

# SSDB spaces and maximal monotonicity

by

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## Abstract

In this paper, we develop some of the theory of SSD spaces and SSDB spaces, and deduce some results on maximally monotone multifunctions on a reflexive Banach space.

## 1 Introduction

In this paper, we develop enough of the theory of SSD spaces and SSDB spaces that we can obtain some significant results on maximally monotone multifunctions on a reflexive Banach space. With a few minor additions, this is a written version of the lecture with the same title delivered at the IX ISORA meeting in Lima, Peru, in October 2009, and we will not attempt to give a comprehensive exposition of the theory of SSD spaces and SSDB spaces. For this, we refer the reader to [14] and [15], from which many of the proofs given here are taken.

Many of the original results on maximally monotone multifunctions on a reflexive Banach space were obtained using Brouwer’s fixed–point theorem either directly or indirectly. The approach given here is based on convex analysis — more specifically the Fenchel duality theorem. In Section 2, we give the three versions of the Fenchel duality theorem that we will use in this paper.

In Section 3, we introduce the concepts of *SSD space*, *q–positive set* and the functions  $\Phi$ ., which are the generalizations to SSD spaces of Fitzpatrick functions of monotone sets. We also introduce the *q–positive sets*  $\mathcal{P}_q(\cdot)$  determined by certain convex functions. Despite the fact that this section uses the idea of conjugate function from convex analysis, the arguments used are essentially algebraic, apart from the disguised differentiability argument of Lemma 3.7.

In Section 4, we introduce the concept of *SSDB space*, which is a SSD space with an appropriate Banach space structure. The main results of this section are the two “pos–neg theorems”, Theorem 4.3 and 4.6 and the criterion for and properties of maximal *q–positivity* contained in Theorem 4.4.

For the rest of this paper, we suppose that  $E$  is a reflexive Banach space. In Section 5, we show how  $E \times E^*$  can be considered as an SSDB space, and how Theorems 2.4 and 4.4 lead to results on maximal monotonicity. Theorem 5.1 and Theorem 5.4 will be used in Section 6.

Up to this point, our discussion of monotonicity has been in terms of monotone subsets of  $E \times E^*$ . In Section 6 we move the emphasis to monotone multifunctions from  $E$  into  $E^*$ . In Theorem 6.3, we show how Theorem 5.1 gives Rockafellar’s surjectivity theorem, and in Theorem 6.5, we show how Theorem 5.4 gives sufficient conditions for the sum of maximally monotone multifunctions to be maximally monotone.

In the final section of this paper, Section 7, we show how the pos–neg theorem, Theorem 4.6, together with the simple properties of the reflection maps  $\rho_1, \rho_2: E \times E^* \rightarrow E \times E^*$  defined by  $\rho_1(x, x^*) := (-x, x^*)$  and  $\rho_2(x, x^*) := (x, -x^*)$  lead to an abstract Hammerstein theorem.

## SSDB spaces and maximal monotonicity

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### 2 Versions of the Fenchel duality theorem

All vector spaces in the paper will be *real*. If  $X$  is a nonzero vector space and  $f: F \mapsto ]-\infty, \infty]$  then we write  $\text{dom } f := \{x \in F: f(x) \in \mathbb{R}\}$ . The set  $\text{dom } f$  is the *effective domain* of  $f$ .  $f$  is said to be *proper* if  $\text{dom } f \neq \emptyset$ . We write  $\mathcal{PC}(X)$  for the set of proper convex functions from  $X$  into  $]-\infty, \infty]$ . If  $X$  is a nonzero normed space we write  $\mathcal{PCLSC}(X)$  for  $\{f \in \mathcal{PC}(X): f \text{ is lower semicontinuous}\}$ . If  $E$  and  $F$  are nonzero vector spaces,  $\langle \cdot, \cdot \rangle: E \times F \rightarrow \mathbb{R}$  is a bilinear form and  $f \in \mathcal{PC}(E)$  then the *Fenchel conjugate*,  $f^*$ , of  $f$  with respect to  $\langle \cdot, \cdot \rangle$  is defined for  $y \in F$  by

$$f^*(y) := \sup_{x \in E} [\langle x, y \rangle - f(x)].$$

The commonest (but not the only) use of this notation is when  $E$  is a normed space,  $F$  is the dual,  $E^*$ , of  $E$  and  $\langle \cdot, \cdot \rangle$  is the duality pairing.

We start off by stating a result that is an immediate consequence of Rockafellar's version of the Fenchel duality theorem (see [11, Theorem 1, p. 82–83] for the original version and Zălinescu, [18, Theorem 2.8.7, p. 126–127] for more general results):

**Theorem 2.1.** *Let  $F$  be a nonzero normed space,  $f: F \mapsto ]-\infty, \infty]$  be proper and convex,  $g: F \mapsto \mathbb{R}$  be convex and continuous, and  $f + g \geq 0$  on  $F$ . Then there exists  $z^* \in F^*$  such that  $f^*(z^*) + g^*(-z^*) \leq 0$ .*

Theorem 2.2 below was first proved by Attouch–Brezis (this follows from [1, Corollary 2.3, pp. 131–132]) – there is a somewhat different proof in Simons, [14, Theorem 15.1, p. 66], and a much more general result was established in Zălinescu, [18, Theorem 2.8.6, pp. 125–126]. We note that the result contained in Simons, [14, Theorem 8.4, p. 46] implies both Theorem 2.1 and Theorem 2.2, and that [14, Theorem 7.4, p. 43] gives a sharp lower bound on the possible values of  $\|z^*\|$ .

**Theorem 2.2.** *Let  $E$  be a nonzero Banach space,  $f, g \in \mathcal{PCLSC}(E)$ ,  $f + g \geq 0$  on  $E$  and  $\bigcup_{\lambda > 0} \lambda[\text{dom } f - \text{dom } g]$  be a closed linear subspace of  $E$ . Then there exists  $z^* \in E^*$  such that  $f^*(z^*) + g^*(-z^*) \leq 0$ .*

**Notation 2.3.** If  $E$  and  $F$  are nonzero normed spaces, we define the dual of  $E \times F$  to be  $E^* \times F^*$  under the pairing  $\langle (x, y), (x^*, y^*) \rangle := \langle x, x^* \rangle + \langle y, y^* \rangle$  ( $(x, y) \in E \times F$ ,  $(x^*, y^*) \in E^* \times F^*$ ). We then define the *projection maps*  $\pi_1, \pi_2$  by  $\pi_1(x, y) := x$  and  $\pi_2(x, y) := y$ .

We end this section with a bivariate generalization of Theorem 2.2, which was first proved in Simons–Zălinescu, [16, Theorem 4.2, pp. 9–10]. There was a simpler proof given in Simons, [14, Theorem 16.4, pp. 68–69]. The hypothesis of Theorem 2.4 is that  $h(x, \cdot)$  is the “inf–convolution” of  $f(x, \cdot)$  and  $g(x, \cdot)$ , and the conclusion is that  $h^*(\cdot, y^*)$  is the “exact inf–convolution” of  $f^*(\cdot, y^*)$  and  $g^*(\cdot, y^*)$ .

### SSDB spaces and maximal monotonicity

**Theorem 2.4.** *Let  $E$  and  $F$  be nonzero Banach spaces,  $f, g: E \times F \mapsto ]-\infty, \infty]$  be proper, convex and lower semicontinuous,  $\bigcup_{\lambda>0} \lambda[\pi_1 \text{ dom } f - \pi_1 \text{ dom } g]$  be a closed linear subspace of  $E$  and, for all  $(x, y) \in E \times F$ ,*

$$h(x, y) := \inf \{f(x, u) + g(x, v): u, v \in F, u + v = y\} > -\infty.$$

Then, for all  $(x^*, y^*) \in E^* \times F^* = (E \times F)^*$ ,

$$h^*(x^*, y^*) = \min \{f^*(s^*, y^*) + g^*(t^*, y^*): s^*, t^* \in E^*, s^* + t^* = x^*\}.$$

### 3 SSD spaces

**Definition 3.1.** We will say that  $(B, [\cdot, \cdot])$  is a *symmetrically self-dual space (SSD space)* if  $B$  is a nonzero real vector space and  $[\cdot, \cdot]: B \times B \rightarrow \mathbb{R}$  is a symmetric bilinear form. In this case, we will *always* write  $q(b) := \frac{1}{2}[b, b]$  ( $b \in B$ ). (“ $q$ ” stands for “quadratic”.)

Now let  $(B, [\cdot, \cdot])$  be an SSD space and  $A \subset B$ . We say that  $A$  is  *$q$ -positive* if  $A \neq \emptyset$  and

$$b, c \in A \implies q(b - c) \geq 0.$$

In this case, since  $q(0) = 0$ ,

$$b \in A \implies \inf q(A - b) = 0. \tag{1}$$

We then define  $\Phi_A: B \rightarrow ]-\infty, \infty]$  by

$$\Phi_A(b) := \sup_A [\cdot, b] - q \quad (b \in B). \tag{2}$$

$\Phi_A$  is a generalization to SSD spaces of the “Fitzpatrick function” of a monotone set, which was originally introduced in [6] in 1988, but lay dormant until it was rediscovered by Martínez-Legaz and Théra in [8] in 2001. We note then that, for all  $b \in B$ ,

$$\left. \begin{aligned} \Phi_A(b) &= q(b) - \inf_{a \in A} [q(a) - [a, b] + q(b)] \\ &= q(b) - \inf_{a \in A} q(a - b) = q(b) - \inf q(A - b). \end{aligned} \right\} \tag{3}$$

From (1),

$$\Phi_A = q \text{ on } A. \tag{4}$$

Thus  $\Phi_A \in \mathcal{PC}(B)$ . We say that  $A$  is *maximally  $q$ -positive* if  $A$  is  $q$ -positive and  $A$  is not properly contained in any other  $q$ -positive set. In this case, if  $b \in B$  and  $\inf q(A - b) \geq 0$  then clearly  $b \in A$ . In other words,  $(b \in B \setminus A \implies \inf q(A - b) < 0)$ . From (1),  $\inf q(A - b) \leq 0$  and  $(\inf q(A - b) = 0 \iff b \in A)$ . Thus, from (3)

$$\Phi_A \geq q \text{ on } B \quad \text{and} \quad (\Phi_A(b) = q(b) \iff b \in A). \tag{5}$$

We make the elementary observation that if  $b \in B$  and  $q(b) \geq 0$  then the linear span  $\mathbb{R}b$  of  $\{b\}$  is  $q$ -positive.

## SSDB spaces and maximal monotonicity

We now give some examples of SSD spaces and their associated  $q$ -positive sets. These examples are taken from [14, pp. 79–80].

**Example 3.2.** Let  $B$  be a Hilbert space with inner product  $(b, c) \mapsto \langle b, c \rangle$ .

(a) If, for all  $b, c \in B$ ,  $[b, c] := \langle b, c \rangle$  then  $q(b) = \frac{1}{2}\|b\|^2$  and every nonempty subset of  $B$  is  $q$ -positive.

(b) If, for all  $b, c \in B$ ,  $[b, c] := -\langle b, c \rangle$  then  $q(b) = -\frac{1}{2}\|b\|^2$  and the  $q$ -positive sets are the singletons.

(c) If  $B = \mathbb{R}^3$  and

$$[(b_1, b_2, b_3), (c_1, c_2, c_3)] := b_1c_2 + b_2c_1 + b_3c_3,$$

then  $q(b_1, b_2, b_3) = b_1b_2 + \frac{1}{2}b_3^2$ . Here, if  $M$  is any nonempty monotone subset of  $\mathbb{R} \times \mathbb{R}$  (in the obvious sense) then  $M \times \mathbb{R}$  is a  $q$ -positive subset of  $B$ . The set  $\mathbb{R}(1, -1, 2)$  is a  $q$ -positive subset of  $B$  which is not contained in a set  $M \times \mathbb{R}$  for any monotone subset of  $\mathbb{R} \times \mathbb{R}$ . The helix  $\{(\cos \theta, \sin \theta, \theta) : \theta \in \mathbb{R}\}$  is a  $q$ -positive subset of  $B$ , but if  $0 < \lambda < 1$  then the helix  $\{(\cos \theta, \sin \theta, \lambda\theta) : \theta \in \mathbb{R}\}$  is not.

**Example 3.3.** Let  $E$  be a nonzero Banach space and  $B := E \times E^*$ . For all  $(x, x^*)$  and  $(y, y^*) \in B$ , we set  $[(x, x^*), (y, y^*)] := \langle x, y^* \rangle + \langle y, x^* \rangle$ . Then  $(B, [\cdot, \cdot])$  is an SSD space with  $q(x, x^*) = \frac{1}{2}[\langle x, x^* \rangle + \langle x, x^* \rangle] = \langle x, x^* \rangle$ . Consequently, if  $(x, x^*), (y, y^*) \in B$  then  $\langle x - y, x^* - y^* \rangle = q(x - y, x^* - y^*) = q((x, x^*) - (y, y^*))$ . Thus if  $A \subset B$  then  $A$  is  $q$ -positive exactly when  $A$  is a nonempty monotone subset of  $B$  in the usual sense, and  $A$  is maximally  $q$ -positive exactly when  $A$  is a maximally monotone subset of  $B$  in the usual sense. We point out that any finite dimensional SSD space of the form described here must have even dimension. Thus cases of Example 3.2 with finite odd dimension cannot be of this form.

**Example 3.4.**  $(\mathbb{R}^3, [\cdot, \cdot])$  is *not* an SSD space with

$$[(b_1, b_2, b_3), (c_1, c_2, c_3)] := b_1c_2 + b_2c_3 + b_3c_1.$$

(The bilinear form  $[\cdot, \cdot]$  is not symmetric.)

**Definition 3.5.** Let  $(B, [\cdot, \cdot])$  be an SSD space. If  $f \in \mathcal{PC}(B)$  and  $f \geq q$  on  $B$ , we write

$$\mathcal{P}_q(f) := \{b \in B : f(b) = q(b)\}.$$

We then note from (5) that

$$\text{if } A \text{ is maximally } q\text{-positive then } A = \mathcal{P}_q(\Phi_A). \tag{6}$$

If  $g \in \mathcal{PC}(B)$  and  $g \geq -q$  on  $B$  then we write  $\mathcal{N}_q(g) := \{b \in B : g(b) = -q(b)\}$ .

We now introduce the concept of *intrinsic conjugate* for an SSD space.

## SSDB spaces and maximal monotonicity

**Definition 3.6.** If  $(B, [\cdot, \cdot])$  is an SSD space and  $f \in \mathcal{PC}(B)$ , we write  $f^\circledast$  for the Fenchel conjugate of  $f$  with respect to the pairing  $[\cdot, \cdot]$ , that is to say,

$$\text{for all } c \in B, \quad f^\circledast(c) := \sup_B [\cdot, c] - f.$$

The concepts introduced in Definitions 3.5 and 3.6 are related in our next result, which uses a disguised differentiability argument.

**Lemma 3.7.** *Let  $(B, [\cdot, \cdot])$  be an SSD space,  $f \in \mathcal{PC}(B)$  and  $f \geq q$  on  $B$ . Then  $f^\circledast = q$  on  $\mathcal{P}_q(f)$ .*

**Proof.** Let  $a \in \mathcal{P}_q(f)$ . Let  $\lambda \in ]0, 1[$ . For simplicity in writing, let  $\mu := 1 - \lambda \in ]0, 1[$ . Then, for all  $b \in B$ ,

$$\begin{aligned} \lambda^2 q(b) + \lambda\mu[b, a] + \mu^2 q(a) &= q(\lambda b + \mu a) \leq f(\lambda b + \mu a) \\ &\leq \lambda f(b) + \mu f(a) = \lambda f(b) + \mu q(a). \end{aligned}$$

Thus  $\lambda^2 q(b) + \lambda\mu[b, a] \leq \lambda f(b) + \lambda\mu q(a)$ . Dividing by  $\lambda$  and letting  $\lambda \rightarrow 0$ , we have  $[b, a] \leq f(b) + q(a)$ , that is to say  $[a, b] - f(b) \leq q(a)$ , and, taking the supremum over  $b$ ,  $f^\circledast(a) \leq q(a)$ . On the other hand,  $f^\circledast(a) \geq [a, a] - f(a) = 2q(a) - q(a) = q(a)$ , completing the proof of Lemma 3.7.  $\square$

The next result gives a basic property of  $\Phi_A^\circledast$ .

**Lemma 3.8.** *Let  $(B, [\cdot, \cdot])$  be an SSD space and  $A$  be a nonempty  $q$ -positive subset of  $B$ . Then  $\Phi_A^\circledast \geq \Phi_A$  on  $B$ .*

**Proof.** Let  $c \in B$ . Then, from (4),

$$\Phi_A^\circledast(c) = \sup_B [[c, \cdot] - \Phi_A] \geq \sup_A [[c, \cdot] - \Phi_A] = \sup_A [[c, \cdot] - q] = \Phi_A(c). \quad \square$$

We now introduce the important concepts of ‘‘BC–function’’ and ‘‘TBC–function’’.

**Definition 3.9.** Let  $(B, [\cdot, \cdot])$  be a SSD space and  $f, g \in \mathcal{PC}(B)$ . We say that  $f$  is a *BC–function* if

$$b \in B \implies f^\circledast(b) \geq f(b) \geq q(b). \quad (7)$$

‘‘BC’’ stands for ‘‘bigger conjugate’’. We say that  $g$  is a *TBC–function* if

$$b \in B \implies g^\circledast(-b) \geq g(b) \geq -q(b). \quad (8)$$

‘‘TBC’’ stands for ‘‘twisted bigger conjugate’’. Of course,  $g$  is a TBC–function if, and only if,  $g$  is a BC–function with respect to the SSD space  $(B, -[\cdot, \cdot])$

**Lemma 3.10.** *Let  $(B, [\cdot, \cdot])$  be a SSD space and  $f \in \mathcal{PC}(B)$  be a BC–function. Then*

$$\mathcal{P}_q(f^\circledast) = \mathcal{P}_q(f).$$

**Proof.** This follows from Lemma 3.7 and the inclusion  $\mathcal{P}_q(f^\circledast) \subset \mathcal{P}_q(f)$ , which is immediate from (7).  $\square$

## SSDB spaces and maximal monotonicity

**Theorem 3.11.** *Let  $(B, [\cdot, \cdot])$  be an SSD space and  $A$  be a maximally  $q$ -positive subset of  $B$ . Then  $\Phi_A$  is a BC-function and  $\mathcal{P}_q(\Phi_A^{\textcircled{a}}) = \mathcal{P}_q(\Phi_A) = A$ .*

**Proof.** The first assertion follows from Lemma 3.8 and (5), and the second assertion is immediate from Lemma 3.10 and (6).  $\square$

**Remark 3.12.** We shall see by combining (9) and Remark 5.5 that there may exist a function that is both a BC-function and a TBC-function but not of the form  $\Phi_A$  for any nonempty  $q$ -positive set  $A$ .

We now give two computational results. In the first of these, we investigate the “translation” of a BC-function. This result will be used in the “pos-neg” theorems, Theorem 4.3 and Theorem 4.6.

**Lemma 3.13.** *Let  $(B, [\cdot, \cdot])$  be a SSD space,  $f \in \mathcal{PC}(B)$  be a BC-function and  $c \in B$ . We define  $f_c \in \mathcal{PC}(B)$  by  $f_c := f(\cdot + c) - [\cdot, c] - q(c)$ . Then  $f_c$  is a BC-function,  $\text{dom } f_c = \text{dom } f - c$  and  $\mathcal{P}_q(f_c) = \mathcal{P}_q(f) - c$ .*

**Proof.** For all  $b \in B$ ,

$$\begin{aligned} f_c^{\textcircled{a}}(b) &= \sup_{d \in B} [[d, b] + [d, c] + q(c) - f(d + c)] \\ &= \sup_{e \in B} [[e - c, b + c] + q(c) - f(e)] \\ &= \sup_{e \in B} [[e, b + c] - [c, b] - f(e)] - q(c) \\ &= f^{\textcircled{a}}(b + c) - [c, b] - q(c). \end{aligned}$$

It follows from (7) that  $f_c^{\textcircled{a}}(b) \geq f(b + c) - [b, c] - q(c) = f_c(b)$  and

$$f_c(b) = f(b + c) - [b, c] - q(c) \geq q(b + c) - [b, c] - q(c) = q(b).$$

Consequently,  $f_c$  is a BC-function. It is obvious that  $\text{dom } f_c = \text{dom } f - c$ . Further, since

$$\begin{aligned} b \in \mathcal{P}_q(f_c) &\iff f(b + c) - [b, c] - q(c) = q(b) \\ &\iff f(b + c) = q(b + c) \iff b + c \in \mathcal{P}_q(f), \end{aligned}$$

we have  $\mathcal{P}_q(f_c) = \mathcal{P}_q(f) - c$ , as required.  $\square$

In the second computational result, we show how we can always obtain a TBC-function from a BC-function by an appropriate linear transformation.

**Lemma 3.14.** *Let  $(B, [\cdot, \cdot])$  be an SSD space,  $\rho: B \rightarrow B$  be a linear surjection such that, for all  $b, c \in B$ ,  $[\rho(b), \rho(c)] = [b, -c]$ , and  $g \in \mathcal{PC}(B)$  be a BC-function. Then  $g \circ \rho$  and  $g \circ (-\rho)$  are TBC-functions,  $\rho \text{dom}(g \circ \rho) = \text{dom } g$ ,  $\rho \mathcal{N}_q(g \circ \rho) = \mathcal{P}_q(g)$ ,  $-\rho \text{dom}(g \circ (-\rho)) = \text{dom } g$  and  $-\rho \mathcal{N}_q(g \circ (-\rho)) = \mathcal{P}_q(g)$ .*

**Proof.** We give the proof for  $g \circ \rho$ . The proof for  $g \circ (-\rho)$  is similar. For all  $c \in B$ ,

$$\begin{aligned} (g \circ \rho)^{\textcircled{a}}(-c) &= \sup_{b \in B} [[b, -c] - g(\rho(b))] = \sup_{b \in B} [[\rho(b), \rho(c)] - g(\rho(b))] \\ &= \sup_{d \in B} [[d, \rho(c)] - g(d)] = g^{\textcircled{a}}(\rho(c)) \geq g(\rho(c)) \\ &\geq q(\rho(c)) = \frac{1}{2}[\rho(c), \rho(c)] = \frac{1}{2}[c, -c] = -q(c). \end{aligned}$$

## SSDB spaces and maximal monotonicity

Thus  $g \circ \rho$  is a TBC– function. Further,

$$\begin{aligned} b \in \rho \operatorname{dom}(g \circ \rho) &\iff \text{there exists } c \in B \text{ such that } g(\rho(c)) \in \mathbb{R} \text{ and } b = \rho(c) \\ &\iff \text{there exists } d \in B \text{ such that } g(d) \in \mathbb{R} \text{ and } b = d \\ &\iff g(b) \in \mathbb{R} \iff b \in \operatorname{dom} g, \end{aligned}$$

from which  $\rho \operatorname{dom}(g \circ \rho) = \operatorname{dom} g$ , and

$$\begin{aligned} b \in \rho \mathcal{N}_q(g \circ \rho) &\iff \text{there exists } c \in B \text{ such that } g(\rho(c)) = -q(c) \text{ and } b = \rho(c) \\ &\iff \text{there exists } c \in B \text{ such that } g(\rho(c)) = q(\rho(c)) \text{ and } b = \rho(c) \\ &\iff \text{there exists } d \in B \text{ such that } g(d) = q(d) \text{ and } b = d \\ &\iff b \in \mathcal{P}_q(g), \end{aligned}$$

from which  $\rho \mathcal{N}_q(g \circ \rho) = \mathcal{P}_q(g)$ . □

Lemma 3.15 is a subtler property of the bilinear form  $q$ . In the situation of Example 3.3, Corollary 3.16 gives us Voisei–Zălinescu [17, Proposition 1].

**Lemma 3.15.** *Let  $(B, [\cdot, \cdot])$  be an SSD space,  $f \in \mathcal{PC}(B)$ ,  $f \geq q$  on  $B$  and  $b, c \in B$ . Then*

$$-q(b - c) \leq \left[ \sqrt{(f - q)(b)} + \sqrt{(f - q)(c)} \right]^2.$$

**Proof.** See [15, Lemma 2.6]. □

**Corollary 3.16.** *Let  $(B, [\cdot, \cdot])$  be an SSD space,  $f \in \mathcal{PC}(B)$ ,  $f \geq q$  on  $B$  and  $b, c \in B$ . Then*

$$-q(b - c) \leq 2(f - q)(b) + 2(f - q)(c).$$

**Proof.** This is immediate from Lemma 3.15 and the Cauchy–Schwarz inequality. □

Lemma 3.17 is suggested by Burachik–Svaiter, [5, Theorem 3.1, pp. 2381–2382] and Penot, [9, Proposition 4(h) $\implies$ (a), pp. 860–861]. Lemma 3.17 will be used in Theorem 4.4.

**Lemma 3.17.** *Let  $(B, [\cdot, \cdot])$  be an SSD space,  $f \in \mathcal{PC}(B)$ ,  $f \geq q$  on  $B$  and  $\mathcal{P}_q(f) \neq \emptyset$ . Then  $\mathcal{P}_q(f)$  is a  $q$ –positive subset of  $B$ .*

**Proof.** This is immediate from Lemma 3.15, or Corollary 3.16, or by using the fact that  $q$  satisfies the *parallelogram law*:  $x, y \in B \implies q(x) + q(y) = \frac{1}{2}q(x + y) + \frac{1}{2}q(x - y)$ . □

The final result of this section can be thought of as a completion of Lemma 3.8. It will not be used in the sequel.

**Lemma 3.18.** *Let  $(B, [\cdot, \cdot])$  be an SSD space and  $A$  be a nonempty  $q$ –positive subset of  $B$ . Then  $\Phi_A^{\textcircled{a}} \leq q$  on  $A$ ,  $\Phi_A^{\textcircled{a}} \geq q$  on  $B$ , and  $\Phi_A^{\textcircled{a}\textcircled{a}} = \Phi_A$  on  $B$ .*

**Proof.** See [15, Lemma 2.11]. □

## SSDB spaces and maximal monotonicity

### 4 SSDB spaces

We now introduce the SSDB spaces, a subclass of the class of SSD spaces. Our treatment of the SSD spaces in Section 3 has been essentially nontopological. The additional norm structure of the SSDB spaces is essentially what makes maximally monotone multifunctions on a reflexive Banach space much more tractable than those on a general Banach space. We will return to this issue when we consider Example 3.3 in Section 5.

**Definition 4.1.** We will say that  $(B, [\cdot, \cdot], \|\cdot\|)$  is a *symmetrically self-dual Banach space* (SSDB space) if  $(B, [\cdot, \cdot])$  is a SSD space,  $(B, \|\cdot\|)$  is a Banach space, and there exists a linear isometry  $\iota$  from  $B$  onto  $B^*$  such that, for all  $b, c \in B$ ,  $\langle b, \iota(c) \rangle = [b, c]$ . We note then that, for all  $f \in \mathcal{PC}(B)$  and  $c \in B$ ,  $f^\circledast(c) := \sup_B [\cdot, c] - f = \sup_B [\langle \cdot, \iota(c) \rangle - f] = f^*(\iota(c))$ , that is to say  $f^\circledast = f^* \circ \iota$ . It is easy to see that the quadratic form  $q$  is continuous and, for all  $b \in B$ ,  $|q(b)| = \frac{1}{2} |[b, b]| \leq \frac{1}{2} \|b\|^2$ .

Let  $g_0 := \frac{1}{2} \|\cdot\|^2$  on  $B$ . Then, for all  $b \in B$ ,

$$g_0^\circledast(b) = g_0^*(\iota(b)) = \frac{1}{2} \|\iota(b)\|^2 = \frac{1}{2} \|b\|^2 = g_0(b).$$

It follows that

$$g_0 \text{ is both a BC-function and a TBC-function.} \tag{9}$$

**Examples 4.2.** (a) In Example 3.2(a),  $(B, [\cdot, \cdot], \|\cdot\|)$  is a SSDB space under the Hilbert space norm,  $q = g_0$  and  $\mathcal{N}_q(g_0) = \{0\}$ . (We recall that the set  $\mathcal{N}_q(g_0)$  was defined in Definition 3.5.)

(b) In Example 3.2(b),  $(B, [\cdot, \cdot], \|\cdot\|)$  is a SSDB space under the Hilbert space norm,  $q = -g_0$  and  $\mathcal{N}_q(g_0) = B$ .

(c) In Example 3.2(c),  $(B, [\cdot, \cdot], \|\cdot\|)$  is a SSDB space under the Euclidean norm and

$$\mathcal{N}_q(g_0) = \{(b_1, b_2, b_3) \in B: b_1 + b_2 = 0, b_3 = 0\}.$$

Our next result is the “pos-neg theorem of Rockafellar type”. This will be used indirectly in Theorem 5.1, which will be used in turn indirectly in our proof of Rockafellar’s surjectivity theorem, Theorem 6.3, and the sum theorem, Theorem 6.5. This result first appeared in Simons, [14, Theorem 19.16, p. 83].

**Theorem 4.3.** Let  $(B, [\cdot, \cdot], \|\cdot\|)$  be a SSDB space,  $f \in \mathcal{PC}(B)$  be a BC-function and  $g: B \rightarrow \mathbb{R}$  be a continuous TBC-function. Then  $\mathcal{P}_q(f) - \mathcal{N}_q(g) = B$ .

**Proof.** Let  $c$  be an arbitrary element of  $B$ . From Lemma 3.13,  $f_c$  is a BC-function and so, using (7) and (8),

$$b \in B \implies f_c(b) + g(b) \geq q(b) - q(b) = 0.$$

Rockafellar’s version of the Fenchel duality theorem, Theorem 2.1, and the surjectivity of  $\iota$  now give  $b \in B$  such that  $f_c^*(\iota(b)) + g^*(-\iota(b)) \leq 0$ , that is to say  $f_c^\circledast(b) + g^\circledast(-b) \leq 0$ . From (7) and (8) again,  $f_c(b) + g(b) \leq 0 = q(b) - q(b)$ . From (7) and (8) for a third time,  $f_c(b) = q(b)$  and  $g(b) = -q(b)$ , that is to say, using Lemma 3.13,  $b \in \mathcal{P}_q(f_c) = \mathcal{P}_q(f) - c$  and also  $b \in \mathcal{N}_q(g)$ . But then  $c = (c + b) - b \in \mathcal{P}_q(f) - \mathcal{N}_q(g)$ . This completes the proof of Theorem 4.3.  $\square$



## SSDB spaces and maximal monotonicity

**Theorem 4.4.** *Let  $(B, [\cdot, \cdot], \|\cdot\|)$  be a SSDB space. Then:*

- (a) *Let  $f \in \mathcal{PC}(B)$  be a BC–function. Then  $\mathcal{P}_q(f) - \mathcal{N}_q(g_0) = B$ .*
- (b) *Let  $A$  be a  $q$ –positive subset of  $B$  and  $A - \mathcal{N}_q(g_0) = B$ . Then  $A$  is maximally  $q$ –positive.*
- (c) *Let  $f \in \mathcal{PC}(B)$  be a BC–function. Then  $\mathcal{P}_q(f)$  is maximally  $q$ –positive. Furthermore,  $\mathcal{P}_q(f^\circledast) = \mathcal{P}_q(f)$ .*
- (d) *Let  $A$  be a  $q$ –positive subset of  $B$ . Then  $A$  is maximally  $q$ –positive if, and only if  $A - \mathcal{N}_q(g_0) = B$ .*

**Proof.** (a) This follows from (9) and Theorem 4.3.

(b) Suppose that  $b \in B$  and  $A \cup \{b\}$  is  $q$ –positive. By hypothesis, there exists  $a \in A$  such that  $a - b \in \mathcal{N}_q(g_0)$ . Thus  $\frac{1}{2}\|a - b\|^2 = -q(a - b)$ . Since  $A \cup \{b\}$  is  $q$ –positive,  $q(a - b) \geq 0$ , and so  $\frac{1}{2}\|a - b\|^2 \leq 0$ . Thus  $b = a \in A$ .

(c) From (a) and Lemma 3.17,  $\mathcal{P}_q(f)$  is nonempty and  $q$ –positive. (c) now follows from (a), (b) and Lemma 3.10.

(d) We have already established ( $\Leftarrow$ ) in (b). Suppose, conversely, that  $A$  is maximally  $q$ –positive. It follows from Theorem 3.11 that  $\Phi_A$  is a BC–function and  $\mathcal{P}_q(\Phi_A) = A$ . Thus, from (a),  $A - \mathcal{N}_q(g_0) = \mathcal{P}_q(\Phi_A) - \mathcal{N}_q(g_0) = B$ .  $\square$

**Remark 4.5.** We note from (9) and Theorem 4.4(c) that  $\mathcal{P}_q(g_0)$  is maximally  $q$ –positive and  $\mathcal{P}_q(g_0^\circledast) = \mathcal{P}_q(g_0)$ . Now suppose that  $a \in \mathcal{P}_q(g_0)$  and  $b \in B$ . From the Fenchel–Young inequality,  $[a, b] - q(a) = [a, b] - g_0^\circledast(a) \leq g_0(b)$ . Taking the supremum over  $a \in \mathcal{P}_q(g_0)$ , we deduce that  $\Phi_{\mathcal{P}_q(g_0)} \leq g_0$  on  $B$ . In particular,  $\text{dom } \Phi_{\mathcal{P}_q(g_0)} = B$ .

The next result in this section is the “pos-neg theorem of Attouch–Brezis type”. A stronger result was proved in Simons, [14, Theorem 21.12, p. 93].

**Theorem 4.6.** *Let  $(B, [\cdot, \cdot], \|\cdot\|)$  be a SSDB space,  $f \in \mathcal{PCLSC}(B)$  be a BC–function, and  $g \in \mathcal{PCLSC}(B)$  be a TBC–function. Then  $\text{dom } f - \text{dom } g = B \iff \mathcal{P}_q(f) - \mathcal{N}_q(g) = B$ .*

**Proof.** Since the implication ( $\Leftarrow$ ) is trivial, we only have to prove ( $\Rightarrow$ ). Let  $c$  be an arbitrary element of  $B$ . Then, from Lemma 3.13,

$$\text{dom } f_c - \text{dom } g = \text{dom } f - c - \text{dom } g = B - c = B,$$

and so  $\bigcup_{\lambda > 0} \lambda[\text{dom } f_c - \text{dom } g] = B$ . From Lemma 3.13,  $f_c$  is a BC–function and so, using (7) and (8),

$$b \in B \implies f_c(b) + g(b) \geq q(b) - q(b) = 0.$$

The Attouch–Brezis theorem, Theorem 2.2, and the surjectivity of  $\iota$  now give  $b \in B$  such that  $f_c^*(\iota(b)) + g^*(-\iota(b)) \leq 0$ , that is to say  $f_c^\circledast(b) + g^\circledast(-b) \leq 0$ . From (7) and (8) again,  $f_c(b) + g(b) \leq 0 = q(b) - q(b)$ . From (7) and (8) for a third time,  $f_c(b) = q(b)$  and  $g(b) = -q(b)$ , that is to say, using Lemma 3.13,  $b \in \mathcal{P}_q(f_c) = \mathcal{P}_q(f) - c$  and also  $b \in \mathcal{N}_q(g)$ . But then  $c = (c + b) - b \in \mathcal{P}_q(f) - \mathcal{N}_q(g)$ . This completes the proof of Theorem 4.6.  $\square$

## SSDB spaces and maximal monotonicity

Corollary 4.7 below is the form in which we will actually apply Theorem 4.6 to abstract Hammerstein theorems in Theorem 7.1.

**Corollary 4.7.** *Let  $(B, [\cdot, \cdot], \|\cdot\|)$  be a SSDB space,  $f, g \in \mathcal{PCLSC}(B)$  and  $f, g$  be BC–functions. Suppose, further, that  $\rho: B \rightarrow B$  is a continuous linear bijection such that, for all  $b, c \in B$ ,  $[\rho(b), \rho(c)] = [b, -c]$ . Then*

$$\text{dom } f - \rho^{-1}\text{dom } g = B \iff \mathcal{P}_q(f) - \rho^{-1}\mathcal{P}_q(g) = B$$

and

$$\text{dom } f + \rho^{-1}\text{dom } g = B \iff \mathcal{P}_q(f) + \rho^{-1}\mathcal{P}_q(g) = B.$$

**Proof.** This is immediate from Theorem 4.6 and Lemma 3.14. □

### 5 BC–functions on $E \times E^*$

From now on,  $E$  is a nonzero reflexive Banach space and  $E^*$  is its topological dual space. As observed in Example 3.3,  $(E \times E^*, [\cdot, \cdot])$  is an SSD space with  $q(x, x^*) = \langle x, x^* \rangle$ , and if  $A \subset E \times E^*$  then  $A$  is  $q$ –positive exactly when  $A$  is a nonempty monotone set in the usual sense, and  $A$  is maximally  $q$ –positive exactly when  $A$  is a maximally monotone set of in the usual sense. We norm  $E \times E^*$  by  $\|(x, x^*)\| := \sqrt{\|x\|^2 + \|x^*\|^2}$ . Then

$$(E \times E^*, \|\cdot\|)^* = (E^* \times E, \|\cdot\|),$$

under the duality  $\langle (x, x^*), (y^*, y) \rangle := \langle x, y^* \rangle + \langle y, x^* \rangle$ . Combining this with the formula in Example 3.3, we have  $\langle (x, x^*), (y^*, y) \rangle = [(x, x^*), (y, y^*)]$ , and so  $\iota(y, y^*) = (y^*, y)$ . It is easily seen from these relationships that  $(E \times E^*, [\cdot, \cdot], \|\cdot\|)$  is an SSDB space. Further,  $(x, x^*) \in \mathcal{P}_q(g_0) \iff \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 = \langle x, x^* \rangle \iff (x, x^*) \in G(J)$ , and  $(x, x^*) \in \mathcal{N}_q(g_0) \iff \frac{1}{2}\|x\|^2 + \frac{1}{2}\|x^*\|^2 = -\langle x, x^* \rangle \iff (x, x^*) \in G(-J)$ , where  $J: E \rightrightarrows E^*$  is the duality map. Thus we obtain the following fundamental result:

**Theorem 5.1.** *Let  $E$  be a nonzero reflexive Banach space. Then:*

- (a) *If  $f \in \mathcal{PC}(E \times E^*)$  and  $f$  is a BC–function then  $\mathcal{P}_q(f)$  is maximally monotone,  $\mathcal{P}_q(f^{\textcircled{a}}) = \mathcal{P}_q(f)$  and  $\mathcal{P}_q(f) - G(-J) = B$ .*
- (b) *If  $A$  is a monotone subset of  $E \times E^*$  then  $A$  is maximally monotone if, and only if,  $A - G(-J) = E \times E^*$ .*

**Proof.** (a) is immediate from Theorem 4.4(a,c), and (b) is immediate from Theorem 4.4(d). □

**Remark 5.2.** If  $f \in \mathcal{PCLSC}(E)$  and, for all  $(x, x^*) \in B$ ,  $h(x, x^*) := f(x) + f^*(x^*)$  then the Fenchel–Moreau theorem and the Fenchel–Young inequality imply that  $h$  is a BC–function on  $E \times E^*$ . Since  $\mathcal{P}_q(h) = G(\partial f)$ , Theorem 5.1(a) gives us that  $\partial f$  is maximally monotone. (Remember that we are assuming that  $E$  is reflexive!)

We note that the “partial episum theorem” of Theorem 5.3 below can also be deduced from Penot–Zălinescu, [10, Corollary 3.7].

## SSDB spaces and maximal monotonicity

**Theorem 5.3.** *Let  $E$  be a nonzero reflexive Banach space,  $f, g \in \mathcal{PCLSC}(E \times E^*)$  be BC-functions,  $\bigcup_{\lambda>0} \lambda[\pi_1 \text{ dom } f - \pi_1 \text{ dom } g]$  be a closed linear subspace of  $E$  and, for all  $(x, x^*) \in E \times E^*$ ,*

$$h(x, x^*) := \inf \{f(x, s^*) + g(x, t^*): s^*, t^* \in E^*, s^* + t^* = x^*\}. \quad (10)$$

*Then  $h$  is a BC-function,  $\mathcal{P}_q(h^\circledast) = \{(x, s^* + t^*): (x, s^*) \in \mathcal{P}_q(f^\circledast), (x, t^*) \in \mathcal{P}_q(g^\circledast)\}$ , and*

$$\mathcal{P}_q(h) = \{(x, s^* + t^*): (x, s^*) \in \mathcal{P}_q(f), (x, t^*) \in \mathcal{P}_q(g)\}.$$

**Proof.** Since  $f \geq q$  and  $g \geq q$  on  $E \times E^*$ , (10) implies that, for all  $(x, x^*) \in E \times E^*$ ,

$$h(x, x^*) \geq \inf \{\langle x, s^* \rangle + \langle x, t^* \rangle: s^*, t^* \in E^*, s^* + t^* = x^*\} = \langle x, x^* \rangle = q(x, x^*),$$

and then Theorem 2.4 and the fact that  $f^\circledast \geq f$  and  $g^\circledast \geq g$  on  $E \times E^*$  give

$$\begin{aligned} h^\circledast(x, x^*) &= h^*(x^*, x) = \min \{f^*(s^*, x) + g^*(t^*, x): s^*, t^* \in E^*, s^* + t^* = x^*\} \\ &= \min \{f^\circledast(x, s^*) + g^\circledast(x, t^*): s^*, t^* \in E^*, s^* + t^* = x^*\} \\ &\geq \inf \{f(x, s^*) + g(x, t^*): s^*, t^* \in E^*, s^* + t^* = x^*\} = h(x, x^*). \end{aligned} \quad (11)$$

Thus  $h$  is a BC-function, the required characterization of  $\mathcal{P}_q(h^\circledast)$  is immediate from (11), and the final assertion now follows from three applications of Theorem 5.1(a).  $\square$

**Theorem 5.4.** *Let  $E$  be a nonzero reflexive Banach space,  $f, g \in \mathcal{PCLSC}(E \times E^*)$  be BC-functions and  $\bigcup_{\lambda>0} \lambda[\pi_1 \text{ dom } f - \pi_1 \text{ dom } g]$  be a closed linear subspace of  $E$ . Then*

$$\{(x, s^* + t^*): (x, s^*) \in \mathcal{P}_q(f), (x, t^*) \in \mathcal{P}_q(g)\}$$

*is a maximally monotone subset of  $E \times E^*$ .*

**Proof.** This is immediate from Theorem 5.1(a) and Theorem 5.3.  $\square$

**Remark 5.5.** The question arises naturally as to whether there can exist a  $q$ -positive subset  $A$  of  $B$  such that  $\Phi_A = g_0$ . It would then follow from (4) that  $g_0 = q$  on  $A$ , and so  $A \subset \mathcal{P}_q(g_0)$ . Thus, for all  $b \in B$ , (2) would give

$$g_0(b) = \Phi_A(b) = \sup_A [\cdot, b] - q = \sup_A [\cdot, b] - g_0 \leq \sup_{\mathcal{P}_q(g_0)} [\cdot, b] - g_0. \quad (12)$$

Now let  $H$  be a nonzero Hilbert space, and consider the SSDB space  $(H \times H, [\cdot, \cdot], \|\cdot\|)$ , as described at the beginning of this section. Then  $(x, x^*) \in \mathcal{P}_q(g_0) \iff x^* = x$ . Let  $y \in H \setminus \{0\}$ . Then (12) would give

$$\|y\|^2 = g_0(y, -y) \leq \sup_{x \in H} [[(x, x), (y, -y)] - g_0(x, x)] = \sup_{x \in H} [-g_0(x, x)] \leq 0.$$

Since this is manifestly impossible, there cannot exist a  $q$ -positive subset  $A$  of  $B$  such that  $\Phi_A = g_0$ . This example is an extension of [14, Example 19.20, p. 85].

## SSDB spaces and maximal monotonicity

### 6 Monotone multifunctions on reflexive Banach spaces

Let  $E$  be a nonzero reflexive Banach space. If  $S: E \rightrightarrows E^*$  is a multifunction then we use the standard notation  $G(S) := \{(x, x^*) \in E \times E^*: x^* \in Sx\}$ . We also write  $D(S) := \{x \in E: Sx \neq \emptyset\} = \pi_1 G(S)$ . If  $S$  is monotone and  $G(S)$  is nonempty, we define the *Fitzpatrick functions*,  $\varphi_S$ , associated with  $S$  by

$$\varphi_S(x, x^*) := \Phi_{G(S)}(x, x^*) = \sup_{(s, s^*) \in G(S)} [\langle x, s^* \rangle + \langle s, x^* \rangle - \langle s, s^* \rangle].$$

The function  $\varphi_A$  was introduced by Fitzpatrick in [6, Definition 3.1, p. 61]. The following result was first proved in [6, Corollary 3.9, p. 62] and [6, Proposition 4.2, pp. 63–64].

**Theorem 6.1.**  *$E$  be a nonzero reflexive Banach space and  $S: E \rightrightarrows E^*$  be maximally monotone. Then  $\varphi_S$  is a BC–function and  $\mathcal{P}_q(\varphi_S^{\circledast}) = \mathcal{P}_q(\varphi_S) = G(S)$ .*

**Proof.** This is immediate from Theorem 3.11. □

The following result will be useful in Corollary 6.6.

**Lemma 6.2.**  *$E$  be a nonzero reflexive Banach space and  $S: E \rightrightarrows E^*$  be maximally monotone. Then  $D(S) \subset \pi_1 \text{dom } \varphi_S$ .*

**Proof.** From Theorem 6.1 and the finite–valuedness of  $q$ ,  $G(S) = \mathcal{P}_q(\varphi_S) \subset \text{dom } \varphi_S$ , from which  $D(S) = \pi_1 G(S) \subset \pi_1 \text{dom } \varphi_S$ . □

If  $S, T: E \rightrightarrows F$  then, for all  $x \in E$ , we use the following standard notation:

$$(S + T)x := \{y + z: y \in Sx, z \in Tx\}.$$

Theorem 6.3 is “Rockafellar’s surjectivity theorem” — see [12, Proposition 1, p. 77] for the original proof depending ultimately on Brouwer’s fixed–point theorem and an Asplund renorming. The proof given here is a simplification of that given in Simons–Zălinescu, [16, Theorem 3.1(b), p. 8], and appeared in Simons, [14, Theorem 29.6, p. 119]. These references also provide a number of formulae for the exact value of  $\min \{\|x\|: x \in E, (S + J)x \ni 0\}$  in terms of  $\varphi_{G(S)}$ . Here is one of them:

$$\min \{\|x\|: x \in E, (S + J)x \ni 0\} = \frac{1}{\sqrt{2}} \sup_{b \in E \times E^*} \left[ \|b\| - \sqrt{2\varphi_S(b) + \|b\|^2} \right] \vee 0.$$

**Theorem 6.3.** *Let  $E$  be a nonzero reflexive Banach space and  $S: E \rightrightarrows E^*$  be maximally monotone. Then  $(S + J)(E) = E^*$ .*

**Proof.** Let  $y^*$  be an arbitrary element of  $E^*$ . From Theorem 5.1(b) (with  $A := G(S)$ ),  $(0, y^*) \in G(S) - G(-J)$ . Thus there exist  $(s, s^*) \in G(S)$  and  $(x, x^*) \in G(J)$  such that  $(0, y^*) = (s, s^*) - (x, -x^*)$ . But then  $x = s$ , and so

$$y^* = s^* + x^* \in Sx + Jx = (S + J)x \subset (S + J)(E). \quad \square$$

**Remark 6.4.** We note that if  $J$  and  $J^{-1}$  are single–valued then the converse of Theorem 6.3 holds, that is to say  $(S + J)(E) = E^* \implies S$  is maximally monotone, however this fails in general. See Simons, [14, Remark 29.7, pp. 120–121]. It is interesting to observe that, even when the converse of Theorem 6.3 fails, the necessary and sufficient condition for maximal monotonicity given in Theorem 5.1(b) remains true.

## SSDB spaces and maximal monotonicity

We now come to the “sum theorem”. More general results can be found in Bot–Csetnek–Wanka, [3] and Bot–Grad–Wanka, [4].

**Theorem 6.5.** *Let  $E$  be a nonzero reflexive Banach space,  $S, T: E \rightrightarrows E^*$  be maximally monotone and  $\bigcup_{\lambda>0} \lambda[\pi_1 \text{dom } \varphi_S - \pi_1 \text{dom } \varphi_T]$  be a closed linear subspace of  $E$ . Then  $S + T$  is maximally monotone.*

**Proof.** From Theorem 6.1,  $\varphi_S$  and  $\varphi_T$  are BC–functions,  $\mathcal{P}_q(\varphi_S) = G(S)$  and  $\mathcal{P}_q(\varphi_T) = G(T)$ . Theorem 5.4 with  $f := \varphi_S$  and  $g := \varphi_T$  now implies that

$$\{(x, s^* + t^*): (x, s^*) \in G(S), (x, t^*) \in G(T)\} = G(S + T)$$

is a maximally monotone subset of  $E \times E^*$ . This completes the proof of Theorem 6.5.  $\square$

The following consequence of Theorem 6.5 is useful in applications:

**Corollary 6.6.** *Let  $E$  be a nonzero reflexive Banach space,  $S, T: E \rightrightarrows E^*$  be maximally monotone and  $\bigcup_{\lambda>0} \lambda[D(S) - D(T)] = E$ . Then  $S + T$  is maximally monotone.*

**Proof.** This is immediate from Lemma 6.2 and Theorem 6.5.  $\square$

Our next result is “Rockafellar’s sum theorem”. It follows immediately from Corollary 6.6.

**Corollary 6.7.** *Let  $E$  be a nonzero reflexive Banach space,  $S, T: E \rightrightarrows E^*$  be maximally monotone and  $D(S) \cap \text{int } D(T) \neq \emptyset$ . Then  $S + T$  is maximally monotone.*

A number of other sufficient conditions have been given for the sum of maximally monotone multifunctions on a reflexive Banach space to be maximally monotone. Many of them are contained in the following “Sandwiched closed subspace theorem”, which first appeared in Simons–Zălinescu, [16, Theorem 5.5, p. 13].

**Theorem 6.8.** *Let  $E$  be a nonzero reflexive Banach space,  $S, T: E \rightrightarrows E^*$  be maximally monotone, and suppose that there exists a closed linear subspace  $F$  of  $E$  such that*

$$D(S) - D(T) \subset F \subset \bigcup_{\lambda>0} \lambda[\pi_1 \text{dom } \varphi_S - \pi_1 \text{dom } \varphi_T].$$

*Then  $S + T$  is maximally monotone. Furthermore, for all  $\varepsilon > 0$ ,*

$$D(S) - D(T) \subset \pi_1 \text{dom } \varphi_S - \pi_1 \text{dom } \varphi_T \subset (1 + \varepsilon)(D(S) - D(T)),$$

*(that is to say, the sets  $\pi_1 \text{dom } \varphi_S - \pi_1 \text{dom } \varphi_T$  and  $D(S) - D(T)$  are almost identical) and*

$$\bigcup_{\lambda>0} \lambda[\pi_1 \text{dom } \varphi_S - \pi_1 \text{dom } \varphi_T] = \bigcup_{\lambda>0} \lambda[D(S) - D(T)].$$

## 7 An abstract Hammerstein theorem

Let  $E$  be a nonzero reflexive Banach space. We define the reflection maps  $\rho_1, \rho_2$  on  $E \times E^*$  by  $\rho_1(x, x^*) := (-x, x^*)$  and  $\rho_2(x, x^*) := (x, -x^*)$ . Our first result is the “ $\rho_1$ –transversality theorem”.

## SSDB spaces and maximal monotonicity

**Theorem 7.1.** *Let  $E$  be a nonzero reflexive Banach space,  $f, g \in \mathcal{PCLSC}(E \times E^*)$  and  $f, g$  be BC-functions. Then:*

$$\text{dom } f + \rho_1 \text{dom } g = E \times E^* \iff \mathcal{P}_q(f) + \rho_1 \mathcal{P}_q(g) = E \times E^*$$

and

$$\text{dom } f - \rho_1 \text{dom } g = E \times E^* \iff \mathcal{P}_q(f) - \rho_1 \mathcal{P}_q(g) = E \times E^*.$$

**Proof.** This follows from Corollary 4.7 with  $\rho := \rho_2$ , and the fact that  $\rho_1 = -\rho_2^{-1}$ .  $\square$

We give the next two results for completeness, though more general results are known. See, for instance, Zălinescu, [19, Theorem 3 and Corollary 4].

**Theorem 7.2.** *Let  $E$  be a nonzero reflexive Banach space,  $S, T: E \rightrightarrows E^*$  be maximally monotone and  $\text{dom } \varphi_S + \rho_1 \text{dom } \varphi_T = E \times E^*$ . Then  $(S + T)(E) = E^*$ .*

**Proof.** Let  $y^*$  be an arbitrary element of  $E^*$ . From Theorem 7.1 (with  $f := \varphi_S$  and  $g := \varphi_T$ ) and Theorem 6.1,  $(0, y^*) \in G(S) + \rho_1 G(T)$ . Thus there exist  $(s, s^*) \in G(S)$  and  $(x, x^*) \in G(T)$  such that  $(0, y^*) = (s, s^*) + (-x, x^*)$ . But then  $x = s$ , and so

$$y^* = s^* + x^* \in Sx + Tx = (S + T)x \subset (S + T)(E). \quad \square$$

Since  $G(J) = \mathcal{P}_q(g_0)$ , it follows that  $\varphi_J = \Phi_{G(J)} = \Phi_{\mathcal{P}_q(g_0)}$ . Thus Remark 4.5 implies that  $\text{dom } \varphi_J = E \times E^*$ . Consequently, the following result generalizes Theorem 6.3. Compare with [7, Theorem 2((1) $\implies$ (2))].

**Corollary 7.3.** *Let  $E$  be a nonzero reflexive Banach space,  $S, T: E \rightrightarrows E^*$  be maximally monotone and  $\text{dom } \varphi_T = E \times E^*$ . Then  $(S + T)(E) = E^*$ .*

**Proof.** This is immediate from Theorem 7.2.  $\square$

**Theorem 7.4.** *Let  $E$  be a nonzero reflexive Banach space,  $f, g \in \mathcal{PCLSC}(E \times E^*)$  be BC-functions, and  $w^* \in E^*$  be such that  $E \times \{w^*\} \subset \text{dom } f$  and  $\pi_2 \text{dom } g = E^*$ . Then:*

- (a)  $\mathcal{P}_q(f) + \rho_1 \mathcal{P}_q(g) = E \times E^*$  and  $\mathcal{P}_q(f) - \rho_1 \mathcal{P}_q(g) = E \times E^*$ .
- (b) If  $x \in E$  then there exist  $(y, y^*) \in \mathcal{P}_q(f)$  and  $(z, z^*) \in \mathcal{P}_q(g)$  such that  $y + z = x$ .
- (c) If  $x^* \in E^*$  then there exist  $(y, y^*) \in \mathcal{P}_q(f)$  and  $(y, z^*) \in \mathcal{P}_q(g)$  such that  $y^* + z^* = x^*$ .

**Proof.** (a) Let  $(x, x^*)$  be an arbitrary element of  $E \times E^*$ . Since  $\pi_2 \text{dom } g = E^*$ , there exist  $y, z \in E$  such that  $(y, x^* - w^*) \in \text{dom } g$  and  $(z, w^* - x^*) \in \text{dom } g$ . But  $(x + y, w^*) \in \text{dom } f$  and  $(x - z, w^*) \in \text{dom } f$ , hence

$$(x, x^*) = (x + y, w^*) + \rho_1(y, x^* - w^*) \in \text{dom } f + \rho_1 \text{dom } g$$

and

$$(x, x^*) = (x - z, w^*) - \rho_1(z, w^* - x^*) \in \text{dom } f - \rho_1 \text{dom } g.$$

Thus we have proved that  $\text{dom } f + \rho_1 \text{dom } g = E \times E^*$  and  $\text{dom } f - \rho_1 \text{dom } g = E \times E^*$ , and (a) follows from Theorem 7.1.

(b) It follows from (a) that there exist  $(y, y^*) \in \mathcal{P}_q(f)$  and  $(z, z^*) \in \mathcal{P}_q(g)$  such that  $(y, y^*) + (z, -z^*) = (x, 0)$ . But then  $z^* = y^*$  and  $y + z = x$ .

(c) It follows from (a) that there exist  $(y, y^*) \in \mathcal{P}_q(f)$  and  $(z, z^*) \in \mathcal{P}_q(g)$  such that  $(y, y^*) - (z, -z^*) = (0, x^*)$ . But then  $z = y$  and  $y^* + z^* = x^*$ .  $\square$

## SSDB spaces and maximal monotonicity

We now reverse the direction of  $T$ .

**Theorem 7.5.** *Let  $E$  be a nonzero reflexive Banach space and  $S: E \rightrightarrows E^*$  and  $T: E^* \rightrightarrows E$  be maximally monotone. Suppose that  $\pi_1 \operatorname{dom} \varphi_T = E^*$  and there exists  $w^* \in E^*$  such that  $E \times \{w^*\} \subset \operatorname{dom} \varphi_S$ . Then:*

- (a) *If  $I_E$  is the identity map on  $E$ ,  $(I_E + TS)(E) = E$ .*
- (b) *If  $I_{E^*}$  is the identity map on  $E^*$ ,  $(I_{E^*} + ST)(E^*) = E^*$ .*

**Proof.** Let  $f := \varphi_S$  and  $g := \varphi_{T^{-1}}$ , so that  $\pi_2 \operatorname{dom} g = E^*$ .

(a) Let  $x$  be an arbitrary element of  $E$ . From Theorem 7.4(b) and Theorem 6.1, there exist  $(y, y^*) \in G(S)$  and  $(y^*, z) \in G(T)$  such that  $y + z = x$ . Thus  $z \in Ty^* \subset TSy$  and  $x = y + z \in (I_E + TS)(y) \subset (I_E + TS)(E)$ .

(b) Let  $x^*$  be an arbitrary element of  $E^*$ . From Theorem 7.4(c) and Theorem 6.1, there exist  $(y, y^*) \in G(S)$  and  $(z^*, y) \in G(T)$  such that  $y^* + z^* = x^*$ . Thus  $y^* \in Sy \subset STz^*$  and  $x^* = z^* + y^* \in (I_{E^*} + ST)(z^*) \subset (I_{E^*} + ST)(E^*)$ .  $\square$

The next result is a considerable generalization of [20, Theorem 32.O, p. 909] which, in turn, was applied to Hammerstein integral equations. See [14, Remark 30.5, p. 124] for a more complete discussion.

**Theorem 7.6.** *Let  $E$  be a nonzero reflexive Banach space and  $S: E \rightrightarrows E^*$  and  $T: E^* \rightrightarrows E$  be maximal monotone. Suppose that **either**  $\pi_1 \operatorname{dom} \varphi_T = E^*$  and there exists  $w^* \in E^*$  such that  $E \times \{w^*\} \subset \operatorname{dom} \varphi_S$  **or**  $\pi_1 \operatorname{dom} \varphi_S = E$  and there exists  $w \in E$  such that  $E \times \{w\} \subset \operatorname{dom} \varphi_T$ . Then  $(I_E + TS)(E) = E$ .*

**Proof.** The first case has already been established in Theorem 7.5(a), while the second case follows from Theorem 7.5(b), with  $E$  replaced by  $E^*$ , and the roles of  $S$  and  $T$  interchanged.  $\square$

**Remark 7.7.** In the recent paper [2], Jonathan Borwein has written a survey of the history of monotonicity (even in the nonreflexive case) over the past fifty years.

## References

- [1] H. Attouch and H. Brezis, *Duality for the sum of convex functions in general Banach spaces.*, Aspects of Mathematics and its Applications, J. A. Barroso, ed., Elsevier Science Publishers (1986), 125–133.
- [2] J. M. Borwein, *Fifty years of maximal monotonicity*, <<http://www.carma.newcastle.edu.au/~jb616/fifty.pdf>>.
- [3] R. I. Boř, E. R. Csetnek and G. Wanka, *A new condition for maximal monotonicity via representative functions*, Nonlinear Anal. **67** (2007), 2390–2402.
- [4] R. I. Boř, S–M Grad and G. Wanka, *Maximal monotonicity for the precomposition with a linear operator*. SIAM J. Optim. **17** (2006), 1239–1252.
- [5] R. S. Burachik and B. F. Svaiter, *Maximal monotonicity, conjugation and the duality product*, Proc. Amer. Math. Soc. **131** (2003), 2379–2383.
- [6] S. Fitzpatrick, *Representing monotone operators by convex functions*, Workshop/Miniconference on Functional Analysis and Optimization (Canberra, 1988), 59–65,

## SSDB spaces and maximal monotonicity

- Proc. Centre Math. Anal. Austral. Nat. Univ., **20**, Austral. Nat. Univ., Canberra, 1988.
- [7] J.-E. Martínez-Legaz, *Some generalizations of Rockafellar's surjectivity theorem*, Pacific J. of Optimization **4** (2008), 527–535.
- [8] J.-E. Martínez-Legaz and M. Théra, *A convex representation of maximal monotone operators*, J. Nonlinear Convex Anal. **2** (2001), 243–247.
- [9] J.-P. Penot, *The relevance of convex analysis for the study of monotonicity*, Nonlinear Anal. **58** (2004), 855–871.
- [10] J.-P. Penot and C. Zălinescu, *Some problems about the representation of monotone operators by convex functions*, ANZIAM J. **47** (2005), 1–20.
- [11] R. T. Rockafellar, *Extension of Fenchel's duality theorem for convex functions*, Duke Math. J. **33** (1966), 81–89.
- [12] —, *On the Maximality of Sums of Nonlinear Monotone Operators*, Trans. Amer. Math. Soc. **149**(1970),75-88.
- [13] S. Simons, *Minimax and monotonicity*, Lecture Notes in Mathematics **1693** (1998), Springer–Verlag.
- [14] —, *From Hahn–Banach to monotonicity*, Lecture Notes in Mathematics, **1693**, second edition, (2008), Springer–Verlag.
- [15] —, *Banach SSD spaces and classes of monotone sets*, <http://arxiv.org/abs/0908.0383v2>, posted August 26, 2009.
- [16] S. Simons and C. Zălinescu, *Fenchel duality, Fitzpatrick functions and maximal monotonicity*, J. of Nonlinear and Convex Anal., **6** (2005), 1–22.
- [17] M. D. Voisei and C. Zălinescu, *Strongly–representable operators*, J. of Convex Anal., **16(3)**, 2009.
- [18] C. Zălinescu, *Convex analysis in general vector spaces*, (2002), World Scientific.
- [19] —, *A new convexity property for monotone operators*, J. Convex Anal. **13** (2006), 883–887.
- [20] E. Zeidler, *Nonlinear functional analysis and its applications, II/B, Nonlinear monotone operators*, Springer-Verlag, New York, 1990.

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