 appears in [10, Theorem 16]. It seems that the dual result, (b), is new.

Theorem 8. *Let E and F be nonzero Banach spaces, π_E map (x, x^*) to x, π_F map (y, y^*) to y, π_{E^*} map (x, x^*) to x^* and π_{F^*} map (y, y^*) to y^*. Let f be a representative function on $E \times E^*$ and g be a representative function on $F \times F^*$.*
(a) *Let $A \in L(E, F)$, $\bigcup_{\lambda>0} \lambda\big[\pi_F \operatorname{dom} g - A(\pi_E \operatorname{dom} f)\big]$ be a closed subspace of F and, for all $(x, x^*) \in E \times E^*$,*

$$h(x, x^*) := \inf\big\{f(x, x^* - A^{\mathbf{T}}y^*) + g(Ax, y^*)\colon\ y^* \in F^*\big\}.$$

Then, for all $(x^*, x^{**}) \in E^* \times E^{**}$,

$$h^*(x^*, x^{**}) = \min\big\{f^*(x^* - A^{\mathbf{T}}y^*, x^{**}) + g^*(y^*, A^{\mathbf{TT}}x^{**})\colon\ y^* \in F^*\big\}.$$

If, further, f is strongly representative on $E \times E^*$ and g is strongly representative on $F \times F^*$ then $\overline{h}$ is strongly representative on $E \times E^*$.
(b) *Let $B \in L(F, E)$, $\bigcup_{\lambda>0} \lambda\big[\pi_{F^*} \operatorname{dom} g - B^{\mathbf{T}}(\pi_{E^*} \operatorname{dom} f)\big]$ be a closed subspace of F^* and, for all $(x, x^*) \in E \times E^*$,*

$$h(x, x^*) := \inf\big\{f(x - By, x^*) + g(y, B^{\mathbf{T}}x^*)\colon\ y \in F\big\}.$$

Then, for all $(x^*, x^{**}) \in E^* \times E^{**}$,

$$h^*(x^*, x^{**}) = \min\big\{f^*(x^*, x^{**} - B^{\mathbf{TT}}y^{**}) + g^*(B^{\mathbf{T}}x^*, y^{**})\colon\ y^{**} \in F^{**}\big\}.$$

If, further, f is strongly representative on $E \times E^*$ and g is strongly representative on $F \times F^*$ then $\overline{h}$ is strongly representative on $E \times E^*$.

Proof. (a) For all $(x, x^*) \in E \times E^*$,

$$h(x, x^*) \ge \inf\big\{\langle x, x^* - A^{\mathbf{T}}y^*\rangle + \langle Ax, y^*\rangle\colon\ y^* \in F^*\big\} = \langle x, x^*\rangle > -\infty.$$

The required formula for $h^*(x^*, x^{**})$ now follows from Theorem 5(a) with $B := A^{\mathbf{T}}$. If f and g are strongly representative then, for all $(x^*, x^{**}) \in E^* \times E^{**}$ and $y^* \in F^*$,

$$\begin{aligned} f^*(x^* - A^{\mathbf{T}}y^*, x^{**}) + g^*(y^*, A^{\mathbf{TT}}x^{**}) &\ge \langle x^* - A^{\mathbf{T}}y^*, x^{**}\rangle + \langle y^*, A^{\mathbf{TT}}x^{**}\rangle \\ &= \langle x^*, x^{**}\rangle. \end{aligned}$$

Consequently, $h^*(x^*, x^{**}) \ge \langle x^*, x^{**}\rangle$, and so $\overline{h}$ is strongly representative.

(b) For all $(x, x^*) \in E \times E^*$,

$$h(x, x^*) \ge \inf\big\{\langle x - By, x^*\rangle + \langle y, B^{\mathbf{T}}x^*\rangle\colon\ y \in F\big\} = \langle x, x^*\rangle > -\infty.$$

The required formula for $h^*(x^*, x^{**})$ now follows from Theorem 5(b) with $A := B^{\mathbf{T}}$. If f and g are strongly representative then, for all $(x^*, x^{**}) \in E^* \times E^{**}$ and $y^{**} \in F^{**}$,

$$\begin{aligned} f^*(x^*, x^{**} - B^{\mathbf{TT}}y^{**}) + g^*(B^{\mathbf{T}}x^*, y^{**}) &\ge \langle x^*, x^{**} - B^{\mathbf{TT}}y^{**}\rangle + \langle B^{\mathbf{T}}x^*, y^{**}\rangle \\ &= \langle x^*, x^{**}\rangle. \end{aligned}$$

Consequently, $h^*(x^*, x^{**}) \ge \langle x^*, x^{**}\rangle$, and so $\overline{h}$ is strongly representative. □

3. Strong maximal monotonicity

We now discuss "strong maximal monotonicity", which is actually defined in terms of two simpler concepts. The main result of this section is Theorem 11, which is implicit in the proof of [6, Theorem 8.2, pp. 877–878], but has not been pointed out explicitly before. In what follows, $\mathcal{G}(\cdot)$ stands for the graph of the multifunction $\cdot$.

Definition 9. Let E be a nonzero Banach space and $S\colon\ E \rightrightarrows E^*$ be monotone. We say that S is *$w(E^*, E)$–cc maximally monotone* if, whenever $y \in E$ and C is a nonempty $w(E^*, E)$–compact convex subset of E^*,

$$(s, s^*) \in \mathcal{G}(S) \quad\Longrightarrow\quad \text{there exists } y^* \in C \text{ such that } \quad \langle s - y, s^* - y^*\rangle \ge 0 \tag{9.1}$$

then

$$Sy \cap C \ne \emptyset. \tag{9.2}$$

We say that S is *$w(E, E^*)$–cc maximally monotone* if, whenever C, a nonempty $w(E, E^*)$–compact convex subset of E, $y^* \in E^*$ and

$$(s, s^*) \in \mathcal{G}(S) \quad\Longrightarrow\quad \text{there exists } w \in C \text{ such that } \quad \langle s - w, s^* - y^*\rangle \ge 0 \tag{9.3}$$

then

$$S^{-1}y^* \cap C \ne \emptyset. \tag{9.4}$$

We say that S is *strongly maximally monotone* if S is both $w(E^*, E)$–cc maximally monotone and $w(E, E^*)$–cc maximally monotone.

Definition 10. Let E be a nonzero Banach space and $S, T\colon\ E \rightrightarrows E^*$ be multifunctions. We define the *parallel sum*, $S \parallel T$, of S and T to be the multifunction defined by $x^* \in (S \parallel T)x$ exactly when there exists $v \in E$ such that $x^* \in S(x - v) \cap Tv$. This terminology comes from the easily verifiable fact that $S \parallel T = \big(S^{-1} + T^{-1}\big)^{-1}$ and some analogy with electric circuits.

Theorem 11. *Let E be a nonzero Banach space and $S\colon\ E \rightrightarrows E^*$ be monotone.*
(a) *Suppose that if $y \in E$ and K is a nonempty $w(E^*, E)$–compact convex subset of E^* then $S + M_{y,K}$ is maximally monotone.* (We recall that $M_{y,K}$ was defined in Example 7.) *Then S is $w(E^*, E)$–cc maximally monotone.*
(b) *Suppose that if K is a nonempty $w(E, E^*)$–compact convex subset of E and $y^* \in E^*$ then $S \parallel N_{K,y^*}$ is maximally monotone.* (We recall that N_{K,y^*} was defined in Example 6.) *Then S is $w(E, E^*)$–cc maximally monotone.*

Proof. (a) Let $y \in E$ and C be a nonempty $w(E^*, E)$–compact convex subset of E^* such that (9.1) is satisfied. Let $K := -C$, and write $T := S + M_{y,K}$. If $(s, t^*) \in \mathcal{G}(T)$ then there exist $(s, s^*) \in \mathcal{G}(S)$ and $w^* \in M_{y,K}s$ such that $t^* = s^* + w^*$. From (9.1), there exists $y^* \in C$ such that $\langle s - y, s^* - y^* \rangle \geq 0$. Since $v^* := -y^* \in K$, we obtain from (7.1) that $\langle s - y, w^* + y^* \rangle \geq 0$, that is to say, $\langle s - y, w^* \rangle \geq \langle s - y, -y^* \rangle$. But then

$$\langle s - y, t^* - 0 \rangle = \langle s - y, s^* + w^* \rangle \geq \langle s - y, s^* - y^* \rangle \geq 0.$$

Since this holds for all $(s, t^*) \in \mathcal{G}(T)$ and T is maximally monotone, $0 \in Ty$, from which there exists $s^* \in Sy$ such that $-s^* \in M_{y,K}y$. From (7.1) again, $-s^* \in K$, and so $s^* \in C$. This establishes (9.2), and completes the proof of (a).

(b) Let C be a nonempty $w(E, E^*)$–compact convex subset of E and $y^* \in E^*$ be such that (9.3) is satisfied. Let $K := -C$, and write $T := S \parallel N_{K,y^*}$. If $(t, s^*) \in \mathcal{G}(T)$ then there exists $v \in E$ such that $s^* \in S(t - v) \cap N_{K,y^*}v$. From (9.3), there exists $w \in C$ such that $\langle t - v - w, s^* - y^* \rangle \geq 0$. Since $-w \in K$, (6.1) implies that $\langle v + w, s^* - y^* \rangle \geq 0$. Thus

$$\langle t - 0, s^* - y^* \rangle = \langle t - v - w, s^* - y^* \rangle + \langle v + w, s^* - y^* \rangle \geq 0.$$

Since this holds for all $(t, s^*) \in \mathcal{G}(T)$ and T is maximally monotone, $y^* \in T0$, from which there exists $s \in S^{-1}y^*$ such that $y^* \in N_{K,y^*}(-s)$. From (6.1) again, $-s \in K$, and so $s \in C$. This establishes (9.4), and completes the proof of (b). □

4. Strongly representable multifunctions

Let E be a nonzero Banach space and $f \in \mathcal{PCLSC}(E \times E^*)$ be a representative function. We then define the multifunction $\mathcal{M}f\colon E \rightrightarrows E^*$ to be the multifunction with graph $\{(x, x^*)\colon f(x, x^*) = \langle x, x^* \rangle\}$. The "$\mathcal{M}$" stands for "monotone" — the justification for this notation is the following result, which was established by Burachik and Svaiter in [3, Theorem 3.1, pp. 2381–2382] and Penot in [5, Proposition 4$\big($h$\big)\Longrightarrow$(a), pp. 860–861]. Monotone multifunctions S of the special form described in Lemma 12 have been investigated thoroughly by Martínez-Legaz and Svaiter in [4], and subsequently in [7]. We also refer the reader to [4] and [7] for more results in the finite–dimensional case. (We note that the references [3] and [5] cited above assume that E is reflexive, but this is not used in the relevant part of the proof.)

Lemma 12. *Let E be a nonzero Banach space and $f \in \mathcal{PCLSC}(E \times E^*)$ be a representative function. Then the multifunction $\mathcal{M}f$ is monotone.*

The following very unexpected result was first proved by Marques Alves and Svaiter in [1, Theorem 4.2], and a different proof was given subsequently by Voisei and Zălinescu in [10, Theorem 8].

Theorem 13. *Let E be a nonzero Banach space and $f \in \mathcal{PCLSC}(E \times E^*)$ be a strongly representative function. Then the multifunction $\mathcal{M}f$ is maximally monotone.*

We now give an important computational result for representative functions (which appears in [1, Theorem 4.2] and [10, Corollary 10]). We use the following notation: if E is a nonzero Banach space and $f \in \mathcal{PCLSC}(E \times E^*)$ is a strongly representative function we define $f^{@}\colon\ E \times E^* \to]{-\infty}, \infty]$ by $f^{@}(x, x^*) := f^*(x^*, \widehat{x})$. The proof of Lemma 14 given here is patterned after that of [8, Lemma 19.12, p. 82].

Lemma 14. *Let E be a nonzero Banach space and $f \in \mathcal{PCLSC}(E \times E^*)$ be a strongly representative function. Then $f^@$ is a representative function and $\mathcal{M}f^@ = \mathcal{M}f$.*

Proof. We first prove that

$$f(x,x^*) = \langle x,x^*\rangle \quad\Longrightarrow\quad f^@(x,x^*) = \langle x,x^*\rangle. \tag{14.1}$$

Let $f(x,x^*) = \langle x,x^*\rangle$, and (y,y^*) be an arbitrary element of $E \times E^*$. Let $\lambda \in]0,1[$. For simplicity in writing, let $\mu := 1-\lambda \in]0,1[$. Then

$$\begin{aligned}\lambda^2\langle y,y^*\rangle + \lambda\mu\big\langle (y,y^*),(x^*,\widehat{x})\big\rangle + \mu^2\langle x,x^*\rangle &= \langle \lambda y+\mu x, \lambda y^*+\mu x^*\rangle \\ \le f(\lambda y+\mu x,\lambda y^*+\mu x^*) \le \lambda f(y,y^*) + \mu f(x,x^*) &= \lambda f(y,y^*) + \mu\langle x,x^*\rangle.\end{aligned}$$

Thus $\lambda\mu\big\langle (y,y^*),(x^*,\widehat{x})\big\rangle - \lambda f(y,y^*) \le \lambda\mu\langle x,x^*\rangle - \lambda^2\langle y,y^*\rangle$. Dividing by λ and letting $\lambda \to 0$, we obtain $\big\langle (y,y^*),(x^*,\widehat{x})\big\rangle - f(y,y^*) \le \langle x,x^*\rangle$. It now follows by taking the supremum over (y,y^*) that $f^*(x^*,\widehat{x}) \le \langle x,x^*\rangle$. On the other hand, the strong representativity of f implies that $f^*(x^*,\widehat{x}) \ge \langle x^*,\widehat{x}\rangle = \langle x,x^*\rangle$, and (14.1) follows by combining these two inequalities. It is clear from Theorem 13 and (14.1) that $f^@ \in \mathcal{PCLSC}(E \times E^*)$, and so Lemma 12, Theorem 13 and (14.1) imply that the monotone multifunction $\mathcal{M}f^@$ is an extension of the maximally monotone multifunction $\mathcal{M}f$. Consequently, the two multifunctions coincide. □

We say that a multifunction $S\colon E \rightrightarrows E^*$ is *strongly representable* if there is a strongly representative function $f \in \mathcal{PCLSC}(E \times E^*)$ such that $\mathcal{M}f = S$. In what follows, $\mathcal{R}(\cdot)$ stands for the range of the multifunction $\cdot$. Theorem 15(a) appears in [10, Theorem 16], but Theorem 15(b,c) seem to be new.

Theorem 15. *Let E and F be nonzero Banach spaces, π_E map (x,x^*) to x, π_F map (y,y^*) to y, π_{E^*} map (x,x^*) to x^* and π_{F^*} map (y,y^*) to y^*. Let f be a strongly representative function on $E \times E^*$ and g be a strongly representative function on $F \times F^*$.*
(a) *Let $A \in L(E,F)$ and $\bigcup_{\lambda>0}\lambda\big[\pi_F\operatorname{dom} g - A(\pi_E\operatorname{dom} f)\big]$ be a closed subspace of F. Then the multifunction $\mathcal{M}f + A^{\mathbf{T}}(\mathcal{M}g)A$ is strongly representable.*
(b) *Let $B \in L(F,E)$ and $\bigcup_{\lambda>0}\lambda\big[\pi_{F^*}\operatorname{dom} g - B^{\mathbf{T}}(\pi_{E^*}\operatorname{dom} f)\big]$ be a closed subspace of F^*. Then the multifunction $x \mapsto \big((\mathcal{M}f^*) + B^{\mathbf{TT}}(\mathcal{M}g^*)B^{\mathbf{T}}\big)^{-1}\widehat{x}$ is strongly representable.*
(c) *Let $B \in L(F,E)$, $\bigcup_{\lambda>0}\lambda\big[\pi_{F^*}\operatorname{dom} g - B^{\mathbf{T}}(\pi_{E^*}\operatorname{dom} f)\big]$ be a closed subspace of F^* and $\mathcal{R}(\mathcal{M}g^*) \subset \widehat{F}$. Then the multifunction $\big((\mathcal{M}f)^{-1} + B(\mathcal{M}g)^{-1}B^{\mathbf{T}}\big)^{-1}$ is strongly representable.*

Proof. (a) Let h be as in Theorem 8(a), so that $\overline{h}$ is a strongly representative function on $E \times E^*$, from which $\mathcal{M}\overline{h}$ is a strongly representable multifunction. Since $\overline{h}^* = h^*$, Theorem 8(a) and the fact that $A^{\mathbf{TT}}\widehat{x} = \widehat{Ax}$ imply that

$$\begin{aligned}\overline{h}^@(x,x^*) = \overline{h}^*(x^*,\widehat{x}) = h^*(x^*,\widehat{x}) &= \min\big\{f^*(x^*-A^{\mathbf{T}}y^*,\widehat{x}) + g^*(y^*,\widehat{Ax})\colon\ y^* \in F^*\big\} \\ &= \min\big\{f^@(x,x^*-A^{\mathbf{T}}y^*) + g^@(Ax,y^*)\colon\ y^* \in F^*\big\}.\end{aligned}$$

Consequently, from three applications of Lemma 14,

$$
\begin{aligned}
x^* \in \big(\mathcal{M}\overline{h}\big)x &\iff x^* \in \big(\mathcal{M}\overline{h}^{@}\big)x \\
&\iff \text{there exists } y^* \in F^* \text{ such that } x^* - A^{\mathbf{T}}y^* \in (\mathcal{M}f^{@})x \text{ and } y^* \in (\mathcal{M}g^{@})Ax \\
&\iff \text{there exists } y^* \in F^* \text{ such that } x^* - A^{\mathbf{T}}y^* \in (\mathcal{M}f)x \text{ and } y^* \in (\mathcal{M}g)Ax \\
&\iff x^* \in \big(\mathcal{M}f + A^{\mathbf{T}}(\mathcal{M}g)A\big)x.
\end{aligned}
$$

This completes the proof of (a).

(b) Let h be as in Theorem 8(b), so that $\overline{h}$ is a strongly representative function on $E \times E^*$, from which $\mathcal{M}\overline{h}$ is a strongly representable multifunction. Since $\overline{h}^* = h^*$, Theorem 8(b) implies that

$$
\begin{aligned}
\overline{h}^{@}(x, x^*) &= \overline{h}^*(x^*, \widehat{x}) \\
&= h^*(x^*, \widehat{x}) = \min\big\{f^*(x^*, \widehat{x} - B^{\mathbf{TT}}y^{**}) + g^*(B^{\mathbf{T}}x^*, y^{**})\colon\ y^{**} \in F^{**}\big\}
\end{aligned}
$$

Consequently, from Lemma 14,

$$
\begin{aligned}
x^* \in \big(\mathcal{M}\overline{h}\big)x &\iff x^* \in \big(\mathcal{M}\overline{h}^{@}\big)x \\
&\iff \text{there exists } y^{**} \in F^{**} \text{ such that } \widehat{x} - B^{\mathbf{TT}}y^{**} \in (\mathcal{M}f^*)x^* \text{ and } y^{**} \in (\mathcal{M}g^*)B^{\mathbf{T}}x^* \\
&\iff \widehat{x} \in \big((\mathcal{M}f^*) + B^{\mathbf{TT}}(\mathcal{M}g^*)B^{\mathbf{T}}\big)x^*.
\end{aligned}
$$

This completes the proof of (b).

(c) We proceed as in (b) up to the statement $y^{**} \in (\mathcal{M}g^*)B^{\mathbf{T}}x^*$. The additional assumption that $\mathcal{R}(\mathcal{M}g^*) \subset \widehat{F}$, the fact that $B^{\mathbf{TT}}\widehat{y} = \widehat{By}$, and two applications of Lemma 14 imply that

$$
\begin{aligned}
x^* \in \big(\mathcal{M}\overline{h}\big)x &\iff x^* \in \big(\mathcal{M}\overline{h}^{@}\big)x \\
&\iff \text{there exists } y \in F \text{ such that } \widehat{x} - \widehat{By} \in (\mathcal{M}f^*)x^* \text{ and } \widehat{y} \in (\mathcal{M}g^*)B^{\mathbf{T}}x^* \\
&\iff \text{there exists } y \in F \text{ such that } x - By \in (\mathcal{M}f^{@})^{-1}x^* \text{ and } y \in (\mathcal{M}g^{@})^{-1}B^{\mathbf{T}}x^* \\
&\iff \text{there exists } y \in F \text{ such that } x - By \in (\mathcal{M}f)^{-1}x^* \text{ and } y \in (\mathcal{M}g)^{-1}B^{\mathbf{T}}x^* \\
&\iff x \in \big((\mathcal{M}f)^{-1} + B(\mathcal{M}g)^{-1}B^{\mathbf{T}}\big)x^*.
\end{aligned}
$$

This completes the proof of (c). $\square$

Corollary 16. *Let E be a nonzero Banach space, π_E map (x, x^*) to x and π_{E^*} map (x, x^*) to x^*. Let f and g be strongly representative functions on $E \times E^*$.*
(a) *If $\bigcup_{\lambda>0} \lambda\big[\pi_E \operatorname{dom} g - \pi_E \operatorname{dom} f\big]$ is a closed subspace of E then the multifunction $\mathcal{M}f + \mathcal{M}g$ is strongly representable.*
(b) *If $\bigcup_{\lambda>0} \lambda\big[\pi_{E^*} \operatorname{dom} g - \pi_{E^*} \operatorname{dom} f\big]$ is a closed subspace of E^* then the multifunction $x \mapsto \big(\mathcal{M}f^* + \mathcal{M}g^*\big)^{-1}\widehat{x}$ is strongly representable.*
(c) *If $\bigcup_{\lambda>0} \lambda\big[\pi_{E^*} \operatorname{dom} g - \pi_{E^*} \operatorname{dom} f\big]$ is a closed subspace of E^* and $\mathcal{R}(\mathcal{M}g^*) \subset \widehat{E}$ then the multifunction $\mathcal{M}f \parallel \mathcal{M}g$ is strongly representable.*

Proof. This follows from Theorem 15, with $F = E$, and A and B identity maps.

Our next result generalizes [6, Theorem 8.2, pp. 877–878].

Theorem 17. *Let E be a nonzero Banach space and $S\colon\ E \rightrightarrows E^*$ be strongly representable. Then S is strongly maximal.*

Proof. Let f be a strongly representative function on $E \times E^*$ such that $\mathcal{M}f = S$.

Let $y \in E$ and K be a nonempty $w(E^*, E)$–compact convex subset of E^*. We recall that $g_{y,K}$ was defined in Example 7, and was shown to be a strongly representative function. Since $\pi_E \mathrm{dom}\, g_{y,K} = E$, Corollary 16(a) implies that the multifunction $\mathcal{M}f + \mathcal{M}g_{y,K} = S + M_{y,K}$ is strongly representable. Thus, from Theorem 13, $S + M_{y,K}$ is maximally monotone. It now follows from Theorem 11(a) that S is $w(E^*, E)$–cc maximal.

Now let K be a nonempty $w(E, E^*)$–compact convex subset of E and $y^* \in E^*$. We recall that h_{K,y^*} was defined in Example 6, and was shown to be a strongly representative function. Since $\pi_{E^*} \mathrm{dom}\, h_{K,y^*} = E^*$ and

$$h^*_{K,y^*}(z^*, z^{**}) = \langle z^*, z^{**}\rangle \Longrightarrow z^{**} \in \widehat{K} \subset \widehat{E},$$

Corollary 16(b) implies that the multifunction $\mathcal{M}f \parallel \mathcal{M}h_{K,y^*} = S \parallel N_{K,y^*}$ is strongly representable. Thus, from Theorem 13, $S \parallel N_{K,y^*}$ is maximally monotone. It now follows from Theorem 11(b) that S is $w(E, E^*)$–cc maximal.

This completes the proof of Theorem 17. $\square$

5. Sandwiched closed subspace theorems

The main result of this section, Theorem 20, shows how we can bootstrap Theorem 8 and obtain different conditions for the formulae for h^* to hold. In what follows, $\mathcal{D}(\cdot)$ stands for the domain of the multifunction $\cdot$. Theorem 20(a) shows that, instead of assuming in Theorem 8(a) that $\bigcup_{\lambda>0} \lambda\big[\pi_F \,\mathrm{dom}\, g - A(\pi_E \,\mathrm{dom}\, f)\big]$ is a closed subspace of F, we could equally well assume that $\bigcup_{\lambda>0} \lambda\big[\mathcal{D}(\mathcal{M}g) - A\big[\mathcal{D}(\mathcal{M}f)\big]\big]$ is a closed subspace of F, or even that $\bigcup_{\lambda>0} \lambda\big[\pi_F \,\mathrm{dom}\, g - A\big[\mathcal{D}(\mathcal{M}f)\big]\big]$ is a closed subspace of F. Similar comments can be made about Theorem 8(b). See [8, Remark 32.4, p. 130] for other possibilities, and the motivitation for this kind of result. The *statement* of Theorem 20 is patterned after [8, Theorem 32.2, p. 129], but the *proof* here is much simpler by virtue of the following Brøndsted–Rockafellar property that was established in [1, Theorem 3.4] as a preliminary result for Theorem 13.

Theorem 18. *Let E be a nonzero Banach space, f be a strongly representative function on $E \times E^*$, $\alpha, \beta > 0$ and $f(x, x^*) < \langle x, x^*\rangle + \alpha\beta$. Then*

there exists $y \in X$ and $y^* \in \mathcal{M}f(x)$ such that $\|y - x\| < \alpha$ and $\|y^* - x^*\| < \beta$.

The following result is then immediate from Theorem 18.

Corollary 19. *Let E be a nonzero Banach space, π_E map (x, x^*) to x and π_{E^*} map (x, x^*) to x^*. Let f be a strongly representative function on $E \times E^*$. Then $\pi_E \,\mathrm{dom}\, f \subset \overline{\mathcal{D}(\mathcal{M}f)}$ and $\pi_{E^*} \,\mathrm{dom}\, f \subset \overline{\mathcal{R}(\mathcal{M}f)}$.*

Theorem 20. *Let E and F be nonzero Banach spaces, π_E map (x,x^*) to x, π_F map (y,y^*) to y, π_{E^*} map (x,x^*) to x^* and π_{F^*} map (y,y^*) to y^*. Let f be a strongly representative function on $E\times E^*$ and g be a strongly representative function on $F\times F^*$.*
(a) *Let $A\in L(E,F)$, and suppose that there exists a closed subspace H of F such that*

$$\mathcal{D}(\mathcal{M}g)-A\big[\mathcal{D}(\mathcal{M}f)\big]\subset H\subset \textstyle\bigcup_{\lambda>0}\lambda\big[\pi_F\operatorname{dom}g-A(\pi_E\operatorname{dom}f)\big].$$

Then $\bigcup_{\lambda>0}\lambda\big[\pi_F\operatorname{dom}g-A(\pi_E\operatorname{dom}f)\big]=H$, and consequently the conclusions of Theorem 8(a) hold.
(b) *Let $B\in L(F,E)$, and suppose that there exists a closed subspace H of F^* such that*

$$\mathcal{R}(\mathcal{M}g)-B^{\mathbf{T}}\big[\mathcal{R}(\mathcal{M}f)\big]\subset H\subset \textstyle\bigcup_{\lambda>0}\lambda\big[\pi_{F^*}\operatorname{dom}g-B^{\mathbf{T}}(\pi_{E^*}\operatorname{dom}f)\big].$$

Then $\bigcup_{\lambda>0}\lambda\big[\pi_{F^*}\operatorname{dom}g-B^{\mathbf{T}}(\pi_{E^*}\operatorname{dom}f)\big]=H$, and consequently the conclusions of Theorem 8(b) hold.

Proof. (a) It is clear from Corollary 19 and the continuity of A that

$$\pi_F\operatorname{dom}g-A(\pi_E\operatorname{dom}f)\subset\overline{\mathcal{D}(\mathcal{M}g)}-\overline{A\big[\mathcal{D}(\mathcal{M}f)\big]}\subset\overline{\mathcal{D}(\mathcal{M}g)-A\big[\mathcal{D}(\mathcal{M}f)\big]}\subset H,$$

which gives (a). (b) follows similarly from the observation that

$$\pi_{F^*}\operatorname{dom}g-B^{\mathbf{T}}(\pi_{E^*}\operatorname{dom}f)\subset\overline{\mathcal{R}(\mathcal{M}g)-B^{\mathbf{T}}\big[\mathcal{R}(\mathcal{M}f)\big]}\subset H. \qquad\square$$

6. A second quadrivariate version of the Attouch–Brezis theorem

We now come to Theorem 21, our second quadrivariate version of the Attouch–Brezis theorem. As explained in the introduction, this seems to be a more basic result than Theorem 3, and we give details in Remark 22 how Theorem 3 can be deduced from Theorem 21. The following chart should help the reader keep track of the various spaces and maps.

$$\begin{array}{ccccccc} & & X & \xrightarrow{C} & W & & \\ ]-\infty,\infty] & \xleftarrow{h} & \times & & \times & \xrightarrow{k} &]-\infty,\infty]. \\ & & U & \xleftarrow{D} & T & & \end{array}$$

Theorem 21. *Let X, W, U and T be nonzero Banach spaces, $C\in L(X,W)$, $D\in L(T,U)$ and π_W map (w,t) to w. Let $k\in\mathcal{PCLSC}(W\times T)$ and*

$$L:=\textstyle\bigcup_{\lambda>0}\lambda\big[\pi_W\operatorname{dom}k-C(X)\big]\ \text{be a closed subspace of } W. \tag{21.1}$$

For all $(x,u)\in X\times U$, let

$$h(x,u):=\inf\big\{k(Cx,t)\colon\ t\in T,\ Dt=u\big\}>-\infty.$$

Let $(x_0^*,u_0^*)\in X^*\times U^*$. Then

$$h^*(x_0^*,u_0^*)=\min\big\{k^*\big(w^*,D^{\mathbf{T}}u_0^*\big)\colon\ w^*\in W^*,\ C^{\mathbf{T}}w^*=x_0^*\big\}.$$

Proof. We first note that it is easy to see that h is convex. Furthermore, (21.1) implies that $\pi_W \operatorname{dom} k \cap C(X) \neq \emptyset$, from which it follows that h is proper. Let $w^* \in W^*$ be such that $C^{\mathbf{T}} w^* = x_0^*$, $(x,u) \in X \times U$, and $t \in T$ be such that $Dt = u$. Then, from the Fenchel–Young inequality,

$$\begin{aligned} k(Cx,t) + k^*\big(w^*, D^{\mathbf{T}} u_0^*\big) &\geq \langle Cx, w^*\rangle + \big\langle t, D^{\mathbf{T}} u_0^*\big\rangle \\ &= \langle x, C^{\mathbf{T}} w^*\rangle + \big\langle Dt, u_0^*\big\rangle = \langle x, x_0^*\rangle + \langle u, u_0^*\rangle = \big\langle (x,u), (x_0^*, u_0^*)\big\rangle. \end{aligned}$$

Taking the infimum over t, $h(x,u) + k^*\big(w^*, D^{\mathbf{T}} u_0^*\big) \geq \big\langle (x,u), (x_0^*, u_0^*)\big\rangle$. It follows from this that $h^*(x_0^*, u_0^*) \leq k^*\big(w^*, D^{\mathbf{T}} u_0^*\big)$. So what we must prove is that there exists $w^* \in W^*$ such that

$$C^{\mathbf{T}} w^* = x_0^* \quad \text{and} \quad k^*\big(w^*, D^{\mathbf{T}} u_0^*\big) \leq h^*(x_0^*, u_0^*). \tag{21.2}$$

Since h is proper, $h^*(x_0^*, u_0^*) > -\infty$, so we can suppose that $h^*(x_0^*, u_0^*) \in \mathbb{R}$. Let P stand for the product space $X \times W \times T$. Define $\sigma \in \mathcal{PCLSC}(P)$ by

$$\sigma(x,w,t) := h^*(x_0^*, u_0^*) - \langle x, x_0^*\rangle - \langle Dt, u_0^*\rangle + k(w,t)$$

and $\tau \in \mathcal{PCLSC}(P)$ by $\tau(x,w,t) := \mathbb{I}_{\{0\}}(w - Cx)$. Now let $(x,w,t) \in P$. Then, from the Fenchel–Young inequality,

$$\begin{aligned} (\sigma + \tau)(x,w,t) &\geq h^*(x_0^*, u_0^*) - \langle x, x_0^*\rangle - \langle Dt, u_0^*\rangle + k(Cx,t) \\ &\geq h^*(x_0^*, u_0^*) - \langle x, x_0^*\rangle - \big\langle Dt, u_0^*\big\rangle + h(x, Dt) \\ &= h(x,Dt) + h^*(x_0^*, u_0^*) - \big\langle (x,Dt), (x_0^*, u_0^*)\big\rangle \geq 0. \end{aligned}$$

We now define $Q \in L(X \times W, W)$ by $Q(x,w) := w - Cx$. Then, from the definitions of σ and τ, and Lemma 2 with $Z = W$, $R = C$ and $G = X \times \pi_W \operatorname{dom} k$,

$$\begin{aligned} \operatorname{dom}\sigma - \operatorname{dom}\tau &= \big\{(x - x_1, w - Cx_1, t - t_1)\colon\ x, x_1 \in X,\ (w,t) \in \operatorname{dom} k,\ t_2 \in T\big\} \\ &= \big\{(x - x_1, w - Cx_1, t_2)\colon\ x, x_2 \in X,\ (w,t) \in \operatorname{dom} k,\ t_2 \in T\big\} \\ &= \big\{(x - x_1, w - Cx_1)\colon\ x, x_1 \in X,\ w \in \pi_W \operatorname{dom} k\big\} \times T \\ &= Q^{-1}\big[Q(X \times \pi_W \operatorname{dom} k)\big] \times T = Q^{-1}\big[\pi_W \operatorname{dom} k - C(X)\big] \times T. \end{aligned}$$

It follows easily from this that $\bigcup_{\lambda > 0} \lambda\big[\operatorname{dom}\sigma - \operatorname{dom}\tau\big] = Q^{-1}(L) \times T$, which is a closed subspace of P. Theorem 1 now gives $(x^*, w^*, t^*) \in P^*$ such that

$$\sigma^*(x^*, w^*, t^*) + \tau^*(-x^*, -w^*, -t^*) \leq 0, \tag{21.3}$$

which clearly implies that $\tau^*(-x^*, -w^*, -t^*) < \infty$. However, we have by direct computation that $\tau^*(-x^*, -w^*, -t^*) = \mathbb{I}_{\{0\}}(x^* + C^{\mathbf{T}} w^*) + \mathbb{I}_{\{0\}}(t^*)$ and so it follows from (21.3) that $\sigma^*(-C^{\mathbf{T}} x^*, x^*, 0) \leq 0$. But

$$\begin{aligned} &\sigma^*(-C^{\mathbf{T}} w^*, w^*, 0) \\ &\quad = \sup_{x,w,t} \big[\big\langle x, -C^{\mathbf{T}} w^*\big\rangle + \langle w, w^*\rangle - h^*(x_0^*, u_0^*) + \langle x, x_0^*\rangle + \big\langle Dt, u_0^*\big\rangle - k(w,t)\big] \\ &\quad = \sup_{x,w,t} \big[\big\langle x, x_0^* - C^{\mathbf{T}} w^*\big\rangle + \langle w, w^*\rangle + \big\langle t, D^{\mathbf{T}} u_0^*\big\rangle - k(w,t)\big] - h^*(x_0^*, u_0^*) \\ &\quad = \mathbb{I}_{\{0\}}\big(x_0^* - C^{\mathbf{T}} w^*\big) + k^*\big(w^*, D^{\mathbf{T}} u_0^*\big) - h^*(x_0^*, u_0^*). \end{aligned}$$

Thus (21.3) reduces to (21.2). This completes the proof of Theorem 21. $\square$

Remark 22. We can deduce Theorem 3 from Theorem 21 by taking $W := X \times Y$ and $T := U \times V$, defining $C \in L(X, W)$ by $Cx := (x, Ax)$, $D \in L(T, U)$ by $D(u, v) := u + Bv$, and $k \in \mathcal{PCLSC}(W \times T)$ by $k\big((x, y), (u, v)\big) := f(x, u) + g(y, v)$.

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