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From Hahn–Banach to Monotonicity

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Preface

A more accurate title for these notes would be: "The Hahn–Banach–Lagrange theorem, Convex analysis, Symmetrically self–dual spaces, Fitzpatrick functions and monotone multifunctions".

The Hahn–Banach–Lagrange theorem is a version of the Hahn–Banach theorem that is admirably suited to applications to the theory of monotone multifunctions, but it turns out that it also leads to extremely short proofs of the standard existence theorems of functional analysis, a minimax theorem, a Lagrange multiplier theorem for constrained convex optimization problems, and the Fenchel duality theorem of convex analysis.

Another feature of the Hahn–Banach–Lagrange theorem is that it can be used to transform problems on the existence of continuous linear functionals into problems on the existence of a single real constant, and then obtain a sharp lower bound on the norm of the linear functional satisfying the required condition. This is the case with both the Lagrange multiplier theorem and the Fenchel duality theorem applications mentioned above.

A multifunction from a Banach space into the subsets of its dual can, of course, be identified with a subset of the product of the space with its dual. Simon Fitzpatrick defined a convex function on this product corresponding with any such multifunction. So part of these notes is devoted to the rather special convex analysis for the product of a Banach space with its dual.

The product of a Banach space with its dual is a special case of a "symmetrically self–dual space". The advantage of going to this slightly higher level of abstraction is not only that it leads to more general results but, more to the point, it cuts the length of each proof approximately in half which, in turn, gives a much greater insight into the nature of the processes involved. Monotone multifunctions then correspond to subsets of the symmetrically self–dual space that are "positive" with respect to a certain quadratic form.

We investigate a particular kind of convex function on a symmetrically self–dual space, which we call a "BC–function". Since the Fitzpatrick function of a maximally monotone multifunction is always a BC–function, these BC–functions turn out to be very successful for obtaining results on maximally monotone multifunctions on reflexive spaces.

The situation for nonreflexive spaces is more challenging. Here, it turns out that we must consider two symmetrically self–dual spaces, and we call the corresponding convex functions “$\widetilde{\mathrm{BC}}$–functions”. In this case, a number of different subclasses of the maximally monotone multifunctions have been introduced over the years — we give particular attention to those that are “of type (ED)”. These have the great virtue that all the common maximally monotone multifunctions are of type (ED), and maximally monotone multifunctions of type (ED) have nearly all the properties that one could desire. In order to study the maximally monotone multifunctions of type (ED), we have to introduce a weird topology on the bidual which has a number of very nice properties, despite that fact that it is not normally compatible with its vector space structure.

These notes are somewhere between a sequel to and a new edition of [99]. As in [99], the essential idea is to reduce questions on monotone multifunctions to questions on convex functions. In [99], this was achieved using a “big convexification” of the graph of the multifunction and the “minimax technique” for proving the existence of linear functionals satisfying certain conditions. The “big convexification” is a very abstract concept, and the analysis is quite heavy in computation. The Fitzpatrick function gives another, more concrete, way of associating a convex functions with a monotone multifunction. The problem is that many of the questions on convex functions that one obtains require an analysis of the special properties of convex functions on the product of a Banach space with its dual, which is exactly what we do in these notes. It is also worth noting that the minimax theorem is hardly used here.

We envision that these notes could be used for four different possible courses/seminars:

• An introductory course in functional analysis which would, at the same time, touch on minimax theorems and give a grounding in convex Lagrange multiplier theory and the main theorems in convex analysis.

• A course in which results on monotonicity on general Banach spaces are established using symmetrically self–dual spaces and Fitzpatrick functions.

• A course in which results on monotonicity on reflexive Banach spaces are established using symmetrically self–dual spaces and Fitzpatrick functions.

• A seminar in which the the more technical properties of maximal monotonicity on general Banach spaces that have been established since 1997 are discussed.

We give more details of these four possible uses at the end of the introduction.

I would like to express my sincerest thanks to Heinz Bausckhe, Patrick Combettes, Michael Crandall, Carl de Boor, Radu Ioan Boţ, Juan Enrique Martínez-Legaz, Xianfu Wang and Constantin Zălinescu for reading preliminary versions of parts of these notes, making a number of excellent suggestions and, of course, finding a number of errors.

Of course, despite all the excellent efforts of the people mentioned above, these notes doubtless still contain errors and ambiguities, and also doubtless have other stylistic shortcomings. At any rate, I hope that there are not too many of these. Those that do exist are entirely my fault.

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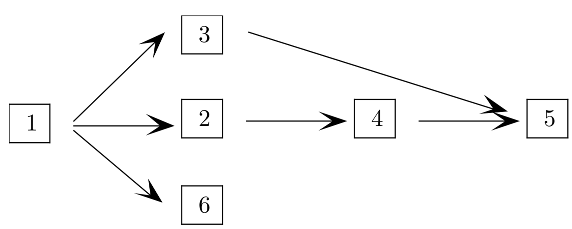
Introduction

These notes fall into three distinct parts. In Chapter I, we discuss the "Hahn–Banach–Lagrange theorem", a new version of the Hahn–Banach theorem, which gives very efficient proofs of the main existence theorems in functional analysis, optimization theory, minimax theory and convex analysis. In Chapter II, we zero in on the applications to convex analysis. In the remaining five chapters, we show how the results of the first two chapters can be used to obtain a large number of results on monotone multifunctions, many of which have not yet appeared in print.

Chapter I: The main result in Chapter I is the "Hahn-Banach-Lagrange" theorem, which first appeared in [103]. We prove this result in Theorem 1.11, discuss the classical functional analytic applications in Section 2 (namely the "Sandwich theorem" in Corollary 2.1, the "extension form of the Hahn–Banach theorem" in Corollary 2.2, and the "one dimensional form of the Hahn–Banach theorem" in Corollary 2.4) and give an application to a classical minimax theorem in Section 3. In Section 4, we introduce the results from classical Banach space theory that we shall need. In Section 5, we prove, among other things, a minimax criterion for a subset of a Banach space to be weakly compact using the concepts of "excess" and "duality gap". The contents of this section first appeared in [102].

In Section 6, we give a necessary and sufficient condition for the existence of Lagrange multipliers for constrained convex optimization problems (generalizing the classical sufficient "Slater condition"), with a sharp lower bound on the norm of the multiplier. We also prove a similar result for Karush–Kuhn–Tucker problems for functions with convex Gâteaux derivatives. Some of the results on Lagrange multipliers first appeared in [104]. In the flowchart below, we show the dependencies of the sections in Chapter I. We note, in particular, that Section 6 does not depend on Sections 2–5.



Chapter II: As explained above, Chapter II is about convex analysis. We start our discussion in Section 7 by using the Hahn-Banach-Lagrange theorem to obtain a necessary *and sufficient* condition for the Fenchel duality theorem to hold for two convex functions on a normed space, with a sharp lower bound on the norm of the functional obtained. (Incidentally, this approach avoids the aggravating problem of the "vertical hyperplane" that so destroys the elegance of the usual approach through the Eidelheit separation theorem.) This sharp version of the Fenchel duality theorem is in Theorem 7.4, and it is explained in Remark 7.6 how the lower bound obtained is of a very geometric character.

While the concept of Fenchel conjugate is introduced in Section 7 with reference to a convex function on a normed space, in fact this causes no end of confusion when dealing with monotone multifunctions on a nonreflexive Banach space. The way out of this problem (as has been observed by many authors) is to define Fenchel conjugates with respect to a dual pair of spaces. This is what we do in Section 8, and it enables a painless transition to the locally convex case. As we will see in Section 22, this is exactly what we need for our discussion of monotone multifunctions on a nonreflexive Banach space. We present a necessary and sufficient condition for the Fenchel duality theorem to be true in this sense in Theorem 8.1, and in Theorem 8.4 we present a unifying sufficient condition that implies the results that are used in practice, the versions due to Rockafellar and Attouch–Brezis. Theorem 8.4 uses the binary operation $\ominus$ defined in Notation 8.3.

In Section 9, we return to the normed case and give some results of a more numerical character, in which we explore the properties of the function $\frac{1}{2}\|\cdot\|^2$. These results will enable us to give a precise expression for the minimum norm of the resolvent of a maximally monotone multifunction on a reflexive Banach space in Theorem 29.5.

We bootstrap Theorem 8.4 in Section 10, and obtain sufficient conditions for the "inf–convolution" formula for the conjugate of a sum to hold, and give as application in Corollary 10.4 a consequence that will be applied in Theorem 21.10 to the existence of autoconjugates in SSDB spaces. This bootstrapping operation exhibits the well known fact that results on the conjugate of a sum are very close to the Fenchel duality theorem. However, these concepts are not interchangeable, and in Section 11 we give examples which should serve to distinguish them (giving examples of the failure of "stability" in duality).

In Section 12, we introduce the concepts of the *biconjugate* of a convex function, and the *Fenchel–Moreau points* of a convex function on a locally convex space. We deduce the Fenchel–Moreau formula in Corollary 12.4 in the case where the function is lower semicontinuous. Some of these results first appeared in [103].

We collect together in Sections 13 and 14 various results on convex functions that depend ultimately on Baire's theorem. The "dom lemma", Lemma 13.3, is a generalization to convex functions of the classical uniform boundedness (Banach Steinhaus) theorem (see Remark 13.6) and the "$\ominus$–theorem", Theorem 14.2 (which uses the operation $\ominus$ already mentioned) is a generalization to convex functions of the classical open mapping theorem (see Remark 14.4). Both of these results will be applied later on to obtain results on monotonicity. We can think of the dom lemma and the $\ominus$–theorem as "quantitative" results, since their main purpose is to provide numerical bounds. Associated with them are two "qualitative" results, the "dom corollary", Corollary 13.5, and the "$\ominus$–corollary", Corollary 14.3, from which the numerics have been removed. The $\ominus$–corollary will also be of use to us later on. In Remark 14.5, we give a brief discussion of convex Borel sets and functions.

In Theorem 15.1, we show how the $\ominus$–theorem leads to the Attouch–Brezis version of the Fenchel duality theorem, which we will use (via the local transversality theorem, Theorem 21.12, and Theorem 30.1) to prove various surjectivity results, including an abstract Hammerstein theorem; and in Theorem 16.4 we obtain a bivariate version of the Attouch-Brezis theorem, which we will use in Theorem 24.1 (via Lemma 22.9) and in Theorem 35.8 (a result that is fundamental for the understanding of maximally monotone multifunctions on nonreflexive Banach spaces). This bivariate version of the Attouch–Brezis theorem first appeared in [109].

Chapter III: In Chapter III, we will discuss the basic result on monotonicity. Section 17 starts off with a conventional discussion of multifunctions, monotonicity and maximal monotonicity. Remark 17.1 is a bridge in which we show that if E is a Banach space then there is a vector space B and a quadratic form q on B such that if $S\colon\ E \rightrightarrows E^*$ is a multifunction then there is a subset A of B such that S is monotone if and only if $b, c \in A \Longrightarrow q(b-c) \geq 0$. Actually $B = E \times E^*$ and $A = G(S)$, but this paradigm leads to a strict generalization of monotonicity, in which the proofs are much more concise.

In Section 18, we digress a little from the general theory in order to give a short proof of Rockafellar's fundamental result that the subdifferential of a proper convex lower semicontinuous function on a Banach space is maximally monotone. In Theorem 18.1 and Theorem 18.2, we give the formula for the subdifferential of the sum of convex functions under two different hypotheses, in Corollary 18.5 and Theorem 18.6, we show how to deduce the Brøndsted–Rockafellar theorem from Ekeland's variational principle and the Hahn–Banach–Lagrange theorem, and then we come finally to our proof of the maximal monotonicity of subdifferentials in Theorem 18.7, which is based on the very elegant one found recently by M. Marques Alves and B. F. Svaiter in [60]. We also give in Corollary 18.3 and Theorem 18.10 two results about normal cones that will be useful later on. Readers who are familiar with the formula for the subdifferential of the sum of convex functions and the Brøndsted–Rockafellar theorem should be able to understand this section without having to read any of the previous sections.

We return to our development of the general theory in Sections 19–21. In Section 19, we introduce the concept of a SSD (symmetrically self–dual) space, a nonzero real vector space with a symmetric bilinear form which separates points. This bilinear form defines a quadratic form, q, in the obvious way. In general, this quadratic form is not positive, but we isolate certain subsets of a SSD space that we will call "q–positive". Appropriate convex functions on the SSD space define q–positive sets. We zero in on a subclass of the convex functions on a SSD space which we call "BC–functions". Critical to this enterprise is the self–dual property, because the conjugate of a convex function has the same domain of definition as the original convex function. Lemma 19.12 contains an unexpected result on BC–functions, but the most important result on BC–functions is undoubtedly the transversality theorem, Theorem 19.16, which leads (via Theorem 21.4) to generalizations of Rockafellar's classical surjectivity theorem for maximally monotone multifunctions on a reflexive Banach space (see Theorem 29.5) together with a sharp lower bound on the norm of solutions in terms of the Fitzpatrick function (see Theorem 29.6), and to sufficient conditions for the sum of maximally monotone multifunctions on a reflexive Banach space to be maximally monotone (see Theorem 24.1). Section 19 concludes with a discussion of how every q–positive set, A, gives rise to a convex function, Φ_A (this construction is an abstraction of the construction of the "Fitzpatrick function" that we will consider in Section 23). In Section 20, we introduce maximally q–positive sets, and show that the convex function determined by a maximally q–positive set is a BC–function.

In Section 21, we introduce the SSDB spaces, which are SSD spaces with an appropriate Banach norm. Roughly speaking, the additional structure that SSDB spaces possess over SSD spaces is ultimately what accounts for the fact that maximally monotone multifunctions on reflexive Banach spaces are much more tractable than maximally monotone multifunctions on general Banach spaces. That is not to say that the SSD space determined by a nonreflexive Banach space does not have a norm structure, the problem is that this norm structure is not "appropriate". Apart from Theorem 21.4, which we have already mentioned, the other important results in this section are Theorem 21.10 on the existence of autoconjugates, Theorem 21.11, which gives a formula for a maximally q–positive superset of a given nonempty q–positive set, and the local transversality theorem, Theorem 21.12, which leads ultimately to a number of surjectivity results, including an abstract Hammerstein theorem in Section 30.

We start considering in earnest the special SSD space $E \times E^*$ (where E is a nonzero Banach space) in Section 22. We first prove some preliminary results which depend ultimately on Rockafellar's version of the Fenchel duality theorem introduced in Corollary 8.6. It is important to realize that, despite the fact that E is a Banach space, we need Corollary 8.6 for *locally convex spaces* since the topology we are using for this result is the topology

$\mathcal{T}_{\|\ \|}(E)\times\ w(E^*,E)$ on $E\times E^*$. Theorem 22.5 has a precise description of the projection on E of the domain of the conjugate of a proper convex function on $E\times E^*$ in terms of a related convex function on E. In Theorem 22.8, we establish the equality of six sets determined by certain proper convex functions on $E\times E^*$, and in Lemma 22.9 we prove a result which will be critical for our treatment of sum theorems for maximally monotone multifunctions in Theorem 24.1.

In Section 23, we show how the concepts introduced in Sections 19–21 specialize to the situation considered in Section 22. The q–positive sets introduced in Section 19 then become the graphs of monotone multifunctions, the maximally q–positive sets introduced in Section 20 then become the graphs of maximally monotone multifunctions, and the function Φ_A determined by a q–positive set A introduced in Section 19 becomes the Fitzpatrick function, φ_S, determined by a monotone multifunction S. The Fitzpatrick function was originally introduced in [42] in 1988, but lay dormant until it was rediscovered by Martínez-Legaz and Théra in [63] in 2001.

This is an appropriate place for us to make a comparison between the analysis presented in these notes with the analysis presented in [99]. In both cases, the essential idea is to reduce questions on monotone multifunctions to questions on convex functions. In [99], this was achieved using a "big convexification" of the graph of the multifunction and the "minimax technique" for proving the existence of linear functionals satisfying certain conditions. This technique is very successful for working back from conjectures, and finding conditions under which they hold. On the other hand, the "big convexification" is a very abstract concept, and the analysis is quite heavy in computation. Now the Fitzpatrick function gives another way of associating a convex functions with a monotone multifunction, and this can also be used to reduce questions on monotone multifunctions to questions on convex functions. The problem is that many of the questions on convex functions that one obtains require an analysis of the special properties of convex functions on $E\times E^*$. This is exactly the analysis that we perform in Section 22, and later on in Section 35. As already explained, the SSD spaces introduced in Sections 19–21 give us a strict generalization of monotonicity. More to the point, the fact that the notation is more concise enables us to get a much better grasp of the underlying structures. A good example of this is Theorem 35.8, a relatively simple result with far–reaching applications to the classification of maximally monotone multifunctions on nonreflexive spaces. Another example is provided by Section 46 on maximally monotone multifunctions with convex graph.

We now return to our discussion of Section 23. We also introduce the "fitzpatrification", S_φ, of a monotone multifunction, S. This is a multifunction with convex graph which is normally much larger than the graph of S. S_φ is, in general, not monotone, but it is very useful since its use shortens the statements of many results considerably. The final result of this section, Lemma 23.9, will be used in our discussion of the sum problem in Theorem 24.1 and the Brezis–Haraux condition in Theorem 31.4.

In Section 24, we give sufficient conditions for the sum of maximally monotone multifunctions to be maximally monotone. These results will be extended in the reflexive case in Section 32, and we will discuss the nonreflexive case in Chapter VII.

Chapter IV: In Chapter IV, we use results from Sections 4, 12, 18 and 23 to establish a number of results on monotone multifunctions on general Banach spaces. Section 25 is devoted to the single result that a maximally monotone multifunction with bounded range has full domain, and in Section 26, we prove a local boundedness theorem for any (not necessarily maximally) monotone multifunction on a Banach space. Specifically, we prove that *a monotone multifunction, S, is locally bounded at any point surrounded by* $D(S_\varphi)$.

In Section 27, we prove the "six set theorem", Theorem 27.1, that if S is maximally monotone then the six sets $\operatorname{int} D(S)$, $\operatorname{int}(\operatorname{co} D(S))$, $\operatorname{int} D(S_\varphi)$, $\operatorname{sur} D(S)$, $\operatorname{sur}(\operatorname{co} D(S))$ and $\operatorname{sur} D(S_\varphi)$ coincide, and its consequence, the "nine set theorem", Theorem 27.3, that, if $\operatorname{sur} D(S_\varphi) \neq \emptyset$, then the nine sets $\overline{D(S)}$, $\overline{\operatorname{co} D(S)}$, $\overline{D(S_\varphi)}$, $\overline{\operatorname{int} D(S)}$, $\overline{\operatorname{int}(\operatorname{co} D(S))}$, $\overline{\operatorname{int} D(S_\varphi)}$, $\overline{\operatorname{sur} D(S)}$, $\overline{\operatorname{sur}(\operatorname{co} D(S))}$ and $\overline{\operatorname{sur} D(S_\varphi)}$ coincide. ("Sur" is defined in Section 13.) The six set theorem and the nine set theorem not only extend the results of Rockafellar that $\operatorname{int} D(S)$ is convex and that, if $\operatorname{int}(\operatorname{co} D(S)) \neq \emptyset$ then $\overline{D(S)}$ is convex, but also answer in the affirmative a question raised by Phelps, namely whether an absorbing point of $D(S)$ is necessarily an interior point. In Theorem 27.5 and Theorem 27.6, we give sufficient conditions that $\overline{D(S)} = \overline{D(S_\varphi)}$ and $\overline{R(S)} = \overline{R(S_\varphi)}$ — these conditions do not have any interiority hypotheses.

Section 28 contains the results that if S is maximally monotone then the closed linear hull of $D(S_\varphi)$ is identical with the closed linear hull of $D(S)$, and the closed affine hull of $D(S_\varphi)$ is identical with the closed affine hull of $D(S)$. The arguments here are quite simple, which is in stark contrast with the similar question for closed convex hulls. This section also contains some results for pairs of multifunctions, which will be used in our analysis of bootstrapped sum theorems for reflexive spaces in Section 32. We also give some results on the "restriction" of a monotone multifunction to a closed subspace. The results in this section depend ultimately on the result of Lemma 20.4 on q–positive sets that are "flattened" by certain elements of a SSD space.

Chapter V: Chapter V is concerned with maximally monotone multifunctions on reflexive Banach spaces. In Section 29, we use the theory of SSDB spaces developed in Section 21 to obtain various criteria for a monotone multifunctions on a reflexive Banach space to be maximally monotone. We deduce in Theorem 29.5 and Theorem 29.6 Rockafellar's surjectivity theorem, together with a sharp lower bound on the norm of solutions in terms of the Fitzpatrick function. Theorem 29.8 contains an expression for a maximally monotone extension of a given nontrivial monotone multifunction on a reflexive space, and Theorem 29.9 gives Torralba's analog in the context of maximally monotone multifunctions of the Brøndsted–Rockafellar theorem for convex lower semicontinuous functions.

In Section 30, we discuss more subtle surjectivity results. The main result here is Theorem 30.1, a general existence theorem for BC–functions, which has as applications (in Theorem 30.2) a nontrivial generalization of Theorem 29.5, and (in Theorem 30.4) an abstract Hammerstein theorem.

Section 31 is devoted to the Brezis–Haraux condition for $R(S+T)$ to be close to $R(S)+R(T)$. In fact, we show in Theorem 31.4(c) that stronger results are true under the condition $R(S)+R(T) \subset R\big(S_\varphi + T_\varphi\big)$, and then in Corollary 31.6, that $R(S)+R(T) \subset R\big(S_\varphi + T_\varphi\big)$ under the original Brezis–Haraux hypotheses.

The final three sections of Chapter V are concerned with various sufficient conditions for the sum of maximally monotone multifunctions on a reflexive Banach space to be maximally monotone, together with certain related identities. In Section 32, we use the results of Section 28 to bootstrap the result of Theorem 24.1(a), obtaining the "sandwiched closed subspace theorem", Theorem 32.2, which first appeared in [109], and unifies sufficient conditions that have been established by various authors for the sum theorem to hold. In Section 33, we again use Theorem 24.1(a), this time to establish that several of the sets appearing in the above mentioned sufficient conditions are, in fact, identical. Finally, in Section 34, we use the theory of BC–functions to obtain various generalization of the Brezis–Crandall–Pazy "perturbation" result on the maximal monotonicity of the sum.

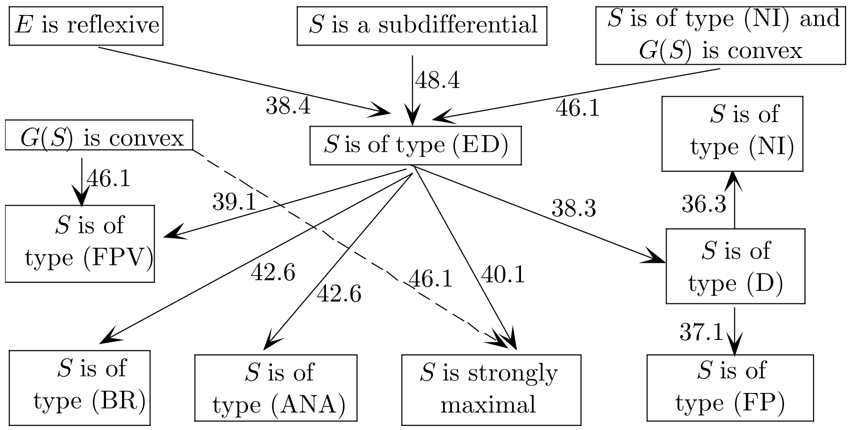
Chapter VI: In Chapter VI, we return to the discussion of monotonicity on possibly nonreflexive Banach spaces that we initiated in Chapters III and IV. Many of the nice results that we established in Chapter V either fail in this context, or the situation is not clear. The precise problem can be traced to the difference between SSD spaces and SSDB spaces.

Section 35 is a continuation of Section 22, but now we use the topology $\mathcal{T}_{\|\ \|}(E \times E^*)$ instead of the topology $\mathcal{T}_{\|\ \|}(E) \times\ w(E^*, E)$ on $E \times E^*$. In this case, we are led to consider two SSD spaces, $E \times E^*$ and $(E \times E^*)^* = E^{**} \times E^*$. We start off Section 35 with some examples, and then give, in Lemma 35.4, Lemma 35.5, Lemma 35.6 and Lemma 35.7, analogs to our present situation of Lemma 19.13, Theorem 19.16, Theorem 21.4(b) and Lemma 22.9. These results are all combined to obtain the main result of this section, Theorem 35.8, which will ultimately be applied in Sections 37, 39, 40 and 41.

Many subclasses of the class of maximally monotone multifunctions have been introduced, the basic idea being to define subclasses for which some of the properties of maximally monotone multifunctions on reflexive spaces continue to hold.

In Section 36, we introduce those that are "of type (D)", "of type (NI)", "of type (FP)", "of type (FPV)", "strongly maximally monotone", "of type (ANA)", and "of type (BR)". The oldest of them, the maximally monotone multifunctions of "type (D)", were introduced by Gossez in 1971, while the others are much more recent. In addition to giving the definitions of these subclasses, we also discuss a number of related open problems. There is an eighth subclass of the class of maximally monotone multifunctions which has a very interesting theory, those that are "of type (ED)". The definition of these requires more preliminary work, and so it will be postponed until Section 38. All of these subclasses share the property that it is hard to find a maximally monotone multifunctions that does *not* belong to the subclass. We now know that there are various inclusions between the subclasses. One of these will be the subject of Section 37, where we will prove that every maximally monotone multifunction of type (D) is automatically of type (FP). The result of Section 37 first appeared in [101].

It is true (and was realized by Gossez) that it is advantageous to replace the topology $w(E^{**},E^*)\times\mathcal{T}_{\| \|}(E^*)$ on $E^{**}\times E^*$ in the definition of "type (D)" by a stronger one. In Section 38, we will define $\mathcal{T}_{\mathcal{CLBN}}(E^{**}\times E^*)$, which is such a replacement, and produces a subclass of the maximally monotone multifunctions that has a number of extremely attractive properties. $\mathcal{T}_{\mathcal{CLBN}}(E^{**}\times E^*)$ is defined in terms of a topology, $\mathcal{T}_{\mathcal{CLB}}(E^{**})$, on E^{**} which lies between the weak* topology $w(E^{**},E^*)$ and the norm topology $\mathcal{T}_{\| \|}(E^{**})$. Despite the fact that $\mathcal{T}_{\mathcal{CLB}}(E^{**})$ has a number of pleasant properties, for reasons explained in the beginning of Section 38, $\big(E^{**},\mathcal{T}_{\mathcal{CLB}}(E^{**})\big)$ will normally fail to be a topological vector space. The corresponding class of maximally monotone multifunctions, those that are of "type (ED)", will be introduced in Definiton 38.3. We now give a graphic that serves to show the central position occupied by maximally monotone multifunctions of type (ED), and will provide a roadmap to some of the results of the following sections. We assume that E is a nonzero Banach space and $S\colon\ E\rightrightarrows E^*$ is maximally monotone.



In Sections 39–42, we use the results of Section 35 to prove that maximally monotone multifunctions of type (ED) are always of type (FPV), strongly maximal, of type (ANA) and of type (BR), and also explore the connection between these multifunctions and coercivity. In particular, we deduce in Corollary 41.4 the classical result that a coercive maximally monotone multifunction on a reflexive space is surjective. The analysis of these four sections is based on results that first appeared in [101] and [107]. It is worth pointing out that we do not know of a maximally monotone multifunction of type (D) which is *not* of type (ED).

In [44] and [45], it was proved that if S is maximally monotone of type (FP), or a certain condition involving *approximate resolvents* holds, then $\overline{R(S)}$ is convex. In Section 43, we use Theorem 27.6 to show that, under either of these hypotheses, in fact $\overline{R(S)} = \overline{R(S_\varphi)}$. In Section 44, we consider the fascinating question (already referred to in Problem 31.3) whether the maximal monotonicity of S implies the convexity of $\overline{D(S)}$. In order to put this question into context, we must now discuss *Rockafellar's sum problem*. This is the following: if S, T are maximally monotone and $D(S) \cap \operatorname{int} D(T) \neq \emptyset$ then is $S + T$ maximally monotone? A solution to this problem has been announced recently, but the jury is still out on this. It follows from Theorem 44.1 that if the solution to this problem is positive then *every maximally monotone multifunction is of type (FPV)*. Furthermore, it would then follow from Theorem 44.2 that, *for every maximally monotone multifunction S*, $\overline{D(S)} = \overline{\operatorname{co} D(S)} = \overline{D(S_\varphi)}$ so, in particular, *for every maximally monotone multifunction S, $\overline{D(S)}$ is convex.*

In Section 45 we prove that, under certain circumstances, the biconjugate of the pointwise maximum of a finite number of functions is the maximum of their biconjugates. (See Corolllary 45.5.) What is curious is that we can establish this result without having a simple explicit formula for the *conjugate* of this pointwise maximum. This result will be applied in Lemma 45.9 to obtain the fundamental property that $\widehat{E}$ is dense in $\big(E^{**}, \mathcal{T}_{\mathcal{CLB}}(E^{**})\big)$, and also a stronger result too complicated to discuss here. Lemma 45.9 will be applied in our work on maximally monotone multifunctions with convex graph in Section 46, and in our proof that subdifferentials are maximally monotone of type (ED) in Section 48. Theorem 45.12 gives an unexpected characterization of the closure of certain convex subsets of $E^{**} \times E^*$ with respect to $\mathcal{T}_{\mathcal{CLBN}}(E^{**} \times E^*)$ — this will also be used in Section 46.

As we have already observed, in Section 46, we discuss maximally monotone multifunctions with convex graph. This is an important subclass of the maximally monotone multifunctions since it includes all affine maximally monotone operators, and all maximally monotone multifunctions whose inverse is an affine function. We prove in Theorem 46.1 that any such multifunction is always strongly maximal and of type (FPV) and, further, if such a multifunction is of type (NI) then it is of type (ED) and, in Theorem 46.3, we give a sufficient condition of "Attouch–Brezis" type for the sum of two such maximally monotone multifunctions to be maximally monotone. In Section 47, we apply the results of Section 46 to possibly discontinuous linear operators, explaining the connections with known results. In addition, Theorem 47.1 contains a necessary and sufficient condition for a positive linear operator to be maximally monotone and we prove in Theorem 47.7 that every *continuous* positive linear operator is of type (ANA).

In Section 48, we first prove in Theorem 48.1 that if $f \in \mathcal{PCLSC}(E)$ then $\iota\big(G(\partial f)\big)$ is dense in $G^{-1}(\partial f^*)$ in $\mathcal{T}_{\mathcal{CLBN}}(E^{**} \times E^*)$, from which we deduce in Theorem 48.4 that subdifferentials are of type (ED), of type (FP), of type (FPV), strongly maximally monotone, of type (ANA) and of type (BR). We also deduce in Corollary 48.8 a result that is approximately a considerable generalization of the Brøndsted–Rockafellar theorem. We emphasize that the results in Section 48 depend on Lemma 45.9(a), which depends ultimately on the formula for the biconjugate of a maximum that we established in Theorem 45.3.

In Section 49, we prove that the "subdifferential" of a closed saddle–function on the product of a Banach space and a reflexive Banach space is maximally monotone of type (ED).

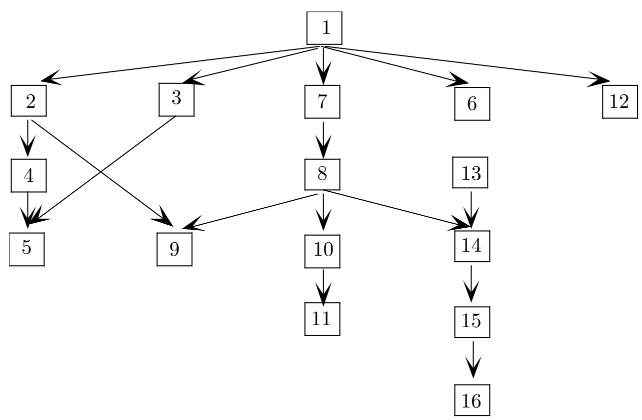
Chapter VII: In this chapter, we give various sufficient conditions for the sum of maximally monotone multifunctions on a general Banach space to be maximally monotone. Andrew Eberhard and Jonathan Borwein have announced the following result: *if E is a nonzero Banach space, $S, T\colon E \rightrightarrows E^*$ are maximally monotone and $D(S) \cap \operatorname{int} D(T) \neq \emptyset$ then $S + T$ is maximally monotone*. While their paper is not in definitive form, in Section 50, we discuss the far–reaching implication of such a result. In Section 51, we use results from Section 24 and Section 28 to prove a mild generalization of Voisei's theorem on the maximal monotonicity of the sum of two maximally monotone multifunctions S and T when $D(S)$ and $D(T)$ are closed and convex. In Section 52, we give sufficient conditions for $S + N_C$ to be maximally monotone, where $S\colon E \rightrightarrows E^*$ is monotone and N_C is the normality map of a nonempty closed convex set and, in Section 53, we give a short proof of the result of Verona–Verona that $S+T$ is maximally monotone when S is a subdifferential and T is maximally monotone with full domain.

Chapters VIII–X: In Chapter VIII, we collect together some of the open problems that have appeared in the body of the text, in Chapter IX, we provide a glossary of the definitions of the various classes of monotone multifunctions that we have introduced in the body of the text and, in Chapter X, we give a selection of the results proved in the body of the text.

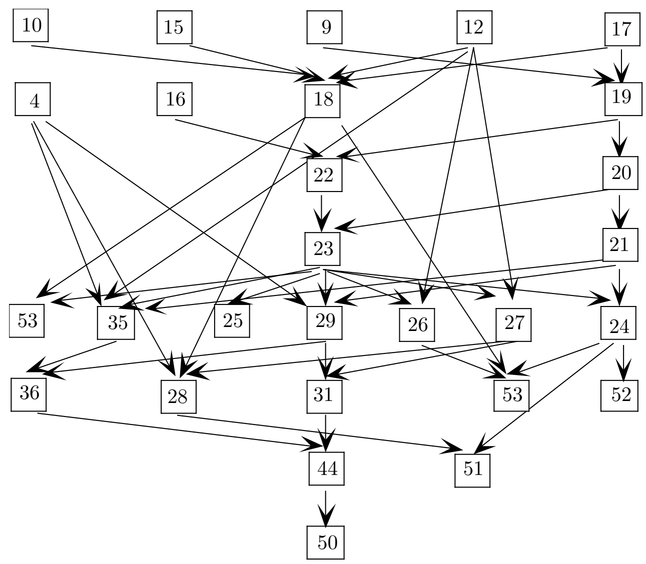
Epilog

We will shortly provide four flowcharts giving various possible uses for this volume. It is worth noting how important Section 23 is in the second, third and fourth of these. This simple idea due to Simon Fitzpatrick has had an enormous impact on the theory of monotonicity. Simon's death in 2004 at the untimely age of 51 was a tremendous loss to mathematics. Some idea of the scope of Simon's work can be obtained from the memorial volume [43], and there is a short history of his life by Borwein et al. in [19].

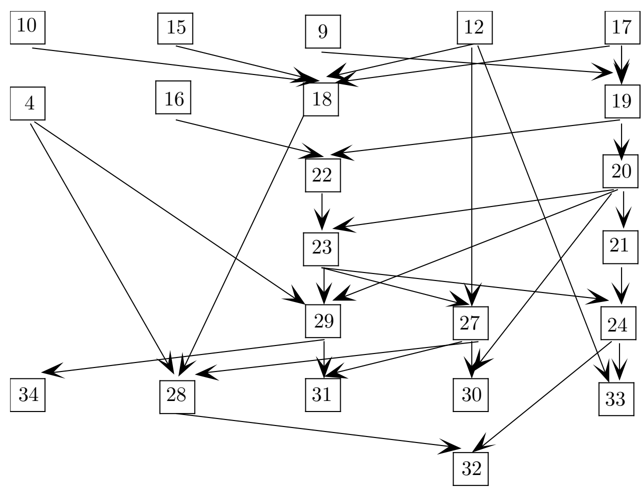
The first of the promised charts shows the flow of logic for Chapters I and II. This material could be used for an introductory course in functional analysis which would, at the same time, touch on minimax theorems and give a grounding in convex Lagrange multiplier theory and the main theorems in convex analysis.



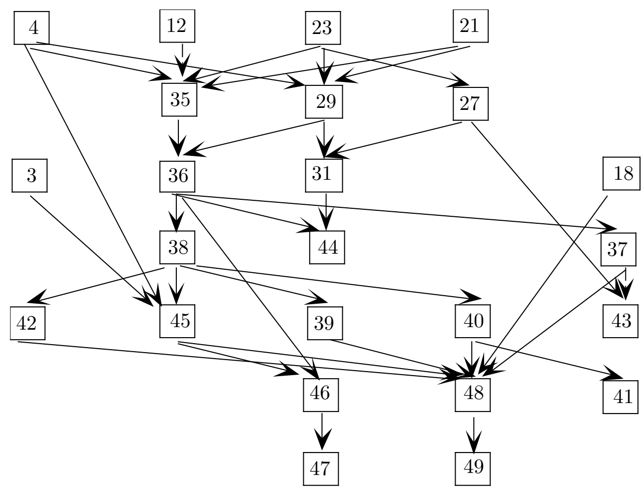
The next chart shows the flow of logic in Chapters III, IV and VII, starting from the appropriate sections in Chapters I and II. This material could be used as a basis for a course in which results on monotonicity on general Banach spaces are established using SSD spaces and Fitzpatrick functions.



The third chart shows the flow of logic in Chapter V, starting from the appropriate sections in Chapters I and II. This material could be used as a basis for a course in which results on monotonicity on reflexive Banach spaces are established using SSD spaces and Fitzpatrick functions.



The final chart shows the flow of logic in Chapter VI, starting from the appropriate sections in Chapters I, II, III, and IV. This contains an exposition of the more technical properties of maximal monotonicity on general Banach spaces that have been established since 1997.



These notes in no way claim to be an exhaustive study of convex analysis or monotonicity. We refer the reader to Zălinescu, [119], for such a study of convex analysis, and to Rockafellar and Wets, [83], for such a study of monotonicity in finite dimensional spaces.

We do not discuss the theory of *cyclical monotonicity* introduced by Rockafellar in [80]. We refer the reader to Bartz, Bauschke, Borwein, Reich and Wang, [5], for recent developments in this direction using a sequence of Fitzpatrick functions. We also do not discuss the *Asplund decomposition* of a maximally monotone multifunction into the sum of a subdifferential and an acyclic maximally monotone multifunction, or of *monotone variational inequalities*. We refer the reader to Borwein, [17, Theorem 3.4, p. 571] and [17, Section 5.3, pp. 580–581] for a discussion of these subjects.

Another topic that we do not discuss is the theory of monotonicity in Hilbert spaces. The connection between Fitzpatrick functions and the Kirszbraun-Valentine extension theorem in Hilbert spaces was explored by Reich–Simons in [73] and Borwein–Zhu in [24, Theorem 5.1.33, p. 179], and further results on this topic have been obtained by Bauschke in [7]. There is also the forthcoming book [10] by Bauschke and Combettes, in which convex analysis and monotonicity are discussed specifically in the Hilbert space setting.

We also do not discuss the semicontinuity of multifunctions, uscos and cuscos. These are treated in [24, Sections 5.1.4–5, pp. 173–177].