Set-Valued Analysis **7:** 255–294, 1999.

Maximal Monotone Multifunctions of Brøndsted–Rockafellar Type

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(Received: 18 March 1999; in final form: 27 August 1999)

Abstract. We consider whether the "inequality-splitting" property established in the Brøndsted–Rockafellar theorem for the subdifferential of a proper convex lower semicontinuous function on a Banach space has an analog for arbitrary maximal monotone multifunctions. We introduce the *maximal monotone multifunctions of type* (*ED*), for which an "inequality-splitting" property does hold. These multifunctions form a subclass of Gossez's maximal monotone multifunctions of type (D); however, *in every case where it has been proved that a multifunction is maximal monotone of type* (*D*) *then it is also of type* (*ED*). Specifically, the following maximal monotone multifunctions are of type (ED):

- *ultramaximal monotone multifunctions*, which occur in the study of certain nonlinear elliptic functional equations;
- single-valued linear operators that are maximal monotone of type (D);
- subdifferentials of proper convex lower semicontinuous functions;
- "subdifferentials" of certain saddle-functions.

We discuss the *negative alignment set* of a maximal monotone multifunction of type (ED) with respect to a point not in its graph – a mysterious continuous curve without end-points lying in the interior of the first quadrant of the plane. We deduce new inequality-splitting properties of subdifferentials, almost giving a substantial generalization of the original Brøndsted–Rockafellar theorem. We develop some mathematical infrastructure, some specific to multifunctions, some with possible applications to other areas of nonlinear analysis:

- the formula for the biconjugate of the pointwise maximum of a finite set of convex functions – in a situation where the "obvious" formula for the conjugate fails;
- a new topology on the bidual of a Banach space – in some respects, quite well behaved, but in other respects, quite pathological;
- an existence theorem for bounded linear functionals – unusual in that it does not assume the existence of any *a priori* bound;
- the 'big convexification' of a multifunction.

Mathematics Subject Classifications (1991): 47H05, 46B10, 49J35, 47N10, 54C08.

Key words: Banach space, convex function, conjugate, biconjugate, minimax theorem, inf-convolution, weak* topology, maximal monotone multifunction, type (D), James's theorem, Tauberian linear operator, Brøndsted–Rockafellar theorem, saddle-function.

1. Introduction

Let E be a (nonzero) real Banach space with dual E^*, $\alpha, \beta > 0$ and $(w, w^*) \in E \times E^*$. The Brøndsted–Rockafellar theorem states that *if* $f\colon E \mapsto \mathbb{R} \cup \{\infty\}$ *is proper, convex and lower semicontinuous, with subdifferential* ∂f, *and*

$$\sup_{x \in \operatorname{dom} f} [f(w) - f(x) - \langle w - x, w^* \rangle] \leqslant \alpha\beta$$

then there exists $(s, s^*) \in G(\partial f)$ *such that* $\|s - w\| \leqslant \alpha$ and $\|s^* - w^*\| \leqslant \beta$. (Here, "$G$" stands for "graph of".) It follows from this that *if*

$$\inf_{(t,t^*) \in G(\partial f)} \langle t - w, t^* - w^* \rangle \geqslant -\alpha\beta$$

then there exists $(s, s^*) \in G(\partial f)$ *such that* $\|s - w\| \leqslant \alpha$ *and* $\|s^* - w^*\| \leqslant \beta$.

These *inequality-splitting* properties of subdifferentials have turned out to be absolutely fundamental in convex analysis. Now in the second of the results mentioned above, f does not appear explicitly, it only appears in the form of ∂f, so it makes sense to ask if a similar inequality-splitting property is true for *any* maximal monotone operator. Thus: *Let* E *be a Banach space,* $T\colon E \mapsto 2^{E^*}$ *be maximal monotone and*

$$\inf_{(t,t^*) \in G(T)} \langle t - w, t^* - w^* \rangle \geqslant -\alpha\beta. \tag{1.0.1}$$

Then does there necessarily exist $(t, t^*) \in G(T)$ *such that* $\|t - w\| \leqslant \alpha$ *and* $\|t^* - w^*\| \leqslant \beta$*?* Indeed, exactly this result has been established recently by Torralba (in connection with problems of scale-change) when E is reflexive. However, Torralba's result does not extend to the nonreflexive case, as we will show by an example. Consequently, it makes sense to investigate if there is a significant subclass of the maximal monotone multifunctions on a general Banach space for which a (possibly slightly different) inequality-splitting property holds. This brings us to the second purpose of this paper: a discussion of the *maximal monotone multifunctions of type* (*ED*), for which a property of the above type holds – the property in question is weaker in that we have to replace the inequalities by the corresponding strict inequalities, but it is much stronger in other respects. We even prove results in which the assumption (1.0.1) is totally absent.

We will complement the above results by showing that several well-studied subclasses of the maximal monotone multifunctions are automatically of type (ED). More precisely, the maximal monotone multifunctions of type (ED) form a subclass of Gossez's maximal monotone multifunctions of type (D); however, we will show that *in every case* (*that is, for the inverses of single-valued everywhere-defined hemicontinuous monotone operators, certain maximal monotone linear operators, and the subdifferentials of proper convex lower semicontinuous functions and certain saddle functions*) *where it has been proved that a multifunction is maximal monotone of type* (*D*) *then it is also of type* (*ED*), *and consequently enjoys the inequality-splitting property in question.* In fact, our results even lead to an unexpected new inequality-splitting property for subdifferentials. A word on notation: the "(D)" in Gossez's notation stands for "dense"; the "(ED)" in our notation stands for "extra dense".

In order to perform the analysis described above, we will develop in Sections 2–7 and 14 a certain amount of mathematical infrastructure: Sections 4, 6 and 7 contain concepts and results specific to multifunctions, while Sections 2, 3, 5 and 14 contain concepts and results that might well have applications to other areas of nonlinear analysis.

The multifunctions of type (D) were defined with reference to the weak* topology on the bidual, E^{**}, of E. Now this topology presents certain problems from our point of view. In order to avoid these problems, we will introduce a new topology, $\mathcal{T}_{\mathcal{CLB}}(E^{**})$, on E^{**}. This will be done in Section 3. It will be seen that, in some respects, $\mathcal{T}_{\mathcal{CLB}}(E^{**})$ is quite well behaved, but in other respects it is quite pathological. For instance, if E is not reflexive then $\mathcal{T}_{\mathcal{CLB}}(E^{**})$ lies *strictly* between the weak* and the norm topology of E^{**}, but $(E^{**}, \mathcal{T}_{\mathcal{CLB}}(E^{**}))$ is not a topological vector space. Maximal monotone multifunctions of type (ED) will then be defined in Section 4. The reader interested in a more complete (and technical) discussion of the deficiencies in the weak* topology on E^{**} and why it seems necessary to introduce this new class of multifunctions can turn to the introduction to Section 4.

Section 5 contains some functional analytic tools – in particular, it contains (in Theorem 5.2) an existence theorem for bounded linear functionals that is somewhat unusual. Typically, such existence theorems assume the existence of an *a priori* bound in order to use some version of the Hahn–Banach theorem, or depend on Baire's theorem. In contrast, Theorem 5.2 does not assume the existence of any such bound and does not depend on Baire's theorem. Another unusual feature of Theorem 5.2 is that it uses *two* applications of a minimax theorem. Sections 6 and 7 contain various technical results on multifunctions that will be required in Section 8. We point, in particular, to Theorem 6.3, which gives a fundamental property of monotone multifunctions, and which we prove using a minimax theorem and the "big convexification" of a multifunction.

We start Section 8 by stating (in Theorem 8.1) the result of Torralba referred to above. We then introduce the concept of a *negative alignment pair*, which is the most convenient vehicle for stating the promised inequality-splitting property for multifunctions of type (ED), which we establish in Theorem 8.6, the main result of this paper. We close Section 8 by discussing the *negative alignment set* of a maximal monotone multifunction of type (ED) with respect to a point not in its graph. This is a rather mysterious continuous curve without end-points that lies in the interior of the first quadrant of the plane. In Section 9, we show that multifunctions of type (ED) have another property that is known to hold for subdifferentials or continuous monotone linear operators.

In Section 10, we change tack and start discussing which maximal monotone *multifunctions* are automatically of type (ED). We discuss the *ultramaximal monotone multifunctions*, a subclass of the maximal monotone *multifunctions* of type

(ED), that occurs in the study of certain nonlinear elliptic functional equations. In Section 11, we shall see (in Theorem 11.1) that if we restrict our attention to *single-valued linear operators* then maximal monotone multifunctions of type (D) in the sense of Gossez are, in fact, automatically of type (ED). It then follows from known results that several other subclasses of the maximal monotone linear operators are automatically of type (ED) – see Theorems 11.2–11.3. We also give (in Example 11.5) a continuous linear example that shows, among other things, that Torralba's result does not extend to the nonreflexive case.

The main result of Section 12 is an approximation result (see Theorem 12.2), which has as a consequence (see Theorem 12.6) that subdifferentials are automatically of type (ED). We show in Section 13 how combining the results of Sections 8 and 12 leads to new inequality-splitting properties of subdifferentials – see Theorem 13.1 and Corollary 13.3. The results of Section 13 can be viewed in the following light. The original result of Brøndsted and Rockafellar was one of the precursors of Ekeland's variational principle for lower semicontinuous functions on a complete metric space, though now one tends to reverse the process and deduce the Brøndsted–Rockafellar theorem from Ekeland's variational principle. At any rate, the functional-analytic content of the Brøndsted–Rockafellar theorem is not more substantial that its metric space content. On the other hand, the results of Section 13 on the existence of negative alignment pairs are truly theorems in functional analysis.

Section 14 contains various technical results on how the topologies $\mathcal{T}_{\mathcal{CLB}}$ react with products. The results of Section 14 will be applied in Section 15, in which we prove that the multifunction associated with a saddle-function is sometimes maximal monotone of type (ED).

Many of the results of Section 3 have fairly simple proofs. The one exception is Lemma 3.3, which depends on the formula for the biconjugate of the pointwise maximum of a finite set of convex functions that we establish in Theorem 2.5. It is worth pointing out that we establish this result in a situation in which the "obvious" formula for the conjugate of the pointwise maximum is false – see Remark 2.3 for more discussion of this. Otherwise, Section 2 contains the notation that we shall use, which is all fairly standard, as well as an example (Example 2.8) showing the pathology possible with continuous convex functions.

2. Biconjugates of Convex Functions

The main result in this section is Theorem 2.5, in which we prove that, under certain circumstances, the biconjugate of the maximum of a finite family of convex functions is the maximum of their biconjugates.

We start with some basic notation. Let $\mathcal{PC}(E)$ stand for the set of all convex functions $f\colon E \mapsto \mathbb{R}\cup\{\infty\}$ such that $\operatorname{dom} f \neq \emptyset$, where the *effective domain* of f, $\operatorname{dom} f$, is defined by

$$\operatorname{dom} f := \{x \in E : f(x) \in \mathbb{R}\}.$$

(The "$\mathcal{P}$" stands for "proper", which is the adjective frequently used to denote the fact that the effective domain of a function is nonempty.) If $f \in \mathcal{PC}(E)$, the *Fenchel conjugate*, f^*, of f is the function from E^* into $\mathbb{R} \cup \{\infty\}$ defined by

$$f^*(x^*) := \sup_E (x^* - f).$$

We write $\mathcal{PCLSC}(E)$ for the set of all those $f \in \mathcal{PC}(E)$ that are lower semicontinuous, and $\mathcal{PCPC}(E)$ for the set of all those $f \in \mathcal{PC}(E)$ such that $\operatorname{dom} f^* \neq \emptyset$. (The extra "$\mathcal{PC}$" stands for "proper conjugate".) It is well known that

$$f \in \mathcal{PCLSC}(E) \Rightarrow f \in \mathcal{PCPC}(E) \Rightarrow f^* \in \mathcal{PCLSC}(E^*).$$

Suppose now that $f \in \mathcal{PCPC}(E)$. Then the *biconjugate*, f^{**}, of f is the function from the bidual, E^{**}, of E into $\mathbb{R} \cup \{\infty\}$ defined by

$$f^{**}(x^{**}) := (f^*)^*(x^{**}) \quad (x^{**} \in E^{**}).$$

Let $w(E^{**}, E^*)$ be the weak* topology on E^{**} and, if $x \in E$, let $\hat{x}$ be the canonical image of x in E^{**}. It follows easily from the definitions that

$$f^{**} \text{ is } w(E^{**}, E^*)\text{-lower semicontinuous,} \tag{2.0.1}$$

$$t^{**} \in E^{**},\ f^{**}(t^{**}) \leqslant 0 \text{ and } w^* \in E^* \Rightarrow \langle w^*, t^{**}\rangle \leqslant f^*(w^*), \tag{2.0.2}$$

$$x^{**} \in E^{**} \Rightarrow f^{**}(x^{**}) \leqslant \sup\big\{f(x)\colon x \in E,\ \|x\| \leqslant \|x^{**}\|\big\} \tag{2.0.3}$$

and

$$x \in E \Rightarrow f^{**}(\hat{x}) \leqslant f(x). \tag{2.0.4}$$

We also have the result (associated by various authors with the names of Legendre, Fenchel, Moreau and Hormander) that if $f \in \mathcal{PCLSC}(E)$ then

$$x \in E \Rightarrow f^{**}(\hat{x}) = f(x). \tag{2.0.5}$$

See Rockafellar, [18, p. 210–211] – there is also a proof of this fact using a minimax theorem in [24, Remark 6.3, p. 26]. While on the subject of minimax theorems, we shall use the following classical minimax theorem in Lemma 2.2, and also twice in Theorem 5.2. It follows from a result of Fan – see [8], or [24, Theorem 3.1, p. 17]. It is important that the set A not be required to have any topological structure.

THEOREM 2.1. *Let A be a nonempty convex subset of a vector space, B be a nonempty convex subset of a vector space and B also be a compact Hausdorff topological space. Let $h\colon A \times B \mapsto \mathbb{R}$ be convex on A, and concave and upper semicontinuous on B. Then*

$$\inf_A \max_B h = \max_B \inf_A h.$$

We write $\mathcal{CC}(E)$ for the set of all real convex continuous functions on E and "$\vee$" for pointwise maximum.

LEMMA 2.2. *Let $f \in \mathcal{PC}(E)$, $g \in \mathcal{CC}(E)$ and $w^* \in E^*$.*

(a) *Let $\rho,\ \sigma > 0$. Then there exist u^*, $v^* \in E^*$ such that*

$$\rho u^* + \sigma v^* = w^* \quad \textit{and} \quad \rho f^*(u^*) + \sigma g^*(v^*) = \sup_{\mathrm{dom}\, f} \big[w^* - \rho f - \sigma g\big].$$

(b) $(f \vee g)^*(w^*) = \min_{\rho\in[0,1]} \sup_{\mathrm{dom}\, f} \big[w^* - \rho f - (1-\rho)g\big].$

Proof. (a) It follows from the "inf-convolution" formula for the conjugate of a sum (see Rockafellar, [20, Theorem 20, p. 56]) that there exists $y^* \in E^*$ such that

$$\begin{aligned}(\rho f)^*(y^*) + (\sigma g)^*(w^* - y^*) &= (\rho f + \sigma g)^*(w^*)\\ &:= \sup_{\mathrm{dom}(\rho f + \sigma g)} \big[w^* - \rho f - \sigma g\big].\end{aligned}$$

We now obtain (a) by setting $u^* := y^*/\rho$ and $v^* := (w^* - y^*)/\sigma$ and noting that $\mathrm{dom}(\rho f + \sigma g) = \mathrm{dom}\, f$. (It is critical for this step that $\mathrm{dom}\, g = E$.)

(b) follows from Theorem 2.1 with $A := \mathrm{dom}\, f$ and $B := [0, 1]$, since

$$\begin{aligned}(f \vee g)^*(w^*) &= \sup_{x\in\mathrm{dom}\, f} \big[\langle x, w^*\rangle - (f \vee g)(x)\big]\\ &= \sup_{x\in\mathrm{dom}\, f} \min_{\rho\in[0,1]} \big[\langle x, w^*\rangle - \rho f(x) - (1-\rho)g(x)\big].\end{aligned}$$ □

Remark 2.3. One might be led to suspect by analogy with the inf-convolution formula for the conjugate of a sum referred to in Lemma 2.2(a) that, in the situation of Lemma 2.2(b), $(f \vee g)^*(w^*)$ is given by the formula

$$\min_{\rho\in[0,1],\ u^*,\ v^*\in E^*,\ \rho u^* + (1-\rho)v^* = w^*} \big[\rho f^*(u^*) + (1-\rho)g^*(v^*)\big], \tag{2.3.1}$$

but this is not necessarily true if $f \notin \mathcal{CC}(E)$ – see Fitzpatrick and Simons, [9]. That paper contains a number of other results on the conjugates and biconjugates of the pointwise maximum of a finite number of convex functions.

LEMMA 2.4. *Let $f \in \mathcal{PCPC}(E)$, $g \in \mathcal{CC}(E)$ and $f^{**}(t^{**}) \vee g^{**}(t^{**}) \leqslant 0$.*

(a) *Let $\rho, \sigma > 0$ and $w^* \in E^*$. Then $\langle w^*, t^{**}\rangle \leqslant \sup_{\mathrm{dom}\, f}[w^* - \rho f - \sigma g]$.*
(b) *Let $w^* \in E^*$. Then $\langle w^*, t^{**}\rangle \leqslant \sup_{\mathrm{dom}\, f}[w^* - g]$.*
(c) *Let $\rho \in [0, 1]$ and $w^* \in E^*$. Then*

$$\langle w^*, t^{**}\rangle \leqslant \sup_{\mathrm{dom}\, f} \big[w^* - \rho f - (1-\rho)g\big]. \tag{2.4.1}$$

(d) *Let $w^* \in E^*$. Then $\langle w^*, t^{**}\rangle \leqslant (f \vee g)^*(w^*)$.*
(e) $(f \vee g)^{**}(t^{**}) \leqslant 0.$

Proof. (a) Choose u^* and v^* as in Lemma 2.2(a). Then, from (2.0.2),

$$\begin{aligned}\langle w^*, t^{**}\rangle &= \langle \rho u^* + \sigma v^*, t^{**}\rangle = \rho\langle u^*, t^{**}\rangle + \sigma\langle v^*, t^{**}\rangle \\ &\leqslant \rho f^*(u^*) + \sigma g^*(v^*),\end{aligned}$$

and the result follows from Lemma 2.2(a).

(b) Since $f \in \mathcal{PCPC}(E)$, we can fix $x^* \in \operatorname{dom} f^*$. For all $\rho > 0$, we apply (a) with w^* replaced by $\rho x^* + w^*$ and $\sigma := 1$ and obtain

$$\begin{aligned}\rho\langle x^*, t^{**}\rangle + \langle w^*, t^{**}\rangle &= \langle \rho x^* + w^*, t^{**}\rangle \\ &\leqslant \sup_{\operatorname{dom} f}\big[\rho x^* + w^* - \rho f - g\big] \\ &\leqslant \sup_{\operatorname{dom} f}\big[\rho x^* - \rho f\big] + \sup_{\operatorname{dom} f}\big[w^* - g\big] \\ &= \rho f^*(x^*) + \sup_{\operatorname{dom} f}\big[w^* - g\big],\end{aligned}$$

and (b) follows by letting $\rho \to 0$.

(c) If $\rho = 0$ then (2.4.1) follows from (b). If $\rho \in (0, 1)$ then (2.4.1) follows from (a). If, finally, $\rho = 1$ then the right-hand side of (2.4.1) is exactly $f^*(w^*)$, and (2.4.1) follows from (2.0.2).

(d) follows from (c) and Lemma 2.2(b), and (e) is immediate from (d). □

Theorem 2.5 generalizes a result proved in [24, Theorem 33.3(c), p. 131], and [9, Theorem 6]. This increased generality (i.e., replacing the assumption "$g_0 \in \mathcal{PCLSC}(E)$" by the assumption "$g_0 \in \mathcal{PCPC}(E)$") is necessitated by the application that we will give in Theorem 11.1.

THEOREM 2.5. *Let $g_0 \in \mathcal{PCPC}(E)$ and $g_1, \ldots, g_m \in \mathcal{CC}(E)$.*

(a) *Let $t^{**} \in E^{**}$. Then $(g_0 \vee g_1)^{**}(t^{**}) \leqslant {g_0}^{**}(t^{**}) \vee {g_1}^{**}(t^{**})$.*
(b) *$(g_0 \vee g_1)^{**} = {g_0}^{**} \vee {g_1}^{**}$ on E^{**}.*
(c) *$(g_0 \vee \cdots \vee g_m)^{**} = {g_0}^{**} \vee \cdots \vee {g_m}^{**}$ on E^{**}.*

Proof. (a) Let $\alpha := {g_0}^{**}(t^{**}) \vee {g_1}^{**}(t^{**})$. Since the result is immediate if $\alpha = \infty$, we can and will suppose that $\alpha \in \mathbb{R}$. We now obtain the result by applying Lemma 2.4(e) with $f := g_0 - \alpha$ and $g := g_1 - \alpha$.

(b) Since $g_0 \vee g_1 \geqslant g_0$ on E, $(g_0 \vee g_1)^{**} \geqslant {g_0}^{**}$ on E^{**}. Similarly, $(g_0 \vee g_1)^{**} \geqslant {g_1}^{**}$ on E^{**}, and so $(g_0 \vee g_1)^{**} \geqslant g_0^{**} \vee {g_1}^{**}$ on E^{**}. The result now follows from (a).

(c) This is immediate from (b) and induction. □

COROLLARY 2.6. *Let $f_0 \in \mathcal{PCPC}(E)$ and $f_1, \ldots, f_m \in \mathcal{CC}(E)$. Let $t^{**} \in E^{**}$ and $\varepsilon > 0$. Then there exists $t \in E$ such that*

$$\text{for all } i = 0, \ldots, m, \quad f_i(t) \leqslant {f_i}^{**}(t^{**}) + \varepsilon.$$

Proof. Since we can remove those values of i for which $f_i^{**}(t^{**}) = \infty$, we can and will suppose that $f_0^{**}(t^{**}), \ldots, f_m^{**}(t^{**}) \in \mathbb{R}$. For all $i = 0, \ldots, m$, let $g_i := f_i - f_i^{**}(t^{**})$. Then $g_i^{**}(t^{**}) = 0$, hence $g_0^{**}(t^{**}) \vee \cdots \vee g_m^{**}(t^{**}) = 0$. From Theorem 2.5(c), $(g_0 \vee \cdots \vee g_m)^{**}(t^{**}) = 0$ and so, from (2.0.2) with $f := g_0 \vee \cdots \vee g_m$, and $w^* := 0$, $(g_0 \vee \cdots \vee g_m)^*(0) \geqslant 0$, that is to say $\inf_E(g_0 \vee \cdots \vee g_m) \leqslant 0$. The result follows by rewriting this inequality in terms of the functions f_i. □

DEFINITION 2.7. We write $\mathcal{CLB}(E)$ for the set of all convex functions $f\colon E \mapsto \mathbb{R}$ that are Lipschitz on the bounded subsets of E, or equivalently bounded on the bounded subsets of E. (If $M > 0$ and $K := \sup\{|f(x)|\colon x \in E,\ \|x\| \leqslant 2M\}$ then $2K/M$ is a Lipschitz constant for f on the set $\{x \in E\colon \|x\| \leqslant M\}$ – see, for instance, the argument of Phelps, [13, Proposition 1.6, p. 4]). The standard example of a function $f \in \mathcal{CC}(\ell^2) \setminus \mathcal{CLB}(\ell^2)$ is defined by $f(x) := \sum_{n=1}^{\infty} n x^{2n}$ $(x = \{x_n\}_{n \geqslant 1} \in \ell^2)$. It was proved by Borwein, Fitzpatrick and Vanderwerff in [4, Theorem 2.2, p. 64] using the deep Josefson–Nissenzweig theorem that if E is infinite-dimensional then $\mathcal{CC}(E) \setminus \mathcal{CLB}(E) \neq \emptyset$.

It follows from (2.0.3) that

$$f \in \mathcal{CLB}(E) \Rightarrow f^{**} \in \mathcal{CLB}(E^{**}). \tag{2.7.1}$$

In general, $\mathcal{CC}(E)$ behaves in a much more pathological fashion. This is illustrated by the example below.

EXAMPLE 2.8. We give an example of a function $f \in \mathcal{CC}(c_0)$ such that f^{**} is not continuous on $c_0^{**} = \ell^\infty$. Define $f \in \mathcal{CC}(c_0)$ by

$$f(x) := \sum_{n \geqslant 1} {x_n}^{2n} \quad (x = \{x_n\}_{n \geqslant 1} \in c_0).$$

(We note from the M-test that if $x \in c_0$ then the series defining f is uniformly convergent on the set $\{y \in c_0\colon \|y - x\| < 1/2\}$. Consequently, f is continuous.) It follows by direct computation that

$$f^{**}(\xi) = \sum_{n \geqslant 1} {\xi_n}^{2n} \quad (\xi = \{\xi_n\}_{n \geqslant 1} \in \ell^\infty).$$

Let $\xi \in \ell^\infty$ be defined by $\xi_n := n^{-1/n}$ $(n \geqslant 1)$ and, for $N \geqslant 1$, $\xi^N \in \ell^\infty$ be defined by

$$\xi^N{}_n := \begin{cases} \xi_n, & \text{if } n \leqslant N; \\ 1, & \text{otherwise.} \end{cases}$$

Since $\lim_{n \to \infty} \xi_n = 1$, $\xi^N \to \xi$ in ℓ^∞ as $N \to \infty$. On the other hand, for all $N \geqslant 1$,

$$f^{**}(\xi^N) \geqslant \sum_{n > N} 1 = \infty$$

and

$$f^{**}(\xi) = \sum_{n\geqslant 1} \frac{1}{n^2} < \infty.$$

So $f^{**}(\xi^N) \not\to f^{**}(\xi)$ as $N \to \infty$. Consequently, f^{**} is not continuous on ℓ^∞.

PROBLEM 2.9. *Let E be a general nonreflexive Banach space. Does there always exist $f \in \mathcal{CC}(E)$ such that f^{**} is not continuous* (*as we proved above for $E := c_0$*)?

3. The Topology $\mathcal{T}_{\mathcal{CLB}}(E^{**})$ on E^{**}

We define the topology $\mathcal{T}_{\mathcal{CLB}}(E^{**})$ on E^{**} to be the coarsest topology on E^{**} making all the functions $h^{**}\colon\ E^{**} \mapsto \mathbb{R}$ ($h \in \mathcal{CLB}(E)$) continuous. In this section, we collect together the basic properties of $\mathcal{T}_{\mathcal{CLB}}(E^{**})$. The "hardest" result in this section is undoubtedly Lemma 3.3(b).

Lemma 3.1 is devoted to the continuity of various naturally defined maps. Lemma 3.1(a) will be used in Theorem 12.6(a), Lemma 3.1(c,d) will be used in Theorem 4.4(b), and Lemma 3.1(e) will be used in Lemma 8.5(a), Theorem 12.2 and Theorem 12.6(a). We recall that if φ is a map from a topological space into E^{**} then φ is continuous into $\mathcal{T}_{\mathcal{CLB}}(E^{**})$ if, and only if, for all $h \in \mathcal{CLB}(E)$, $h^{**} \circ \varphi$ is continuous into $\mathbb{R}$; further, if x^{**}_γ is a net of elements of E^{**} and $x^{**} \in E^{**}$ then $x^{**}_\gamma \to x^{**}$ in $\mathcal{T}_{\mathcal{CLB}}(E^{**})$ if, and only if, for all $h \in \mathcal{CLB}(E)$, $h^{**}(x^{**}_\gamma) \to h^{**}(x^{**})$ in $\mathbb{R}$.

We write "$\mathcal{T}_{\|\ \|}$" for "norm topology of".

LEMMA 3.1.

(a) *If $x^* \in E^*$ then the map $x^{**} \mapsto \langle x^*, x^{**}\rangle$ is continuous from $(E^{**}, \mathcal{T}_{\mathcal{CLB}}(E^{**}))$ into $\mathbb{R}$.*
(b) *If $x \in E$ then the map $x^{**} \mapsto \|x^{**} - \hat{x}\|$ is continuous from $(E^{**}, \mathcal{T}_{\mathcal{CLB}}(E^{**}))$ into $\mathbb{R}$.*
(c) *If $w \in E$ then the map $x^{**} \mapsto x^{**} - \widehat{w}$ is continuous from $(E^{**}, \mathcal{T}_{\mathcal{CLB}}(E^{**}))$ into itself.*
(d) *If $\lambda \in \mathbb{R}$ then the map $x^{**} \mapsto \lambda x^{**}$ is continuous from $(E^{**}, \mathcal{T}_{\mathcal{CLB}}(E^{**}))$ into itself.*
(e) *The map $(x^{**}, x^*) \mapsto \langle x^*, x^{**}\rangle$ is continuous from $(E^{**} \times E^*, \mathcal{T}_{\mathcal{CLB}}(E^{**}) \times \mathcal{T}_{\|\ \|}(E^*))$ into $\mathbb{R}$.*

Proof. (a) follows since, if $h := x^* \in \mathcal{CLB}(E)$, then $h^{**} = \langle x^*, \cdot\rangle$. Likewise, (b) follows since, if $h := \|\cdot - x\| \in \mathcal{CLB}(E)$, then $h^{**} = \|\cdot - \hat{x}\|$.

(c) follows since, if $h \in \mathcal{CLB}(E)$ and we define $g \in \mathcal{CLB}(E)$ by $g := h(\cdot - w)$, then $g^{**} = h^{**}(\cdot - \widehat{w})$. Likewise, (d) follows since, if $h \in \mathcal{CLB}(E)$

and we define $g \in \mathcal{CLB}(E)$ by $g := h(\lambda\,\cdot)$, then $g^{**} = h^{**}(\lambda\,\cdot)$. (Here, the cases $\lambda \neq 0$ and $\lambda = 0$ must be handled separately.)

(e) Let $\{(x_\gamma^{**}, x_\gamma^*)\}$ be a net of elements of $E^{**} \times E^*$, $(x^{**}, x^*) \in E^{**} \times E^*$, $x_\gamma^{**} \to x^{**}$ in $\mathcal{T}_{\mathcal{CLB}}(E^{**})$ and $x_\gamma^* \to x^*$ in $\mathcal{T}_{\|\ \|}(E^*)$. From (a), $\langle x^*, x_\gamma^{**} - x^{**}\rangle \to 0$ and, from (b) with $x := 0$, $\|x_\gamma^{**}\| \to \|x^{**}\|$ and so $\{x_\gamma^{**}\}$ is eventually bounded. Since

$$\begin{aligned} |\langle x_\gamma^*, x_\gamma^{**}\rangle - \langle x^*, x^{**}\rangle| &= |\langle x_\gamma^* - x^*, x_\gamma^{**}\rangle + \langle x^*, x_\gamma^{**} - x^{**}\rangle| \\ &\leqslant \|x_\gamma^* - x^*\|\|x_\gamma^{**}\| + |\langle x^*, x_\gamma^{**} - x^{**}\rangle|, \end{aligned}$$

it follows that $\langle x_\gamma^*, x_\gamma^{**}\rangle \to \langle x^*, x^{**}\rangle$ in $\mathbb{R}$. □

We can extract from the proof of Lemma 3.1(e) the following result, which will be used in Theorem 7.2(a), Lemma 8.5(a) and Theorem 12.2:

LEMMA 3.2. *Let $\{x_\gamma^{**}\}$ be a net of elements of E^{**}, $x^{**} \in E^{**}$ and $x_\gamma^{**} \to x^{**}$ in $\mathcal{T}_{\mathcal{CLB}}(E^{**})$. Then $x_\gamma^{**} \to x^{**}$ in $w(E^{**}, E^*)$, $\|x_\gamma^{**}\| \to \|x^{**}\|$ and $\{x_\gamma^{**}\}$ is eventually bounded.*

Lemma 3.3(b) will be used in Theorems 11.1 and 12.2, and Lemma 3.3(c) contains a fundamental density property of $\mathcal{T}_{\mathcal{CLB}}(E^{**})$ that we will use in Lemma 14.4.

LEMMA 3.3. (a) *Let $\{x_\gamma^{**}\}$ be a net of elements of E^{**} and $x^{**} \in E^{**}$. Then $x_\gamma^{**} \to x^{**}$ in $\mathcal{T}_{\mathcal{CLB}}(E^{**})$ if, and only if,*

$$\text{for all } h \in \mathcal{CLB}(E), \quad \limsup_\gamma h^{**}(x_\gamma^{**}) \leqslant h^{**}(x^{**}). \tag{3.3.1}$$

(b) *Let $f \in \mathcal{PCPC}(E)$ and $t^{**} \in E^{**}$. Then there exists a net $\{t_\gamma\}$ of elements of E such that $\widehat{t_\gamma} \to t^{**}$ in $\mathcal{T}_{\mathcal{CLB}}(E^{**})$, $f(t_\gamma) \to f^{**}(t^{**})$ and $f^{**}(\widehat{t_\gamma}) \to f^{**}(t^{**})$.*

(c) *$\widehat{E}$ is a dense subset of $(E^{**}, \mathcal{T}_{\mathcal{CLB}}(E^{**}))$.*

Proof. “Only if” in (a) is immediate. Suppose, conversely, that (3.3.1) is true. Let $x^* \in E^*$. Then, since $x^* \in \mathcal{CLB}(E)$ and $-x^* \in \mathcal{CLB}(E)$, we have from (3.3.1) and the argument of Lemma 3.1(a) that

$$\limsup_\gamma \langle x^*, x_\gamma^{**}\rangle \leqslant \langle x^*, x^{**}\rangle \quad \text{and} \quad \liminf_\gamma \langle x^*, x_\gamma^{**}\rangle \geqslant \langle x^*, x^{**}\rangle,$$

thus $x_\gamma^{**} \to x^{**}$ in $w(E^{**}, E^*)$. From the $w(E^{**}, E^*)$-lower semicontinuity of h^{**} mentioned in (2.0.1),

$$\text{for all } h \in \mathcal{CLB}(E) \subset \mathcal{PCPC}(E), \quad \liminf_\gamma h^{**}(x_\gamma^{**}) \geqslant h^{**}(x^{**}).$$

Combining this with (3.3.1),

$$\text{for all } h \in \mathcal{CLB}(E), \quad h^{**}(x_\gamma^{**}) \to h^{**}(x^{**}),$$

that is to say, $x_\gamma^{**} \to x^{**}$ in $\mathcal{T}_{\mathcal{CLB}}(E^{**})$. This completes the proof of “if” of (a).

(b) From Corollary 2.6, for each nonempty finite subset H of $\mathcal{CC}(E)$ and $\varepsilon > 0$, there exists $t_{H,\varepsilon} \in E$ such that,

$$f(t_{H,\varepsilon}) \leqslant f^{**}(t^{**}) + \varepsilon \quad \text{and,} \quad \text{for all } h \in H, \quad h(t_{H,\varepsilon}) \leqslant h^{**}(t^{**}) + \varepsilon.$$

If we direct (H, ε) in the usual (product) way, we can construct a net $\{t_\gamma\}$ of elements of E such that,

$$\limsup_\gamma f(t_\gamma) \leqslant f^{**}(t^{**}) \quad \text{and,} \tag{3.3.2}$$

$$\text{for all } h \in \mathcal{CC}(E), \quad \limsup_\gamma h(t_\gamma) \leqslant h^{**}(t^{**}).$$

It follows from the second inequality in (3.3.2) above and (2.0.5) that,

$$\text{for all } h \in \mathcal{CC}(E), \quad \limsup_\gamma h^{**}(\widehat{t_\gamma}) \leqslant h^{**}(t^{**}).$$

Since $\mathcal{CLB}(E) \subset \mathcal{CC}(E)$, we derive from (a) that $\widehat{t_\gamma} \to t^{**}$ in $\mathcal{T}_{\mathcal{CLB}}(E^{**})$. From Lemma 3.2, $\widehat{t_\gamma} \to t^{**}$ in $w(E^{**}, E^*)$ and so, from the $w(E^{**}, E^*)$-lower semicontinuity of f^{**} mentioned in (2.0.1), $f^{**}(t^{**}) \leqslant \liminf_\gamma f^{**}(\widehat{t_\gamma})$. (b) follows by combining this with (2.0.4) and the first inequality in (3.3.2).

(c) follows from (b) by simply taking $f := 0$. □

In Theorem 3.4, we explore the relationship between $\mathcal{T}_{\mathcal{CLB}}(E^{**})$ and the two classical topologies on E^{**}. Theorem 3.4(b) will be used in Lemma 14.4 and Theorem 14.6.

THEOREM 3.4. (a) $w(E^{**}, E^*) \subset \mathcal{T}_{\mathcal{CLB}}(E^{**}) \subset \mathcal{T}_{\| \|}(E^{**})$.

(b) *Let $\{x_\gamma\}$ be a net of elements of E and $x \in E$. Then*

$$\widehat{x_\gamma} \to \hat{x} \textit{ in } \mathcal{T}_{\mathcal{CLB}}(E^{**}) \Leftrightarrow x_\gamma \to x \textit{ in } \mathcal{T}_{\| \|}(E).$$

(c) $\mathcal{T}_{\mathcal{CLB}}(E^{**}) = w(E^{**}, E^*) \Leftrightarrow E$ *is finite-dimensional.*

(d) $\mathcal{T}_{\mathcal{CLB}}(E^{**}) = \mathcal{T}_{\| \|}(E^{**}) \Leftrightarrow E$ *is reflexive.*

Proof. (a) It is clear from Lemma 3.1(a) that $w(E^{**}, E^*) \subset \mathcal{T}_{\mathcal{CLB}}(E^{**})$, and from the norm-continuity of f^{**} implied in (2.7.1) that $\mathcal{T}_{\mathcal{CLB}}(E^{**}) \subset \mathcal{T}_{\| \|}(E^{**})$.

(b) ($\Rightarrow$) It follows from Lemma 3.1(b) that

$$\|x_\gamma - x\| = \|\widehat{x_\gamma} - \hat{x}\| \to \|\hat{x} - \hat{x}\| = 0,$$

i.e., $x_\gamma \to x$ in $\mathcal{T}_{\| \|}(E)$.

($\Leftarrow$) If $h \in \mathcal{CLB}(E)$ then, from (2.0.5) and the continuity of h,

$$h^{**}(\widehat{x_\gamma}) = h(x_\gamma) \to h(x) = h^{**}(\hat{x}),$$

and so $\widehat{x_\gamma} \to \hat{x}$ in $\mathcal{T}_{\mathcal{CLB}}(E^{**})$.

(c) ($\Leftarrow$) follows from (a), and ($\Rightarrow$) from the observation that if $\mathcal{T}_{\mathcal{CLB}}(E^{**}) = w(E^{**}, E^*)$ then, from (b), $\mathcal{T}_{\| \|}(E)$ is identical with the weak topology of E. It is well known that this implies that E is finite-dimensional.

(d) ($\Leftarrow$) follows from (b). We now establish ($\Rightarrow$). If $\mathcal{T}_{\mathcal{CLB}}(E^{**}) = \mathcal{T}_{\|\ \|}(E^{**})$, then, from Lemma 3.3(c), $\widehat{E}$ is dense in $(E^{**}, \mathcal{T}_{\|\ \|}(E^{**}))$. Since $\widehat{E}$ is closed in $(E^{**}, \mathcal{T}_{\|\ \|}(E^{**}))$, $\widehat{E} = E^{**}$ and so E is reflexive. □

We will use Remark 3.5 below in Section 14.

Remark 3.5. Despite all the nice properties of $\mathcal{T}_{\mathcal{CLB}}(E^{**})$ established above, it is quite a pathological topology if E is not reflexive. In this case, there exists $x^{**} \in E^{**} \setminus \widehat{E}$. From Lemma 3.3(c), there exists a net $\{x_\gamma\}$ of elements of E such that $\widehat{x_\gamma} \to x^{**}$ in $\mathcal{T}_{\mathcal{CLB}}(E^{**})$. Since $\widehat{E}$ is norm-closed in E^{**} and $x^{**} \notin \widehat{E}$, $\|\widehat{x_\gamma} - x^{**}\| \not\to 0 = \|0\|$. From Lemma 3.2, $\widehat{x_\gamma} - x^{**} \not\to 0$ in $\mathcal{T}_{\mathcal{CLB}}(E^{**})$. Thus we have shown that *if E is not reflexive then there exists a net $\{x_\gamma\}$ of elements of E and $x^{**} \in E^{**}$ such that*

$$\widehat{x_\gamma} \to x^{**} \textit{ in } \mathcal{T}_{\mathcal{CLB}}(E^{**}) \quad \textit{and} \quad \widehat{x_\gamma} - x^{**} \not\to 0 \textit{ in } \mathcal{T}_{\mathcal{CLB}}(E^{**}).$$

It follows from this observation that

if E is not reflexive then $(E^{**}, \mathcal{T}_{\mathcal{CLB}}(E^{**}))$ is not a topological vector space.

Remark 3.6. We define the topology $\mathcal{T}_{\mathcal{CC}}(E^{**})$ on E^{**} to be the coarsest topology on E^{**} making all the functions $h^{**}\colon E^{**} \mapsto \mathbb{R}\cup\{\infty\}$ $(h \in \mathcal{CC}(E))$ continuous. Then, with one significant exception, all the results proved so far in this section remain true with $\mathcal{T}_{\mathcal{CLB}}(E^{**})$ replaced by $\mathcal{T}_{\mathcal{CC}}(E^{**})$ throughout. The exception is the inclusion $\mathcal{T}_{\mathcal{CLB}}(E^{**}) \subset \mathcal{T}_{\|\ \|}(E^{**})$ in Theorem 3.4(a). Let $f \in \mathcal{CC}(c_0)$ be as in Example 2.8. Then, from the definition of $\mathcal{T}_{\mathcal{CC}}(\ell^\infty)$, f^{**} is $\mathcal{T}_{\mathcal{CC}}(\ell^\infty)$-continuous. Since f^{**} is not $\mathcal{T}_{\|\ \|}(\ell^\infty)$-continuous, it follows that $\mathcal{T}_{\mathcal{CC}}(\ell^\infty) \not\subset \mathcal{T}_{\|\ \|}(\ell^\infty)$.

The final result of this section will be used in our main result on subdifferentials, Theorem 12.2, and also in Lemma 14.3.

LEMMA 3.7. *Let $\{t_\gamma^{**}\}$ be a net of elements of E^{**}, $t^{**} \in E^{**}$ and $t_\gamma^{**} \to t^{**}$ in $\mathcal{T}_{\mathcal{CLB}}(E^{**})$. Let $\{s_\gamma^{**}\}$ be a net of elements of E^{**} and $\|s_\gamma^{**} - t_\gamma^{**}\| \to 0$. Then $s_\gamma^{**} \to t^{**}$ in $\mathcal{T}_{\mathcal{CLB}}(E^{**})$.*

Proof. Let $B := \{x^{**} \in E^{**} : \|x^{**}\| \leqslant \|t^{**}\| + 2\}$. It follows from Lemma 3.2 that eventually $\|t_\gamma^{**}\| \leqslant \|t^{**}\| + 1$. Also, eventually $\|s_\gamma^{**} - t_\gamma^{**}\| \leqslant 1$. Thus, eventually both t_γ^{**} and s_γ^{**} are in B. Now let $h \in \mathcal{CLB}(E)$. Since h^{**} is Lipschitz on B and $\|s_\gamma^{**} - t_\gamma^{**}\| \to 0$, $|h^{**}(s_\gamma^{**}) - h^{**}(t_\gamma^{**})| \to 0$. Further, since $t_\gamma^{**} \to t^{**}$ in $\mathcal{T}_{\mathcal{CLB}}(E^{**})$, $h^{**}(t_\gamma^{**}) \to h^{**}(t^{**})$. We now obtain by addition that $h^{**}(s_\gamma^{**}) \to h^{**}(t^{**})$. This gives the required result. □

Lemma 3.7 suggests the following problem:

PROBLEM 3.8. *Let $\{t_\gamma^{**}\}$ be a net of elements of E^{**}, $t^{**} \in E^{**}$ and $t_\gamma^{**} \to t^{**}$ in $\mathcal{T}_{\mathcal{CC}}(E^{**})$. Let $\{s_\gamma^{**}\}$ be a net of elements of E^{**} and $\|s_\gamma^{**} - t_\gamma^{**}\| \to 0$. Does it necessarily follow that $s_\gamma^{**} \to t^{**}$ in $\mathcal{T}_{\mathcal{CC}}(E^{**})$?*

In fact, for the later results in this paper it is enough to consider (instead of $\mathcal{T}_{\mathcal{CLB}}(E^{**})$) the topology $\mathcal{T}_{\mathcal{UB}}(E^{**})$ on E^{**}, which we now define: $\mathcal{T}_{\mathcal{UB}}(E^{**})$ is the least upper bound of $w(E^{**}, E^*)$ and the coarsest topology on E^{**} making all the functions $\| \cdot -\hat{x}\|$ ($x \in E$) continuous. Clearly $\mathcal{T}_{\mathcal{UB}}(E^{**}) \subset \mathcal{T}_{\mathcal{CLB}}(E^{**})$. Do we have equality? In other words, we have the following problem:

PROBLEM 3.9. *Let $\{x_\gamma^{**}\}$ be a net of elements of E^{**}, $x^{**} \in E^{**}$, $x_\gamma^{**} \to x^{**}$ in $w(E^{**}, E^*)$ and, for all $x \in E$, $\|x_\gamma^{**} - \hat{x}\| \to \|x^{**} - \hat{x}\|$. Does it necessarily follow that, for all $f \in \mathcal{CLB}(E)$, $f^{**}(x_\gamma^{**}) \to f^{**}(x^{**})$?*

4. Maximal Monotone Multifunctions of Type (ED)

If $S\colon E \mapsto 2^{E^*}$ is a multifunction, we write

$$G(S) := \{(x, x^*)\colon x \in E,\ x^* \in Sx\}.$$

$G(S)$ is the *graph* of S. We say that S is *nontrivial* if $G(S) \neq \emptyset$. A nontrivial multifunction $S\colon E \mapsto 2^{E^*}$ is *monotone* if

$$(x, x^*) \text{ and } (y, y^*) \in G(S) \Rightarrow \langle x - y, x^* - y^*\rangle \geqslant 0.$$

S is *maximal monotone* if S is monotone, and S has no proper monotone extension.

In this section, we introduce a new subclass of the class of maximal monotone multifunctions, those that are *of type* (*ED*), and explain the relations between this subclass and two older subclasses, those that are *of dense type* and those that are *of type* (*D*).

Maximal monotone multifunctions of dense type were introduced by Gossez in [10, Lemme 2.1, p. 375] in order to generalize to nonreflexive spaces some of the results previously known for reflexive spaces. However, maximal monotone multifunctions of dense type seem to lack a fundamental stability property (which we will use in the proof in Theorem 8.6 that maximal monotone multifunctions of type (ED) always possess a certain inequality-splitting property) – see the remark preceding Theorem 4.4 below. Maximal monotone multifunctions of type (D), a larger class of multifunctions, were also introduced by Gossez – see Phelps, [14, Section 3] for an exposition. These have many of the properties of maximal monotone multifunctions of dense type, and also the stability property corresponding to Theorem 4.4. However, they do not seem to have all the properties required for the kind of the analysis we are performing in this paper: roughly speaking, the problem is that it does not follow from $x_\gamma^{**} \to x^{**}$ in $w(E^{**}, E^*)$ that $\|x_\gamma^{**}\| \to \|x^{**}\|$ – compare with Lemma 3.2, which will be used in the proof of Lemma 8.5. (Lemma 8.5 is the “normalized” version of Theorem 8.6, which we have already mentioned above.) The maximal monotone multifunctions of type (ED), a subclass of the multifunctions of dense type, seem to have all the properties that we require. (“ED” stands for “extra dense”. There were some preliminary results on maximal monotone multifunctions of type (ED) in [24], where they were called maximal monotone multifunctions of type (DS).)

We must now introduce another concept due to Gossez: if $S\colon E \mapsto 2^{E^*}$, we define the multifunction $\overline{S}\colon E^{**} \mapsto 2^{E^*}$ by:

$$x^* \in \overline{S}x^{**} \quad \Leftrightarrow \quad \inf_{(s,s^*)\in G(S)} \langle s^* - x^*, \hat{s} - x^{**}\rangle \geqslant 0.$$

DEFINITION 4.1. $S\colon E \mapsto 2^{E^*}$ is *maximal monotone of type* (*ED*) if S is maximal monotone and, for all $(x^{**}, x^*) \in G(\overline{S})$, there exists a net $\{(s_\gamma, s_\gamma^*)\}$ of elements of $G(S)$ such that $(\widehat{s_\gamma}, s_\gamma^*) \to (x^{**}, x^*)$ in $\mathcal{T}_{\mathcal{CLB}}(E^{**}) \times \mathcal{T}_{\|\ \|}(E^*)$.

For comparison purposes, we give the definition of *type* (*D*).

DEFINITION 4.2. $S\colon E \mapsto 2^{E^*}$ is *maximal monotone of type* (*D*) if S is maximal monotone and, for all $(x^{**}, x^*) \in G(\overline{S})$, there exists a bounded net $\{(s_\gamma, s_\gamma^*)\}$ of elements of $G(S)$ such that $(\widehat{s_\gamma}, s_\gamma^*) \to (x^{**}, x^*)$ in $w(E^{**}, E^*) \times \mathcal{T}_{\|\ \|}(E^*)$.

In view of Lemma 4.5, Section 10 and Theorems 11.1, 12.6(b) and 15.2, the following problem might be quite difficult.

PROBLEM 4.3. *Is every maximal monotone multifunction of type* (*D*) *of type* (*ED*)?

Theorem 4.4(b) contains a stability property of maximal monotone multifunctions of type (ED) that will be important for us in Theorem 8.6. The analogous property presumably does not hold for multifunctions of dense type, though since we do not know of a multifunction of dense type that is not of type (ED), we do not have an example of the failure of the corresponding property for multifunctions of dense type!

THEOREM 4.4. *Let* $T\colon E \mapsto 2^{E^*}$ *be nontrivial,* $(w, w^*) \in E \times E^*$, $\alpha, \beta > 0$ *and* $S\colon E \mapsto 2^{E^*}$ *be defined by*

$$G(S) := \left\{\left(\frac{t-w}{\alpha}, \frac{t^*-w^*}{\beta}\right) : (t, t^*) \in G(T)\right\}.$$

Then:

(a) $(x^{**}, x^*) \in G(\overline{S}) \Leftrightarrow (\alpha x^{**} + \widehat{w}, \beta x^* + w^*) \in G(\overline{T})$.
(b) *If* T *is maximal monotone of type* (*ED*) *then so is* S.

Proof. (a) is immediate from the definitions of $\overline{S}$ and $\overline{T}$, and it is also immediate that S is maximal monotone in (b). It remains to prove that S is of type (ED). To this end, let $(x^{**}, x^*) \in G(\overline{S})$. Since T is of type (ED), we derive from (a) that there exists a net (t_γ, t_γ^*) of elements of $G(T)$ such that

$$(\widehat{t_\gamma}, t_\gamma^*) \to (\alpha x^{**} + \widehat{w}, \beta x^* + w^*) \quad \text{in } \mathcal{T}_{\mathcal{CLB}}(E^{**}) \times \mathcal{T}_{\|\ \|}(E^*).$$

It now follows from Lemma 3.1(c,d) and the standard properties of $\mathcal{T}_{\|\ \|}(E^*)$ that

$$\left(\frac{\widehat{t_\gamma} - \widehat{w}}{\alpha}, \frac{t_\gamma^* - w^*}{\beta}\right) \to (x^{**}, x^*) \quad \text{in } \mathcal{T}_{\mathcal{CLB}}(E^{**}) \times \mathcal{T}_{\|\ \|}(E^*).$$

Since, for each γ,

$$\left(\frac{t_\gamma - w}{\alpha}, \frac{t_\gamma^* - w^*}{\beta}\right) \in G(S),$$

it follows from the linearity of the canonical map $\widehat{\ }$ that S is of type (ED). □

We close this section with a simple result – we leave the proof to the reader.

LEMMA 4.5. *If E is reflexive then every maximal monotone multifunction S: $E \mapsto 2^{E^*}$ is of type* (*ED*).

5. Functional Analytic Tools

In this section, we give two results that we shall use later on. The main result is Theorem 5.2, which has other applications to the theory of multifunctions (see [24, Lemma 20.1, p. 77] and [25, Lemma 16]). Theorem 5.2 is an existence theorem for bounded linear functionals that does not assume the existence of any *a priori* bounds.

LEMMA 5.1. *Let $x \in E$ and $x^* \in E^*$.*

(a) *Then*

$$\|x\|^2 + \|x^*\|^2 + 2\langle x, x^*\rangle \geqslant 0. \tag{5.1.1}$$

(b) *If we have equality in* (5.1.1) *then $\|x\| = \|x^*\|$, and so $\langle x, x^*\rangle = -\|x\|\|x^*\|$.*

Proof. $\|x\|^2 + \|x^*\|^2 + 2\langle x, x^*\rangle \geqslant \|x\|^2 + \|x^*\|^2 - 2\|x\|\|x^*\| = (\|x\| - \|x^*\|)^2$. □

We give the proof of Theorem 5.2 below for completeness – it can also be found in [24, Theorem 7.2, p. 27] or [25, Theorem 3].

THEOREM 5.2. *Let A be a nonempty convex subset of a vector space, F be a Banach space, f: $A \mapsto \mathbb{R}$ be convex and g: $A \mapsto F$ be affine. Then* (5.2.1) $\Leftrightarrow$ (5.2.2).

$$a \in A \Rightarrow f(a) + \|g(a)\|^2 \geqslant 0. \tag{5.2.1}$$

There exists $y^* \in F^*$ such that (5.2.2)

$$a \in A \Rightarrow f(a) - 2\langle g(a), y^*\rangle \geqslant \|y^*\|^2.$$

Proof. (⇒) Let $a_0 \in A$, and n be an integer such that $n \geqslant \|g(a_0)\|$. Put

$$A_n := \{a \in A\colon \ \|g(a)\| \leqslant n\}.$$

Since $a_0 \in A_n$, A_n is not empty, and A_n is clearly convex. Define $h\colon\ A \times [0, \infty) \mapsto \mathbb{R}$ by

$$h(a, \beta) := f(a) + 2\beta\|g(a)\| - \beta^2.$$

Using (5.2.1), we have

$$\inf_{A_n} \max_{[0,n]} h \geqslant \inf_{a \in A_n} h(a, \|g(a)\|) \geqslant 0.$$

The function h is convex on A_n, and concave and continuous on $[0, n]$. Since $[0, n]$ is compact, from Theorem 2.1,

$$\max_{[0,n]} \inf_{A_n} h \geqslant 0,$$

from which

$$C_n := \bigcap_{a \in A_n} \{\beta \in [0, \infty)\colon\ h(a, \beta) \geqslant 0\} \neq \emptyset.$$

C_n is clearly compact. Further, the sets C_n decrease as n increases. Consequently,

$$\bigcap_{n \geqslant \|g(a_0)\|} C_n \neq \emptyset.$$

Since

$$A = \bigcup_{n \geqslant \|g(a_0)\|} A_n,$$

it now follows that

$$\text{there exists } \beta \geqslant 0 \text{ such that} \quad a \in A \Rightarrow f(a) + 2\beta\|g(a)\| - \beta^2 \geqslant 0. \tag{5.2.3}$$

Now define $h\colon\ A \times F^* \mapsto \mathbb{R}$ by

$$h(a, y^*) := f(a) - 2\langle g(a), y^*\rangle - \|y^*\|^2.$$

Let $B := \{y^* \in F^*\colon\ \|y^*\| \leqslant \beta\}$, with the topology $w(F^*, F)$. From the Banach–Alaoglu theorem, B is compact. Let $a \in A$. Using the Hahn–Banach theorem, we can find $y^* \in B$ such that $\langle g(a), y^*\rangle = -\beta\|g(a)\|$. From (5.2.3),

$$h(a, y^*) = f(a) - 2\langle g(a), y^*\rangle - \|y^*\|^2 \geqslant f(a) + 2\beta\|g(a)\| - \beta^2 \geqslant 0.$$

Thus we have proved that

$$\inf_A \max_B h \geqslant 0.$$

Since A and B are convex, and h is convex on A and concave and upper semicontinuous on B, from Theorem 2.1 again,

$$\max_B \inf_A h \geqslant 0,$$

which gives (5.2.2).

($\Leftarrow$) From Lemma 5.1, for all $a \in A$ and $y^* \in F^*$,

$$\|g(a)\|^2 \geqslant -2\langle g(a), y^*\rangle - \|y^*\|^2. \qquad \square$$

6. The Big Convexification of a Multifunction

In this section, we define the “big convexification” of a multifunction, and then apply the results of Section 5 to obtain in Theorem 6.3 a fundamental existence result for monotone multifunctions. One of the intermediate results, Lemma 6.2, is an equivalence valid for *any* nontrivial multifunction from E into 2^{E^*}.

We write $\mathbb{R}^{(E\times E^*)}$ for the direct sum of $E \times E^*$ copies of $\mathbb{R}$, that is the set of functions μ: $E \times E^* \mapsto \mathbb{R}$ such that

$$\{(s, s^*) \in E \times E^* : \mu(s, s^*) \neq 0\} \quad \text{is finite.}$$

$\mathbb{R}^{(E\times E^*)}$ is a vector space. We define the three linear operators p: $\mathbb{R}^{(E\times E^*)} \mapsto E$, q: $\mathbb{R}^{(E\times E^*)} \mapsto E^*$ and r: $\mathbb{R}^{(E\times E^*)} \mapsto \mathbb{R}$ by

$$p(\mu) := \sum_{(s,s^*)\in E\times E^*} \mu(s, s^*)s,$$
$$q(\mu) := \sum_{(s,s^*)\in E\times E^*} \mu(s, s^*)s^*$$

and

$$r(\mu) := \sum_{(s,s^*)\in E\times E^*} \mu(s, s^*)\langle s, s^*\rangle.$$

If $(y, y^*) \in E \times E^*$ then $\delta_{(y,y^*)} \in \mathbb{R}^{(E\times E^*)}$, where $\delta_{(y,y^*)}$ is defined by

$$\delta_{(y,y^*)}(s, s^*) := \begin{cases} 1, & \text{if } (s, s^*) = (y, y^*); \\ 0, & \text{otherwise.} \end{cases}$$

If S: $E \mapsto 2^{E^*}$ is a nontrivial multifunction, then we define the *big convexification, $\mathcal{CO}(S)$* of S to be the convex hull in $\mathbb{R}^{(E\times E^*)}$ of $\{\delta_{(y,y^*)} : (y, y^*) \in G(S)\}$. Explicitly, if $\mu \in \mathbb{R}^{(E\times E^*)}$ then $\mu \in \mathcal{CO}(S)$ if, and only if

$$\mu \geqslant 0 \text{ on } E \times E^*, \quad \mu(s, s^*) > 0 \Rightarrow (s, s^*) \in G(S) \quad \text{and}$$
$$\sum_{(s,s^*)\in G(S)} \mu(s, s^*) = 1.$$

This concept dates back to the paper [7] by Coodey and Simons, though the notation there was somewhat different.

We observe that,

for all $(y, y^*) \in E \times E^*$, $\quad p(\delta_{(y,y^*)}) = y$, $\quad q(\delta_{(y,y^*)}) = y^*$
and $r(\delta_{(y,y^*)}) = \langle y, y^* \rangle$.

The manipulations contained in the proof of (6.1.1) below are part of the folklore of monotonicity.

LEMMA 6.1. *Let* $S\colon E \mapsto 2^{E^*}$ *be a nontrivial monotone multifunction. Then*

$$\mu \in \mathcal{CO}(S) \Rightarrow 2r(\mu) + \|p(\mu)\|^2 + \|q(\mu)\|^2 \geqslant 0.$$

Proof. We first prove that

$$\mu \in \mathcal{CO}(S) \Rightarrow r(\mu) \geqslant \langle p(\mu), q(\mu) \rangle. \tag{6.1.1}$$

To this end, let $(s_1, s_1^*), \ldots, (s_m, s_m^*)$ be an enumeration of those elements (s, s^*) of $G(S)$ for which $\mu(s, s^*) > 0$, and write α_i for $\mu(s_i, s_i^*)$. Then, with the summations going from 1 to m,

$$\begin{aligned}
r(\mu) - \langle p(\mu), q(\mu) \rangle &= \sum_i \alpha_i \langle s_i, s_i^* \rangle - \left\langle \sum_i \alpha_i s_i, \sum_i \alpha_i s_i^* \right\rangle \\
&= \sum_{i,j} \alpha_i \alpha_j \langle s_i, s_i^* \rangle - \sum_{i,j} \alpha_i \alpha_j \langle s_i, s_j^* \rangle \\
&= \sum_{i,j} \alpha_i \alpha_j \langle s_i, s_i^* - s_j^* \rangle \\
&= \sum_{i<j} \alpha_i \alpha_j \langle s_i, s_i^* - s_j^* \rangle + \sum_{j<i} \alpha_i \alpha_j \langle s_i, s_i^* - s_j^* \rangle \\
&= \sum_{i<j} \alpha_i \alpha_j \langle s_i, s_i^* - s_j^* \rangle + \sum_{i<j} \alpha_i \alpha_j \langle s_j, s_j^* - s_i^* \rangle \\
&= \sum_{i<j} \alpha_i \alpha_j \langle s_i - s_j, s_i^* - s_j^* \rangle \geqslant 0.
\end{aligned}$$

This gives (6.1.1). Using (6.1.1) and Lemma 5.1(a), for all $\mu \in \mathcal{CO}(S)$,

$$2r(\mu) + \|p(\mu)\|^2 + \|q(\mu)\|^2 \geqslant \|p(\mu)\|^2 + \|q(\mu)\|^2 + 2\langle p(\mu), q(\mu) \rangle \geqslant 0. \ \square$$

We now use Theorem 5.2 to prove an equivalence for *arbitrary* nontrivial multifunctions $S\colon E \mapsto 2^{E^*}$.

LEMMA 6.2. *Let* $S\colon E \mapsto 2^{E^*}$ *be nontrivial. Then the conditions* (6.2.1) *and* (6.2.2) *are equivalent*:

$$\mu \in \mathcal{CO}(S) \Rightarrow 2r(\mu) + \|p(\mu)\|^2 + \|q(\mu)\|^2 \geqslant 0. \tag{6.2.1}$$

$$\begin{aligned}&\textit{There exists } (x^*, x^{**}) \in E^* \times E^{**} \textit{ such that } (s, s^*) \in G(S)\\ &\quad \Rightarrow 2\langle s^* - x^*, \hat{s} - x^{**}\rangle \geqslant \|x^*\|^2 + \|x^{**}\|^2 + 2\langle x^*, x^{**}\rangle.\end{aligned} \tag{6.2.2}$$

Proof. We shall establish the equivalence of (6.2.1) and (6.2.2) by proving their equivalence with the intermediate conditions (6.2.3)–(6.2.4) below:

$$\begin{aligned}&\text{There exists } (x^*, x^{**}) \in E^* \times E^{**} \text{ such that } \mu \in \mathcal{CO}(S)\\ &\quad \Rightarrow 2r(\mu) - 2\langle p(\mu), x^*\rangle - 2\langle q(\mu), x^{**}\rangle \geqslant \|x^*\|^2 + \|x^{**}\|^2.\end{aligned} \tag{6.2.3}$$

$$\begin{aligned}&\text{There exists } (x^*, x^{**}) \in E^* \times E^{**} \text{ such that } (s, s^*) \in G(S)\\ &\quad \Rightarrow \langle s, s^*\rangle - 2\langle s, x^*\rangle - 2\langle s^*, x^{**}\rangle \geqslant \|x^*\|^2 + \|x^{**}\|^2.\end{aligned} \tag{6.2.4}$$

((6.2.1) ⇔ (6.2.3)) This follows from Theorem 5.2, with $F := E \times E^*$ normed by $\|(x, x^*)\| := \sqrt{\|x\|^2 + \|x^*\|^2}$, $A := \mathcal{CO}(S)$ and, for all $\mu \in A$,

$$f(\mu) := 2r(\mu) \quad \text{and} \quad g(\mu) := \big(p(\mu), q(\mu)\big).$$

Then any element y^* of F^* can be written in the form (x^*, x^{**}) for some $(x^*, x^{**}) \in E^* \times E^{**}$, and $\|y^*\| = \sqrt{\|x^*\|^2 + \|x^{**}\|^2}$.

((6.2.3) ⇔ (6.2.4)) If (6.2.3) is satisfied then (6.2.4) follows by restricting μ to the values $\delta_{(s,s^*)}$. If, conversely, (6.2.4) is satisfied and $\mu \in \mathcal{CO}(S)$ then (6.2.3) follows by multiplying the left hand side of the inequality in (6.2.4) by $\mu(s, s^*)$ and summing up over all $(s, s^*) \in G(S)$.

((6.2.4) ⇔ (6.2.2)) This can be seen by rearranging the terms and adding $\pm 2\langle x, x^*\rangle$ to each side.

This completes the proof of Lemma 6.2. □

If we combine Lemmas 6.1 and 6.2, we obtain in Theorem 6.3 a fundamental property of monotone multifunctions. There is a (much more complicated) proof of Theorem 6.3 in [22, Lemma 9, p. 183].

THEOREM 6.3. *Let S: $E \mapsto 2^{E^*}$ be nontrivial and monotone. Then there exists $(x^*, x^{**}) \in E^* \times E^{**}$ such that*

$$2 \inf_{(s,s^*) \in G(S)} \langle s^* - x^*, \hat{s} - x^{**}\rangle \geqslant \|x^*\|^2 + \|x^{**}\|^2 + 2\langle x^*, x^{**}\rangle.$$

7. Maximal Monotone Multifunctions of Type (NI)

In this section we discuss maximal monotone multifunctions of type (NI). These were introduced in [22, Definition 10, p. 183]. Their introduction was motivated by some questions about the range of maximal monotone operators in nonreflexive spaces. “NI” stands for “negative infimum”.

DEFINITION 7.1. Let S: $E \mapsto 2^{E^*}$ be maximal monotone. We say that S is *of type (NI)* if

$$(x^{**}, x^*) \in E^{**} \times E^* \Rightarrow \inf_{(s,s^*) \in G(S)} \langle s^* - x^*, \hat{s} - x^{**}\rangle \leqslant 0. \tag{7.1.1}$$

Theorem 7.2 summarizes the relations between the various classes of multifunctions, as well as giving a fundamental property of multifunctions of type (NI). Theorem 7.2(c) can be stated in terms of a *duality mapping*, but it is not necessary to introduce this additional complexity. (See [22, Theorem 12(a), p. 184.])

THEOREM 7.2. *Let* $S\colon E \mapsto 2^{E^*}$. *Then*:

(a) *If* S *is maximal monotone of type* (*ED*) *then* S *is maximal monotone of type* (*D*).
(b) *If* S *is maximal monotone of type* (*D*) *then* S *is maximal monotone of type* (*NI*).
(c) *Let* $S\colon E \mapsto 2^{E^*}$ *be maximal monotone of type* (*NI*). *Then there exists* $(x^{**}, x^*) \in G(\overline{S})$ *such that* $\|x^*\| = \|x^{**}\|$ *and* $\langle x^*, x^{**}\rangle = -\|x^*\|\|x^{**}\|$.

Proof. (a) follows from Lemma 3.2.
(b) is straightforward (see [22, Lemma 15, pp. 187–188]).
(c) It follows from Theorem 6.3 that there exists $(x^*, x^{**}) \in E^* \times E^{**}$ such that

$$2 \inf_{(s,s^*)\in G(S)} \langle s^* - x^*, \hat{s} - x^{**}\rangle \geqslant \|x^*\|^2 + \|x^{**}\|^2 + 2\langle x^*, x^{**}\rangle. \tag{7.2.1}$$

From Lemma 5.1(a),

$$2 \inf_{(s,s^*)\in G(S)} \langle s^* - x^*, \hat{s} - x^{**}\rangle \geqslant 0,$$

that is to say, $(x^{**}, x^*) \in G(\overline{S})$. Since S is of type (NI), it follows by taking the infimum over $(s, s^*) \in G(S)$ in (7.2.1) and using Lemma 5.1(a) again that

$$\|x^*\|^2 + \|x^{**}\|^2 + 2\langle x^*, x^{**}\rangle = 0.$$

We now obtain (c) from Lemma 5.1(b). □

8. Negative Alignment Pairs for Multifunctions

We start this section by giving some background. The following result was proved by Torralba in [26, Proposition 6.17]:

THEOREM 8.1. *Let* E *be a reflexive Banach space,* $T\colon E \mapsto 2^{E^*}$ *be maximal monotone,* $(w, w^*) \in E \times E^*$, $\alpha,\ \beta > 0$ *and*

$$\inf_{(t,t^*)\in G(T)} \langle t - w, t^* - w^*\rangle \geqslant -\alpha\beta.$$

Then there exists $(t, t^*) \in G(T)$ *such that* $\|t - w\| \leqslant \alpha$ *and* $\|t^* - w^*\| \leqslant \beta$.

This inequality-splitting result was motivated by the Brøndsted–Rockafellar theorem for subdifferentials (see Theorem 12.1 below). Theorem 8.1 does not generally extend to the nonreflexive case – see Example 11.5 below. We will, in fact, give a generalization of Theorem 8.1 to the nonreflexive case in Theorem 10.2 below, however it holds for an extremely restricted subclass of the maximal monotone multifunctions. So it makes sense to ask if, by modifying the question slightly, we can obtain a useful inequality-splitting result that is true for a significant subclass of the maximal monotone multifunctions in the nonreflexive case. The first such result was established by Revalski and Théra in [16, Theorem 2.8]. They proved the following:

THEOREM 8.2. *Let* $T\colon E \mapsto 2^{E^*}$ *be maximal monotone of type* (*D*), $(w, w^*) \in E \times E^*$, $\alpha,\ \beta > 0$ *and*

$$\inf_{(t,t^*)\in G(T)} \langle t - w, t^* - w^*\rangle \geqslant -\alpha\beta.$$

Then there exists $(t^{**}, t^*) \in G(\overline{T})$ *such that* $\|t^{**} - \widehat{w}\| \leqslant \alpha$ *and* $\|t^* - w^*\| \leqslant \beta$.

The main result of this section is Theorem 8.6, in which we show that if we consider multifunctions of type (ED) rather than of type (D) and change the inequalities from "$\geqslant$" and "$\leqslant$" to "$>$" and "$<$", respectively, then there is a result analogous to Theorem 8.2 in which the approximation to (w, w^*) can be taken in $G(T)$ rather than in $G(\overline{T})$. Further, if $(w, w^*) \notin G(T)$ then we can control both the ratios

$$\frac{\|t - w\|}{\|t^* - w^*\|} \quad \text{and} \quad \frac{\langle t - w, t^* - w^*\rangle}{\|t - w\|\|t^* - w^*\|}.$$

This control is best explained using the concept of *negative alignment pair*, which we now describe.

DEFINITION 8.3. Let $T\colon E \mapsto 2^{E^*}$ and $\rho,\ \sigma \geqslant 0$. We say that (ρ, σ) is a *negative alignment pair* for T with respect to (w, w^*) if there exists a sequence $\{(t_m, t_m^*)\}_{m\geqslant 1}$ of elements of $G(T)$ such that

$$\lim_{m\to\infty} \|t_m - w\| = \rho, \qquad \lim_{m\to\infty} \|t_m^* - w^*\| = \sigma$$

and

$$\lim_{m\to\infty} \langle t_m - w, t_m^* - w^*\rangle = -\rho\sigma.$$

Theorem 8.4(a) contains an "antimonotone" property of negative alignment pairs, and Theorem 8.4(b) contains a uniqueness theorem for negative alignment pairs – both for the case when T is monotone.

THEOREM 8.4. *Let* $T\colon E \mapsto 2^{E^*}$ *be monotone and* $(w, w^*) \in E \times E^*$.

(a) *Let* (ρ, σ) *and* $(\widetilde{\rho}, \widetilde{\sigma})$ *be negative alignment pairs for* T *with respect to* (w, w^*). *Then*

$$(\rho - \widetilde{\rho})(\sigma - \widetilde{\sigma}) \leqslant 0.$$

(b) *Suppose now that* $\alpha,\ \beta > 0$. *Then there exists at most one value of* $\tau \geqslant 0$ *such that* $(\tau\alpha, \tau\beta)$ *is a negative alignment pair for* T *with respect to* (w, w^*).

Proof. (a) Let $\{(t_m, t_m^*)\}_{m\geqslant 1}$ and $\{((\widetilde{t}_n, \widetilde{t}_n^*)\}_{n\geqslant 1}$ be sequences of elements of $G(T)$ such that

$$\lim_{m\to\infty} \|t_m - w\| = \rho, \quad \lim_{m\to\infty} \|t_m^* - w^*\| = \sigma,$$
$$\lim_{m\to\infty} \langle t_m - w, t_m^* - w^*\rangle = -\rho\sigma,$$

$$\lim_{n\to\infty} \|\widetilde{t}_n - w\| = \widetilde{\rho}, \quad \lim_{n\to\infty} \|\widetilde{t}_n^* - w^*\| = \widetilde{\sigma} \quad \text{and}$$
$$\lim_{n\to\infty} \langle \widetilde{t}_n - w, \widetilde{t}_n^* - w^*\rangle = -\widetilde{\rho}\widetilde{\sigma}.$$

Then, since T is monotone, for all $m,\ n \geqslant 1$,

$$\begin{aligned} 0 &\leqslant \langle t_m - \widetilde{t}_n, t_m^* - \widetilde{t}_n^*\rangle \\ &= \langle t_m - w, t_m^* - w^*\rangle - \langle t_m - w, \widetilde{t}_n^* - w^*\rangle - \\ &\quad - \langle \widetilde{t}_n - w, t_m^* - w^*\rangle + \langle \widetilde{t}_n - w, \widetilde{t}_n^* - w^*\rangle \\ &\leqslant \langle t_m - w, t_m^* - w^*\rangle + \|t_m - w\|\|\widetilde{t}_n^* - w^*\| + \\ &\quad + \|\widetilde{t}_n - w\|\|t_m^* - w^*\| + \langle \widetilde{t}_n - w, \widetilde{t}_n^* - w^*\rangle. \end{aligned}$$

Letting $m \to \infty$,

$$0 \leqslant -\rho\sigma + \rho\|\widetilde{t}_n^* - w^*\| + \|\widetilde{t}_n - w\|\sigma + \langle \widetilde{t}_n - w, \widetilde{t}_n^* - w^*\rangle,$$

and then, letting $n \to \infty$,

$$0 \leqslant -\rho\sigma + \rho\widetilde{\sigma} + \widetilde{\rho}\sigma - \widetilde{\rho}\widetilde{\sigma}.$$

This completes the proof of (a).

(b) Suppose that $\tau,\ \widetilde{\tau} \geqslant 0$ and $(\tau\alpha, \tau\beta)$ and $(\widetilde{\tau}\alpha, \widetilde{\tau}\beta)$ are negative alignment pairs for T with respect to (w, w^*). We have from (a) that

$$(\tau\alpha - \widetilde{\tau}\alpha)(\tau\beta - \widetilde{\tau}\beta) \leqslant 0.$$

It follows easily from this that $\tau = \widetilde{\tau}$, which gives (b). □

We now give an existence theorem for negative alignment pairs for maximal monotone multifunctions of type (ED). Its proof was suggested by that of Revalski and Théra, [16, Proposition 2.4]. Lemma 8.5 is a simplified version of our main result, Theorem 8.6.

LEMMA 8.5. *Let* S: $E \mapsto 2^{E^*}$ *be maximal monotone of type* (*ED*). *Then*:

(a) *There exists a unique value of* $\sigma \geqslant 0$ *such that* (σ, σ) *is a negative alignment pair for* S *with respect to* $(0, 0)$.

(b) *If* $(0, 0) \notin G(S)$ *then* $\sigma > 0$.
(c) *If* $\inf_{(s,s^*)\in G(S)}\langle s, s^*\rangle > -1$ *then* $\sigma < 1$.

Proof. (a) From Theorem 7.2, there exists $(x^{**}, x^*) \in G(\overline{S})$ such that $\|x^*\| = \|x^{**}\|$ and $\|x^*\|\|x^{**}\| = -\langle x^*, x^{**}\rangle$. Now let $\sigma := \|x^*\| = \|x^{**}\|$. Then we have $\sigma \geqslant 0$ and $\langle x^*, x^{**}\rangle = -\sigma^2$. Since S is of type (ED), there exists a net (s_γ, s_γ^*) of elements of $G(S)$ such that $(\widehat{s_\gamma}, s_\gamma^*) \to (x^{**}, x^*)$ in $\mathcal{T}_{\mathcal{CLB}}(E^{**}) \times \mathcal{T}_{\|\ \|}(E^*)$. Then

$$\|s_\gamma^*\| \to \|x^*\| = \sigma,$$

from Lemma 3.2,

$$\|s_\gamma\| = \|\widehat{s_\gamma}\| \to \|x^{**}\| = \sigma$$

and, from Lemma 3.1(e),

$$\langle s_\gamma, s_\gamma^*\rangle = \langle s_\gamma^*, \widehat{s_\gamma}\rangle \to \langle x^*, x^{**}\rangle = -\sigma^2. \tag{8.5.1}$$

It is now easy to see that (σ, σ) is a negative alignment pair for S with respect to $(0, 0)$, and the "uniqueness" is immediate from Theorem 8.4(b).

(b) If $(0, 0) \notin G(S)$ then, from the maximal monotonicity of S, $(0, 0) \notin G(\overline{S})$ and so $(x^*, x^{**}) \neq (0, 0)$. It follows that $\sigma = \|x^*\| = \|x^{**}\| > 0$.

(c) We follow the argument of (a) up to (8.5.1). The additional hypothesis gives that, $\inf_\gamma\langle s_\gamma, s_\gamma^*\rangle > -1$. Passing to the limit, $-\sigma^2 > -1$. Hence, $\sigma < 1$, as required. □

We now bootstrap Lemma 8.5 to obtain our main result on the existence of negative alignment pairs, and give some simple consequences. We will show in Remark 11.4 that the conclusion of Theorem 8.6(c) may, indeed, be true even if T is *not* of type (ED), and in Example 11.5 that if T is not of type (ED) then the conclusion of Theorem 8.6(c) may fail. Both these examples are single-valued, continuous and linear.

THEOREM 8.6. *Let* $T\colon E \mapsto 2^{E^*}$ *be maximal monotone of type* (*ED*), $(w, w^*) \in E \times E^*$ *and* $\alpha,\ \beta > 0$. *Then*:

(a) *There exists a unique value of* $\tau \geqslant 0$ *such that* $(\tau\alpha, \tau\beta)$ *is a negative alignment pair for* T *with respect to* (w, w^*).
(b) *If* $(w, w^*) \notin G(T)$ *then* $\tau > 0$, *and there exists* $(t, t^*) \in G(T)$ *such that* $t \neq w$, $t^* \neq w^*$,

$$\frac{\|t - w\|}{\|t^* - w^*\|} \text{ is as near as we please to } \frac{\alpha}{\beta}$$

and

$$\frac{\langle t - w, t^* - w^*\rangle}{\|t - w\|\|t^* - w^*\|} \text{ is as near as we please to } -1.$$

(c) *If, further,* $\inf_{(t,t^*)\in G(T)}\langle t - w, t^* - w^*\rangle > -\alpha\beta$ *then* $\tau < 1$, *and we can take* (t, t^*) *so that, in addition,* $\|t - w\| < \alpha$ *and* $\|t^* - w^*\| < \beta$.

Proof. We define S as in Theorem 4.4(a). From Theorem 4.4(b), S is maximal monotone of type (ED). The result now follows from Lemma 8.5. $\square$

Now suppose that $T\colon E \mapsto 2^{E^*}$ is maximal monotone of type (ED), and $(w, w^*) \in E \times E^* \setminus G(T)$. We close this section by investigating the "negative alignment set" (of T with respect to (w, w^*)), defined by

$$\mathcal{NAS} := \{(\tau\alpha, \tau\beta)\colon \alpha > 0,\ \beta > 0,\ \tau \text{ is as in Theorem 8.6(a,b)}\}.$$

We will see in Theorem 8.8 that $\mathcal{NAS}$ is a continuous curve with certain monotonicity and maximality properties. We will need the following elementary lemma:

LEMMA 8.7. *Let* $\theta \in (0, \pi/2)$, $\varphi \in (0, \pi/2)$, $\lambda \in \mathbb{R}$ *and*

$$(\lambda \cos\theta - \cos\varphi)(\lambda \sin\theta - \sin\varphi) \leqslant 0.$$

Then

$$\frac{\cos\varphi}{\cos\theta} \wedge \frac{\sin\varphi}{\sin\theta} \leqslant \lambda \leqslant \frac{\cos\varphi}{\cos\theta} \vee \frac{\sin\varphi}{\sin\theta}.$$

Proof. This is simply a restatement of the assertion that λ lies between the zeros of the quadratic function $\mathbb{R} \mapsto \mathbb{R}$ defined by $\nu \mapsto (\nu\cos\theta - \cos\varphi)(\nu\sin\theta - \sin\varphi)$, which is true since the leading term $\cos\theta\sin\theta$ of the quadratic is strictly positive. $\square$

THEOREM 8.8. (a) *There is a continuous function* $g\colon (0, \pi/2) \mapsto (0, \infty)$ *such that*

$$\mathcal{NAS} = \{(g(\theta)\cos\theta, g(\theta)\sin\theta)\colon 0 < \theta < \pi/2\}. \tag{8.8.1}$$

(b) *The "x-projection"* $\theta \mapsto g(\theta)\cos\theta$ *is decreasing* (*i.e. nonincreasing*) *on* $(0, \pi/2)$, *and the "y-projection"* $\theta \mapsto g(\theta)\sin\theta$ *is increasing* (*i.e. nondecreasing*) *on* $(0, \pi/2)$.

(c) *If* $\gamma > 0$, $\delta > 0$ *and*

$$(\alpha, \beta) \in \mathcal{NAS} \Rightarrow (\gamma - \alpha)(\delta - \beta) \leqslant 0$$

then $(\gamma, \delta) \in \mathcal{NAS}$. *In other words,* $\mathcal{NAS}$ *is a "maximal antimonotone" subset of the first quadrant.*

Proof. First, let $\theta \in (0, \pi/2)$. Apply Theorem 8.6(b) with $(\alpha, \beta) := (\cos\theta, \sin\theta)$, and let $g(\theta) > 0$ be the value of τ obtained. We then have:

$$(g(\theta)\cos\theta, g(\theta)\sin\theta) \in \mathcal{NAS}. \tag{8.8.2}$$

On the other hand, let $(\gamma, \delta) \in \mathcal{NAS}$. Choose $\alpha > 0$, $\beta > 0$ and $\tau > 0$ as in Theorem 8.6(a,b) so that $(\tau\alpha, \tau\beta) = (\gamma, \delta)$. So we have

$$(\tau\alpha, \tau\beta) \in \mathcal{NAS}. \tag{8.8.3}$$

Choose $\rho > 0$ and $\theta \in (0, \pi/2)$ so that $(\alpha, \beta) = (\rho\cos\theta, \rho\sin\theta)$. Note then that

$$(\tau\alpha, \tau\beta) = (\tau\rho\cos\theta, \tau\rho\sin\theta). \tag{8.8.4}$$

Thus, from (8.8.3),

$$(\tau\rho\cos\theta, \tau\rho\sin\theta) \in \mathcal{NAS}. \tag{8.8.5}$$

If we now compare (8.8.2) and (8.8.5) and use the "uniqueness" part of Theorem 8.6(a), we obtain that $g(\theta) = \tau\rho$. Substituting this into (8.8.4), we have

$$(\gamma, \delta) = (\tau\alpha, \tau\beta) = (g(\theta)\cos\theta, g(\theta)\sin\theta).$$

This completes the proof of (8.8.1). It remains to show for (a) that g is continuous on $(0, \pi/2)$. Let $\theta \in (0, \pi/2)$ and $\varphi \in (0, \pi/2)$. It follows from Theorem 8.4(a) that

$$(g(\theta)\cos\theta - g(\varphi)\cos\varphi)(g(\theta)\sin\theta - g(\varphi)\sin\varphi) \leqslant 0.$$

Thus, from Lemma 8.7,

$$\frac{\cos\varphi}{\cos\theta} \wedge \frac{\sin\varphi}{\sin\theta} \leqslant \frac{g(\theta)}{g(\varphi)} \leqslant \frac{\cos\varphi}{\cos\theta} \vee \frac{\sin\varphi}{\sin\theta}, \tag{8.8.6}$$

from which it is clear by letting $\varphi \to \theta$ that g is continuous on $(0, \pi/2)$, which completes the proof of (a).

(b) Suppose now that $0 < \theta \leqslant \varphi < \pi/2$. Since $\cos\theta \geqslant \cos\varphi$ and $\sin\theta \leqslant \sin\varphi$, it follows from (8.8.6) that

$$\frac{\cos\varphi}{\cos\theta} \leqslant \frac{g(\theta)}{g(\varphi)} \leqslant \frac{\sin\varphi}{\sin\theta}.$$

Thus $g(\theta)\cos\theta \geqslant g(\varphi)\cos\varphi$ and $g(\theta)\sin\theta \leqslant g(\varphi)\sin\varphi$, which give the required results.

(c) Let $\gamma > 0$, $\delta > 0$ and $(\alpha, \beta) \in \mathcal{NAS} \Rightarrow (\gamma - \alpha)(\delta - \beta) \leqslant 0$. Choose $\rho > 0$ and $\theta \in (0, \pi/2)$ so that $(\gamma, \delta) = (\rho\cos\theta, \rho\sin\theta)$. Let $\varphi \in (0, \pi/2)$ be arbitrary. Then, since $(g(\varphi)\cos\varphi, g(\varphi)\sin\varphi) \in \mathcal{NAS}$ we have by hypothesis that

$$(\rho\cos\theta - g(\varphi)\cos\varphi)(\rho\sin\theta - g(\varphi)\sin\varphi) \leqslant 0.$$

Thus, from Lemma 8.7,

$$\frac{\cos\varphi}{\cos\theta} \wedge \frac{\sin\varphi}{\sin\theta} \leqslant \frac{\rho}{g(\varphi)} \leqslant \frac{\cos\varphi}{\cos\theta} \vee \frac{\sin\varphi}{\sin\theta}.$$

Letting $\varphi \to \theta$ and using the continuity of g, we obtain $\rho = g(\theta)$, thus

$$(\gamma, \delta) = (\rho\cos\theta, \rho\sin\theta) = (g(\theta)\cos\theta, g(\theta)\sin\theta) \in \mathcal{NAS},$$

as required. □

There are various questions that come to mind about $\mathcal{NAS}$ and g:

- Can the set $\mathcal{NAS}$ have horizontal or vertical segments?

- What can be said about the behavior of g near 0 and $\pi/2$?
- At a more general level, what functions g are possible, and what insight does the set $\mathcal{NAS}$ give about T?

9. Maximal Monotone Multifunctions of Type (ANA)

Just as we did in Section 8, we start the discussion in this section with reference to a result that was proved in the reflexive case. The following was established in [24, Corollary 10.4, p. 36] and [25, Corollary 10]:

THEOREM 9.1. *Let E be reflexive, $T\colon E \mapsto 2^{E^*}$ be a maximal monotone multifunction and $(w, w^*) \in E \times E^* \setminus G(T)$. Then there exists $(t, t^*) \in G(T)$ such that*

$$t \neq w,\ t^* \neq w^* \quad \textit{and} \quad \langle t - w, t^* - w^*\rangle = -\|t - w\|\|t^* - w^*\|.$$

It is easily seen from James's theorem that if E is not reflexive then this result fails even for T the zero multifunction, which is a continuous linear subdifferential. This suggests the following subclass of the maximal monotone multifunctions (see [24, Definition 25.10, p. 101]) – "ANA" stands for "almost negative alignment".

DEFINITION 9.2. We say that $T\colon E \mapsto 2^{E^*}$ is *maximal monotone of type* (*ANA*) if T is maximal monotone and, whenever $(w, w^*) \in E \times E^* \setminus G(T)$, there exists $(t, t^*) \in G(T)$ such that $t \neq w$, $t^* \neq w^*$ and

$$\frac{\langle t - w, t^* - w^*\rangle}{\|t - w\|\|t^* - w^*\|} \text{ is as near as we please to } -1.$$

It was proved in [23] that subdifferentials are maximal monotone of type (ANA), and in [24, Theorem 38.6, p. 150] and [3] that continuous or surjective maximal monotone linear operators are of type (ANA). The following problem is unsolved:

PROBLEM 9.3. *Find an example of a maximal monotone multifunction that is not of type* (*ANA*)*.*

We will prove in Theorem 9.4 below that there is no point in looking for an example of a maximal monotone multifunction that is not of type (ANA) among the maximal monotone multifunctions of type (ED).

THEOREM 9.4. *Let $T\colon E \mapsto 2^{E^*}$ be maximal monotone of type* (*ED*)*. Then T is of type* (*ANA*)*.*

Proof. This is immediate from Theorem 8.6(b). □

Of course, Theorem 8.6 gives much more information than Theorem 9.4 – it tells us that we can also control the values of $\|t-w\|$ and $\|t^*-w^*\|$.

Remark 9.5. As we have observed above, every continuous monotone linear (hence maximal monotone) operator is of type (ANA). On the other hand, Gossez has given in [11] an example of a continuous monotone linear operator that is not of type (D), hence not of type (ED) – see Remark 11.4 below for more details of this example. We will continue our discussion of maximal monotone linear operators in Section 11.

10. Ultramaximal Monotone Multifunctions

DEFINITION 10.1. We say that $T\colon E \mapsto 2^{E^*}$ is *ultramaximal monotone* if T is monotone and, for all $(x^{**}, x^*) \in G(\overline{T})$, there exists $(t, t^*) \in G(T)$ such that $(\hat{t}, t^*) = (x^{**}, x^*)$. An ultramaximal monotone multifunction is clearly maximal monotone of type (ED). By appropriately modifying the proofs of Lemma 8.5 and Theorem 8.6, one can prove the following result:

THEOREM 10.2. *Let* $T\colon E \mapsto 2^{E^*}$ *be ultramaximal monotone,* $(w, w^*) \in E \times E^*$ *and* $\alpha,\ \beta > 0$. *Then*:

(a) *There exists a unique value of* $\tau \geqslant 0$ *for which there exists* $(t, t^*) \in G(T)$ *such that* $\|t-w\| = \tau\alpha$, $\|t^*-w^*\| = \tau\beta$ *and* $\langle t-w, t^*-w^*\rangle = -\tau^2\alpha\beta$.

(b) *If* $(w, w^*) \notin G(T)$ *then* $\tau > 0$, *and there exists* $(t, t^*) \in G(T)$ *such that* $t \neq w$, $t^* \neq w^*$,

$$\frac{\|t-w\|}{\|t^*-w^*\|} = \frac{\alpha}{\beta} \quad \textit{and} \quad \frac{\langle t-w, t^*-w^*\rangle}{\|t-w\|\|t^*-w^*\|} = -1.$$

(c) *If, further,* $\inf_{(t,t^*)\in G(T)}\langle t-w, t^*-w^*\rangle \geqslant -\alpha\beta$ *then* $\tau \leqslant 1$, *and we can take* (t, t^*) *so that, in addition,* $\|t-w\| \leqslant \alpha$ *and* $\|t^*-w^*\| \leqslant \beta$.

If E is reflexive then every maximal monotone multifunction on E is ultramaximal, and so Theorem 10.2 generalizes Theorem 8.1. The proof of Theorem 10.2 is clearly much simpler than that of Theorem 8.6 (since it does not involve any nets), nevertheless it can be applied in a situation that occurs in the study of certain nonlinear elliptic functional equations – see Browder, [6, Theorem 1, pp. 90–91]. Here is a description of the situation: Let $S\colon E^* \mapsto E$ be single-valued, hemicontinuous and monotone and $T := S^{-1}$. Then, arguing as in Gossez, [10, Proposition 3.1, p. 378], T is ultramaximal and so Theorem 10.2 applies to T. We shall explain in Remark 11.6 why certain nonreflexive Banach spaces cannot support *continuous linear* ultramaximal monotone operators.

11. Maximal Monotone Linear Operators

The main result of this section, Theorem 11.1, is self explanatory. Theorem 11.2, which is an easy consequence, provides an affirmative answer to [24, Problem 38.1, p. 145].

THEOREM 11.1. *Let $D(T)$ be a subspace of E and $T\colon\ D(T) \mapsto E^*$ be a maximal monotone linear operator of type* (*D*). *Then T is of type* (*ED*).

Proof. Let $(t^{**}, t^*) \in G(\overline{T})$, and define $f \in \mathcal{PCPC}(E)$ by

$$f(x) := \begin{cases} \|Tx - t^*\|, & \text{if } x \in D(T); \\ \infty, & \text{otherwise.} \end{cases}$$

We first show that

$$f^{**}(t^{**}) = 0. \tag{11.1.1}$$

Since $f \geqslant 0$ on E, $f^*(0) = -\inf_E f \leqslant 0$, and so $f^{**}(t^{**}) \geqslant \langle 0, t^{**}\rangle - f^*(0) \geqslant 0$. On the other hand, by hypothesis, there exists a bounded net $\{s_\gamma\}$ of elements of $D(T)$ such that $(\widehat{s_\gamma}, Ts_\gamma) \to (t^{**}, t^*)$ in $w(E^{**}, E^*) \times \mathcal{T}_{\|\ \|}(E^*)$. Using (2.0.4), we have

$$f^{**}(\widehat{s_\gamma}) \leqslant f(s_\gamma) = \|Ts_\gamma - t^*\| \to 0$$

so, from the $w(E^{**}, E^*)$-lower semicontinuity of f^{**} mentioned in (2.0.1), $f^{**}(t^{**}) \leqslant 0$. This completes the proof of (11.1.1). From Lemma 3.3(b), there exists a net $\{t_\gamma\}$ of elements of E such that $\widehat{t_\gamma} \to t^{**}$ in $\mathcal{T}_{\mathcal{CLB}}(E^{**})$ and $f(t_\gamma) \to f^{**}(t^{**})$. Using (11.1.1), $f(t_\gamma) \to 0$, so eventually $f(t_\gamma) < \infty$, in which case $t_\gamma \in D(T)$ and $f(t_\gamma) = \|Tt_\gamma - t^*\|$. We now truncate the net $\{t_\gamma\}$ to exclude those values of γ for which $t_\gamma \notin D(T)$. Thus $\|Tt_\gamma - t^*\| \to 0$, and so

$$(\widehat{t_\gamma}, Tt_\gamma) \to (t^{**}, t^*) \quad \text{in } \mathcal{T}_{\mathcal{CLB}}(E^{**}) \times \mathcal{T}_{\|\ \|}(E^*). \qquad \square$$

If we combine the results of Theorems 7.2 and 11.1 with those of Phelps and Simons, [15, Theorem 6.4((c) ⇔ (d))] and Bauschke and Borwein, [1, Theorem 4.1] (see also [15, Theorem 8.1]), we obtain Theorems 11.2 and 11.3 below:

THEOREM 11.2. *Let $D(T)$ be a subspace of E and $T\colon\ D(T) \mapsto E^*$ be a maximal monotone linear operator. Then the four conditions below are equivalent*:

T is of type (*ED*),
T is of dense type,
T is of type (*D*)

and

T is of type (*NI*).

THEOREM 11.3. *Let* $T\colon\ E \mapsto E^*$ *be a* (*continuous*) *monotone linear operator. Then the conditions of Theorem* 11.2 *above are equivalent to the additional two below*:

T *is of type* (*FP*) (*i.e., locally maximal monotone*)

and

T^* *is monotone.*

Remark 11.4. Let $T\colon\ E \mapsto E^*$ be a continuous *skew* linear operator, that is to say, for all $x \in E$, $\langle x, Tx\rangle = 0$, or equivalently, for all $x,\ w \in E$, $\langle w, Tx\rangle = -\langle x, Tw\rangle$. Then T is maximal monotone. Suppose now that $(w, w^*) \in E \times E^*$, $\alpha,\ \beta > 0$ and

$$\inf_{(t,t^*)\in G(T)} \langle t - w, t^* - w^*\rangle > -\alpha\beta.$$

Using the fact that T is skew, this last inequality can be rewritten:

$$\sup_{x\in E}\langle x, w^* - Tw\rangle < \langle w, w^*\rangle + \alpha\beta,$$

which implies that $w^* - Tw = 0$, i.e., $(w, w^*) \in G(T)$. Thus if $\tau = 0$ then $(\tau\alpha, \tau\beta)$ is the unique negative alignment pair for T with respect to (w, w^*). Now let T be the example of Gossez referred to in Remark 9.5. Specifically, $E := \ell^1$ and $T\colon\ \ell^1 \mapsto \ell^\infty = E^*$ is defined by $(Tx)_n := \sum_{k>n} x_k - \sum_{k<n} x_k$. Since T is skew, we have an example where the conclusion of Theorem 8.6(c) is true, but T is not of type (ED).

In our next example, the "tail" operator, we give an example of a continuous linear monotone operator for which the *conclusions* of Theorems 8.1 and 8.6(c) fail.

EXAMPLE 11.5. Let $E = \ell^1$, and define $T\colon\ \ell^1 \mapsto \ell^\infty = E^*$ by $(Tx)_n = \sum_{k\geqslant n} x_k$. We leave it to the reader to check that T is continuous, linear and monotone (hence maximal monotone). Define $\widetilde{T}\colon\ \ell^1 \mapsto \ell^\infty$ by

$$\widetilde{T}y := T^*\hat{y} \quad (y \in \ell^1),$$

where $T^*\colon\ (\ell^\infty)^* \mapsto \ell^\infty$ is the adjoint of T. Then,

$$\text{for all } x,\ y \in \ell^1, \quad \langle x, \widetilde{T}y\rangle = \langle x, T^*\hat{y}\rangle = \langle Tx, \hat{y}\rangle = \langle y, Tx\rangle.$$

Using this substitution, it follows easily that, for all $x,\ y \in \ell^1$,

$$\langle x, Tx - (\widetilde{T} + T)y\rangle + \langle y, Ty\rangle = \langle x - y, Tx - Ty\rangle \geqslant 0. \tag{11.5.1}$$

Fix y to have the value $(1, 0, 0, \ldots) \in \ell^1$. Then $Ty = (1, 0, 0, \ldots) \in \ell^\infty$, $\widetilde{T}y = (1, 1, 1, \ldots) \in \ell^\infty$ and $\langle y, Ty\rangle = 1$. Thus, putting $w^* := (\widetilde{T} + T)y = (2, 1, 1, \ldots) \in \ell^\infty$, we have from (11.5.1) that $\inf_{x\in\ell^1}\langle x, Tx - w^*\rangle \geqslant -1$. Now if the inequality-splitting conclusion of Theorem 8.1 were true for this maximal monotone linear operator T then there would exist $x \in \ell^1$ such that

$$\|x - 0\| \leqslant 2 \quad \text{and} \quad \|Tx - w^*\| \leqslant 1/2.$$

This is clearly impossible since, for all $x \in E$, $Tx \in c_0$ and so $\|Tx - w^*\| \geqslant 1$. We also have $\inf_{x\in\ell^1}\langle x, Tx - w^*\rangle > -2$, so if T were of type (ED) then the inequality-splitting conclusion of Theorem 8.6(c) would give $x \in \ell^1$ such that

$$\|x - 0\| < 4 \quad \text{and} \quad \|Tx - w^*\| < 1/2.$$

Since this is also impossible we have shown that T is not of type (ED). In fact, taking into account Theorem 11.3, we have strengthened the result proved explicitly in [15] that T^* is not monotone, and the result proved explicitly in [24, Remark 25.3, pp. 98–99] that T is not of type (FP). On the other hand, since T is continuous and linear, T is nevertheless maximal monotone of type (ANA).

Remark 11.6. The author is grateful to Heinz Bauschke for pointing out to him that certain Banach spaces cannot support continuous linear ultramaximal monotone operators. If $T\colon E \mapsto E^*$ is continuous, linear and ultramaximal monotone then it can be seen using the monotonicity of T^* established in Theorem 11.3 that

$$x^{**} \in E^{**} \text{ and } T^{**}x^{**} \in \widehat{E^*} \Rightarrow x^{**} \in \widehat{E},$$

that is, T is *Tauberian* – see Wilansky, [27, p. 175]. It follows from [27, Theorem 11-4-2, pp. 174–175] that if E is not reflexive then the closure in E^* of the image under T of the unit ball of E is not weakly compact in E^* hence, in the notation of Saab and Saab, [21, Definition 6, p. 378], E does not have “property (w)”. Spaces with this property are discussed in [21, pp. 378–380 and Proposition 47, p. 386]. In particular, E cannot be of the form $c_0(\Gamma)$ or $C(\Omega)$ (Ω compact Hausdorff). See also the discussion in Bauschke, [2, pp. 167–169].

12. Subdifferentials Are of Type (ED)

Suppose that $f \in \mathcal{PCLSC}(E)$. If $x \in E$, the *subdifferential* of f at x is defined by

$$\partial f(x) := \{z^* \in E^*\colon \quad y \in E \Rightarrow f(x) + \langle y - x, z^*\rangle \leqslant f(y)\}.$$

The main result of this section is Theorem 12.2, in which we show that if $f \in \mathcal{PCLSC}(E)$ then $G(\partial f)$ is dense in $G(\partial f^*)$ in the topology $\mathcal{T}_{\mathcal{CLB}}(E^{**}) \times \mathcal{T}_{\|\ \|}(E^*)$. Theorem 12.2 represents a sharpening of a result proved by Gossez in [10, Théorème 3.1 and Lemme 3.1, pp. 376–378] which was, in turn, a sharpening of a result proved by Rockafellar in [18, Proposition 1, pp. 211–212]. The latter proofs used some rather delicate functional analysis. Our proof depends on Lemma 3.3(b), which depends ultimately on the formula for the biconjugate of a maximum that we established in Theorem 2.5. It is then but a short step from Theorem 12.2 to Theorem 12.6, in which we prove that subdifferentials are maximal monotone of type (ED), thus extending the result proved in [10, Théorème 3.1] that subdifferentials are maximal monotone of dense type. Coupled with Theorem 9.4, it also extends the result from [23] already referred to that subdifferentials are maximal monotone of type (ANA).

We will need the Brøndsted–Rockafellar theorem, which we state as Theorem 12.1 below (see [5, p. 608] or Phelps, [13, Theorem 3.17, p. 48]):

THEOREM 12.1. *Let* $f \in \mathcal{PCLSC}(E)$, $\alpha, \beta > 0$, $(w, w^*) \in E \times E^*$ *and*

$$\sup_{x \in \mathrm{dom}\, f} [f(w) - f(x) - \langle w - x, w^* \rangle] \leqslant \alpha\beta.$$

Then there exists $(s, s^*) \in G(\partial f)$ *such that* $\|s - w\| \leqslant \alpha$ *and* $\|s^* - w^*\| \leqslant \beta$.

We do not know if the analog of Theorem 12.2 with $\mathcal{CLB}(E)$ replaced by $\mathcal{CC}(E)$ is true. Indeed, if it were, then it would have been unnecessary to introduce the set of functions $\mathcal{CLB}(E)$, and we could have worked with $\mathcal{CC}(E)$ all through this paper. See Problem 12.4 below for a precise formulation of this question.

THEOREM 12.2. *Let* $f \in \mathcal{PCLSC}(E)$, *and* $(t^*, t^{**}) \in G(\partial f^*)$. *Then there exists a net* $\{(s_\gamma, s_\gamma^*)\}$ *of elements of* $G(\partial f)$ *such that*

$$(\widehat{s_\gamma}, s_\gamma^*) \to (t^{**}, t^*) \quad \text{in } \mathcal{T}_{\mathcal{CLB}}(E^{**}) \times \mathcal{T}_{\|\ \|}(E^*) \tag{12.2.1}$$

and

$$f(s_\gamma) \to f^{**}(t^{**}). \tag{12.2.2}$$

Proof. From Lemma 3.3(b), there exists a net $\{t_\gamma\}$ of elements of E such that

$$\widehat{t_\gamma} \to t^{**} \text{ in } \mathcal{T}_{\mathcal{CLB}}(E^{**}) \quad \text{and} \quad f(t_\gamma) \to f^{**}(t^{**}). \tag{12.2.3}$$

Since $f^{**}(t^{**}) \in \mathbb{R}$, we can and will suppose that, for all γ, $f(t_\gamma) \in \mathbb{R}$. For each γ, let

$$\eta_\gamma := \sup_{x \in \mathrm{dom}\, f} [f(t_\gamma) - f(x) - \langle t_\gamma - x, t^* \rangle] = f(t_\gamma) - \langle t_\gamma, t^* \rangle + f^*(t^*) \geqslant 0.$$

Now if $\eta_\gamma > 0$ then it follows from Theorem 12.1 that there exists $(s_\gamma, s_\gamma^*) \in G(\partial f)$ such that $\|s_\gamma - t_\gamma\| \leqslant \sqrt{\eta_\gamma}$ and $\|s_\gamma^* - t^*\| \leqslant \sqrt{\eta_\gamma}$; if, on the other hand, $\eta_\gamma = 0$, this remains true with $(s_\gamma, s_\gamma^*) := (t_\gamma, t^*)$. From (12.2.3) and Lemma 3.2, $\eta_\gamma \to f^{**}(t^{**}) - \langle t^*, t^{**} \rangle + f^*(t^*)$ thus, since $t^{**} \in \partial f^*(t^*)$, $\eta_\gamma \to 0$. Now we have $\|\widehat{s_\gamma} - \widehat{t_\gamma}\| = \|s_\gamma - t_\gamma\| \to 0$ and so, from the first part of (12.2.3) and Lemma 3.7,

$\widehat{s_\gamma} \to t^{**}$ in $\mathcal{T}_{\mathcal{CLB}}(E^{**})$. Since $s_\gamma^* \to t^*$, this completes the proof of (12.2.1). Using Lemma 3.2, we have $\widehat{s_\gamma} \to t^{**}$ in $w(E^{**}, E^*)$ and so, from the $w(E^{**}, E^*)$-lower semicontinuity of f^{**} mentioned in (2.0.1), and (2.0.5),

$$f^{**}(t^{**}) \leqslant \liminf_\gamma f^{**}(\widehat{s_\gamma}) = \liminf_\gamma f(s_\gamma). \tag{12.2.4}$$

For each γ, $(s_\gamma, s_\gamma^*) \in G(\partial f)$ and so $f(s_\gamma) = \langle s_\gamma, s_\gamma^* \rangle - f^*(s_\gamma^*)$, hence

$$\limsup_\gamma f(s_\gamma) \leqslant \limsup_\gamma \langle s_\gamma, s_\gamma^* \rangle - \liminf_\gamma f^*(s_\gamma^*). \tag{12.2.5}$$

Now, from Lemma 3.1(e), $\limsup_\gamma \langle s_\gamma, s_\gamma^* \rangle = \limsup_\gamma \langle s_\gamma^*, \widehat{s_\gamma} \rangle = \langle t^*, t^{**} \rangle$. Furthermore, $f^* \in \mathcal{PCLSC}(E^*)$ and $s_\gamma^* \to t^*$ in $\mathcal{T}_{\|\ \|}(E^*)$, consequently $\liminf_\gamma f^*(s_\gamma^*) \geqslant f^*(t^*)$. Substituting back in (12.2.5) and using the fact that $(t^*, t^{**}) \in G(\partial f^*)$,

$$\limsup_\gamma f(s_\gamma) \leqslant \langle t^*, t^{**} \rangle - f^*(t^*) = f^{**}(t^{**}),$$

and (12.2.2) follows by combining this with (12.2.4). □

Remark 12.3. We recall that the topology $\mathcal{T}_{\mathcal{CC}}(E^{**})$ on E^{**} was defined in Remark 3.6 to be the coarsest topology on E^{**} making all the functions h^{**} $(h \in \mathcal{CC}(E))$ continuous.

- As pointed out in Remark 3.6, the proof of Lemma 3.3(b) can be modified to yield the following result: *let $f \in \mathcal{PCPC}(E)$ and $t^{**} \in E^{**}$. Then there exists a net $\{t_\gamma\}$ of elements of E such that $\widehat{t_\gamma} \to t^{**}$ in $\mathcal{T}_{\mathcal{CC}}(E^{**})$, $f(t_\gamma) \to f^{**}(t^{**})$ and $f^{**}(\widehat{t_\gamma}) \to f^{**}(t^{**})$.*

These comments are illustrated by the following special case – obtained by taking f to be the indicator function of C. Let C be a nonempty proper convex subset of E and B be the $w(E^{**}, E^*)$-closure of $\hat{C}$ in E^{**}.

- *If $t^{**} \in B$ then there exists a net $\{t_\gamma\}$ of elements of C such that $\widehat{t_\gamma} \to t^{**}$ in $\mathcal{T}_{\mathcal{CC}}(E^{**})$*. In other words the $w(E^{**}, E^*)$-closure of $\hat{C}$ is identical with its $\mathcal{T}_{\mathcal{CC}}(E^{**})$-closure – compare with Gossez, [10, Corollaire 3.2, p. 379].
- *If C is closed, $t^{**} \in B$, $t^* \in E^*$ and $\langle t^*, t^{**} \rangle = \sup_C t^*$ then there exists a net $\{(s_\gamma, s_\gamma^*)\}$ of elements of $C \times E^*$ such that $(\widehat{s_\gamma}, s_\gamma^*) \to (t^{**}, t^*)$ in $\mathcal{T}_{\mathcal{CLB}}(E^{**}) \times \mathcal{T}_{\|\ \|}(E^*)$ and, for all γ, $\langle s_\gamma, s_\gamma^* \rangle = \max_C s_\gamma^*$* – compare with Gossez, [10, Corollaire 3.1, pp. 378–379].

Here are some situations in which we do not know whether $\mathcal{T}_{\mathcal{CC}}(E^{**})$ can be substituted for $\mathcal{T}_{\mathcal{CLB}}(E^{**})$ in results described above:

PROBLEM 12.4. *Let $f \in \mathcal{PCLSC}(E)$ and $(t^*, t^{**}) \in G(\partial f^*)$. Does there necessarily exist a net $\{(s_\gamma, s_\gamma^*)\}$ of elements of $G(\partial f)$ such that $(\widehat{s_\gamma}, s_\gamma^*) \to (t^{**}, t^*)$ in $\mathcal{T}_{\mathcal{CC}}(E^{**}) \times \mathcal{T}_{\|\ \|}(E^*)$ and $f(s_\gamma) \to f^{**}(t^{**})$?*

PROBLEM 12.5. *Let C be a nonempty proper closed convex subset of E, B be the $w(E^{**}, E^*)$-closure of $\hat{C}$ in E^{**}, $t^{**} \in B$, $t^* \in E^*$ and $\langle t^*, t^{**}\rangle = \sup_C t^*$. Does there there necessarily exist a net $\{(s_\gamma, s_\gamma^*)\}$ of elements of $C \times E^*$ such that $(\widehat{s_\gamma}, s_\gamma^*) \to (t^{**}, t^*)$ in $\mathcal{T}_{\mathcal{CC}}(E^{**}) \times \mathcal{T}_{\|\ \|}(E^*)$ and, for all γ, $\langle s_\gamma, s_\gamma^*\rangle = \max_C s_\gamma^*$?*

The result of Theorem 12.6(a) below can be found in Rockafellar, [18, Proposition 1, pp. 211–212] and Gossez, [10, Théorème 3.1 and Lemme 3.1, pp. 376–378].

THEOREM 12.6. *Let $f \in \mathcal{PCLSC}(E)$.*

(a) *Let $(x^*, x^{**}) \in E^* \times E^{**}$. Then the conditions* (12.6.1)–(12.6.3) *are equivalent*:

$$(x^{**}, x^*) \in G\big(\overline{\partial f}\big) \tag{12.6.1}$$

$$\inf_{(t^*, t^{**}) \in G(\partial f^*)} \langle t^* - x^*, t^{**} - x^{**}\rangle \geqslant 0 \tag{12.6.2}$$

$$(x^*, x^{**}) \in G(\partial f^*). \tag{12.6.3}$$

(b) $\partial f\colon E \mapsto 2^{E^*}$ *is maximal monotone of type* (*ED*).

Proof. (a) ((12.6.1) $\Rightarrow$ (12.6.2)) Let (t^*, t^{**}) be an arbitrary element of $G(\partial f^*)$. From Theorem 12.2, there exists a exists a net $\{(s_\gamma, s_\gamma^*)\}$ of elements of $G(\partial f)$ such that $(\widehat{s_\gamma}, s_\gamma^*) \to (t^{**}, t^*)$ in $\mathcal{T}_{\mathcal{CLB}}(E^{**}) \times \mathcal{T}_{\|\ \|}(E^*)$. From (12.6.1),

for all γ,

$$\langle s_\gamma^*, \widehat{s_\gamma}\rangle - \langle s_\gamma^*, x^{**}\rangle - \langle x^*, \widehat{s_\gamma}\rangle + \langle x^*, x^{**}\rangle = \langle s_\gamma^* - x^*, \widehat{s_\gamma} - x^{**}\rangle \geqslant 0$$

hence, by passing to the limit and using Lemma 3.1(a, e),

$$\langle t^* - x^*, t^{**} - x^{**}\rangle = \langle t^*, t^{**}\rangle - \langle t^*, x^{**}\rangle - \langle x^*, t^{**}\rangle + \langle x^*, x^{**}\rangle \geqslant 0.$$

Thus we have established (12.6.2).

((12.6.2) $\Rightarrow$ (12.6.1)) This is immediate since, from (2.0.5),

$$(s, s^*) \in G(\partial f) \Leftrightarrow (s^*, \hat{s}) \in G(\partial f^*).$$

((12.6.2) $\Leftrightarrow$ (12.6.3)) This equivalence follows since, from Rockafellar's maximal monotonicity theorem (see [18] or [23] or [24]), $\partial f^*\colon\ E^* \mapsto 2^{E^{**}}$ is maximal monotone.

(b) follows from (a) and Theorem 12.2. □

13. A new Inequality-Splitting Property of Subdifferentials

Our first result is immediate from Theorem 12.6(b) and Theorem 8.6. Of course, Theorem 13.1(a) extends Rockafellar's original result that subdifferentials are maximal monotone.

THEOREM 13.1. *Let* $f \in \mathcal{PCLSC}(E)$, $(w, w^*) \in E \times E^* \setminus G(\partial f)$ *and* $\alpha,\ \beta > 0$. *Then*:

(a) *There exists a unique value of* $\tau > 0$ *such that* $(\tau\alpha, \tau\beta)$ *is a negative alignment pair for* ∂f *with respect to* (w, w^*), *and there exists* $(t, t^*) \in G(\partial f)$ *such that* $t \neq w$, $t^* \neq w^*$,

$$\frac{\|t - w\|}{\|t^* - w^*\|} \text{ as near as we please to } \frac{\alpha}{\beta}$$

and

$$\frac{\langle t - w, t^* - w^* \rangle}{\|t - w\| \|t^* - w^*\|} \text{ as near as we please to } -1.$$

(b) *If, further,* $\inf_{(t,t^*) \in G(\partial f)} \langle t - w, t^* - w^* \rangle > -\alpha\beta$, *then* $\tau < 1$, *and we can take* (t, t^*) *so that, in addition,* $\|t - w\| < \alpha$ *and* $\|t^* - w^*\| < \beta$.

Theorem 13.1 leads to a new version of the Brøndsted–Rockafellar theorem for subdifferentials, which we shall state as Corollary 13.3. The bridge between these two results is provided by Lemma 13.2 below, which was essentially proved in [12]. It was also shown in [12] that the inequality (13.2.1) can easily be strict. For instance, we can take $E := \mathbb{R}$, and $f(t) := t^2$. Then, for all $(w, w^*) \in \mathbb{R}^2$,

$$\inf_{(t,t^*) \in G(\partial f)} \langle t - w, t^* - w^* \rangle = -\frac{(2w - w^*)^2}{8} = -\frac{(f'(w) - w^*)^2}{8}$$

and

$$- \sup_{x \in \operatorname{dom} f} [f(w) - f(x) - \langle w - x, w^* \rangle]$$
$$= -\frac{(2w - w^*)^2}{4} = -\frac{(f'(w) - w^*)^2}{4}.$$

So the inequality (13.2.1) is always strict in this case when $(w, w^*) \notin G(\partial f)$.

LEMMA 13.2. *Let* $f \in \mathcal{PCLSC}(E)$ *and* $(w, w^*) \in E \times E^*$. *Then*

$$\inf_{(t,t^*) \in G(\partial f)} \langle t - w, t^* - w^* \rangle \geqslant - \sup_{x \in \operatorname{dom} f} [f(w) - f(x) - \langle w - x, w^* \rangle]. \tag{13.2.1}$$

Proof. This follows from the observation that if $(t, t^*) \in G(\partial f)$ then $t \in \operatorname{dom} f$ and

$$\langle t - w, t^* - w^* \rangle + [f(w) - f(t) - \langle w - t, w^* \rangle]$$
$$= \langle t - w, t^* \rangle + f(w) - f(t) \geqslant 0.$$

□

Corollary 13.3 tells us that, provided that we replace "$\leqslant$" by "$<$" in the appropriate places, in addition to the other conclusions that we had in the classical Theorem 12.1, we can exert considerable control over the values of $\|t - w\|$, $\|t^* - w^*\|$ and $\langle t - w, t^* - w^*\rangle$. Of course, in any case when the inequality in (13.2.1) is *strict*, we can obtain a generalization of Theorem 12.1 by applying Theorem 13.1 and Lemma 13.2 with slightly smaller values of α and β. We leave details of this to the reader.

COROLLARY 13.3. *Let $f \in \mathcal{PCLSC}(E)$, $(w, w^*) \in E \times E^* \setminus G(\partial f)$, $\alpha,\ \beta > 0$ and*

$$\sup_{x \in \operatorname{dom} f} [f(w) - f(x) - \langle w - x, w^*\rangle] < \alpha\beta.$$

Then there exists a unique value of $\tau \in (0, 1)$ such that $(\tau\alpha, \tau\beta)$ is a negative alignment pair for ∂f with respect to (w, w^*). In particular, there exists $(t, t^*) \in G(\partial f)$ such that $\|t - w\| \in (0, \alpha)$, $\|t^* - w^*\| \in (0, \beta)$,

$$\frac{\|t - w\|}{\|t^* - w^*\|} \text{ as near as we please to } \frac{\alpha}{\beta}$$

and

$$\frac{\langle t - w, t^* - w^*\rangle}{\|t - w\|\|t^* - w^*\|} \text{ as near as we please to } -1.$$

Proof. This is immediate from Theorem 13.1 and Lemma 13.2. □

Remark 13.4. Let $f \in \mathcal{PCLSC}(E)$, and $f(0) > \inf_E f$. Then $0 \notin \partial f(0)$ hence, from Theorem 13.1(a) with $(w, w^*) := (0, 0)$ and $\alpha := \beta := 1$, there exists a unique value of $\tau > 0$ for which there exists a sequence $\{(t_m, t_m^*)\}_{m \geqslant 1}$ of elements of $G(\partial f)$ such that

$$\lim_{m\to\infty} \|t_m\| = \tau, \qquad \lim_{m\to\infty} \|t_m^*\| = \tau \quad \text{and} \quad \lim_{m\to\infty} \langle t_m, t_m^*\rangle = -\tau^2.$$

It would be interesting to find a direct way (without using properties of E^{**}) of computing this value of τ. In this connection, it is worth comparing the proof of the maximal monotonicity of subdifferentials given in [18, Theorem A] with the proof given in [23, Corollary 5].

14. $\mathcal{T}_{\mathcal{CLB}}$ and Product Spaces

Let H also be a real Banach space. In what follows, we will give the product space $E \times H$ the norm $\|(x, z)\| := \sqrt{\|x\|^2 + \|z\|^2}$, and we will identify $(E \times H)^*$ with $E^* \times H^*$, and $(E \times H)^{**}$ with $E^{**} \times H^{**}$ in the usual way. The main result of this section is Theorem 14.6, in which we characterize the convergence of particular kinds of nets in $\mathcal{T}_{\mathcal{CLB}}((E \times H)^{**})$. We will apply Corollary 14.7 on the permuting of spaces through $\mathcal{T}_{\mathcal{CLB}}$ and $\mathcal{T}_{\| \, \|}$ in our analysis of the multifunction associated with a saddle function in the next section.

LEMMA 14.1. *Let $h \in \mathcal{CLB}(E)$. Define $g \in \mathcal{CLB}(E \times H)$ by the formula $g(x, z) := h(x)$ for $(x, z) \in E \times H$. Then*:

(a) *For all $(x^*, z^*) \in (E \times H)^*$, $g^*(x^*, z^*) = \begin{cases} h^*(x^*), & \text{if } z^* = 0; \\ \infty, & \text{otherwise.} \end{cases}$*

(b) *For all $(x^{**}, z^{**}) \in (E \times H)^{**}$, $g^{**}(x^{**}, z^{**}) = h^{**}(x^{**})$.*

Proof. This is immediate by direct computation. □

LEMMA 14.2. *Let $\{(x_\gamma^{**}, z_\gamma^{**})\}$ be a net of elements of $(E \times H)^{**}$, $(x^{**}, z^{**}) \in (E \times H)^{**}$ and $(x_\gamma^{**}, z_\gamma^{**}) \to (x^{**}, z^{**})$ in $\mathcal{T}_{\mathcal{CLB}}((E \times H)^{**})$. Then $(x_\gamma^{**}, z_\gamma^{**}) \to (x^{**}, z^{**})$ in $\mathcal{T}_{\mathcal{CLB}}(E^{**}) \times \mathcal{T}_{\mathcal{CLB}}(H^{**})$.*

Proof. Let h be an arbitrary element of $\mathcal{CLB}(E)$, and define $g \in \mathcal{CLB}(E \times H)$ as in Lemma 14.1. It then follows from the definition of $\mathcal{T}_{\mathcal{CLB}}((E \times H)^{**})$ that $g^{**}(x_\gamma^{**}, z_\gamma^{**}) \to g^{**}(x^{**}, z^{**})$. We now obtain from Lemma 14.1(b) that $h^{**}(x_\gamma^{**}) \to h^{**}(x^{**})$. It then follows from the definition of $\mathcal{T}_{\mathcal{CLB}}(E^{**})$ that $x_\gamma^{**} \to x^{**}$ in $\mathcal{T}_{\mathcal{CLB}}(E^{**})$. The proof that $z_\gamma^{**} \to z^{**}$ in $\mathcal{T}_{\mathcal{CLB}}(H^{**})$ is similar. □

The converse of Lemma 14.2 is false in general. To see this, define $h \in \mathcal{CLB}(E \times E)$ by $h(x, y) := \|x - y\|$. Then, by direct computation, $h^{**}(x^{**}, y^{**}) = \|x^{**} - y^{**}\|$. Suppose now that E is not reflexive, and consider the net x_γ of elements of E defined in Remark 3.5. Since $\widehat{x_\gamma} \to x^{**}$ in $\mathcal{T}_{\mathcal{CLB}}(E^{**})$, $(\widehat{x_\gamma}, x^{**}) \to (x^{**}, x^{**})$ in $\mathcal{T}_{\mathcal{CLB}}(E^{**}) \times \mathcal{T}_{\mathcal{CLB}}(E^{**})$. However, $\|\widehat{x_\gamma} - x^{**}\| \not\to 0 = \|x^{**} - x^{**}\|$ and so, from the comments above, $(\widehat{x_\gamma}, x^{**}) \not\to (x^{**}, x^{**})$ in $\mathcal{T}_{\mathcal{CLB}}((E \times E)^{**})$. We will, however, prove a restricted converse for Lemma 14.2 in Theorem 14.6 below.

LEMMA 14.3. *Let $\{(x_\gamma^{**}, z_\gamma)\}$ be a net of elements of $E^{**} \times H$, $(x^{**}, z) \in E^{**} \times H$ and $z_\gamma \to z$ in $\mathcal{T}_{\| \, \|}(H)$. Then*

$$\begin{aligned} &(x_\gamma^{**}, \widehat{z_\gamma}) \to (x^{**}, \hat{z}) \text{ in } \mathcal{T}_{\mathcal{CLB}}((E \times H)^{**}) \\ &\quad \Leftrightarrow (x_\gamma^{**}, \hat{z}) \to (x^{**}, \hat{z}) \text{ in } \mathcal{T}_{\mathcal{CLB}}((E \times H)^{**}). \end{aligned}$$

Proof. This follows from Lemma 3.7, since

$$\|(x_\gamma^{**}, \widehat{z_\gamma}) - (x_\gamma^{**}, \hat{z})\| = \|(0, \widehat{z_\gamma} - \hat{z})\| = \|\widehat{z_\gamma} - \hat{z}\| = \|z_\gamma - z\| \to 0. \qquad \square$$

It is worth pointing out that we give an indirect proof of Lemma 14.4 below since we do not have an explicit formula for g^* in terms of h^*.

LEMMA 14.4. *Let $z \in H$ and $h \in \mathcal{CLB}(E \times H)$. Define $g \in \mathcal{CLB}(E)$ by the formula $g(x) := h(x, z)$ for $x \in E$. Let $x^{**} \in E^{**}$. Then $g^{**}(x^{**}) = h^{**}(x^{**}, \hat{z})$.*

Proof. It follows from Lemma 3.3(c) that there exists a net $\{(x_\gamma, z_\gamma)\}$ of elements of $E \times H$ such that

$$(\widehat{x_\gamma}, \widehat{z_\gamma}) = \widehat{(x_\gamma, z_\gamma)} \to (x^{**}, \hat{z}) \quad \text{in } \mathcal{T}_{\mathcal{CLB}}((E \times H)^{**}). \tag{14.4.1}$$

Using Lemma 14.2, we derive from this that $\widehat{z_\gamma} \to \hat{z}$ in $\mathcal{T}_{\mathcal{CLB}}(H^{**})$ so, from Theorem 3.4(b), $z_\gamma \to z$ in $\mathcal{T}_{\| \|}(H)$. It now follows from Lemma 14.3(⇒) that

$$(\widehat{x_\gamma}, \hat{z}) \to (x^{**}, \hat{z}) \quad \text{in } \mathcal{T}_{\mathcal{CLB}}((E \times H)^{**}) \tag{14.4.2}$$

hence, from Lemma 14.2 again,

$$\widehat{x_\gamma} \to x^{**} \quad \text{in } \mathcal{T}_{\mathcal{CLB}}(E^{**}). \tag{14.4.3}$$

It now follows from (14.4.3), (14.4.2) and the definitions of $\mathcal{T}_{\mathcal{CLB}}(E^{**})$ and $\mathcal{T}_{\mathcal{CLB}}((E \times H)^{**})$ that $g^{**}(\widehat{x_\gamma}) \to g^{**}(x^{**})$ and $h^{**}(\widehat{x_\gamma}, \hat{z}) \to h^{**}(x^{**}, \hat{z})$. However, from two applications of (2.0.5), for all γ, $g^{**}(\widehat{x_\gamma}) = g(x_\gamma) := h(x_\gamma, z) = h^{**}(\widehat{(x_\gamma, z)}) = h^{**}(\widehat{x_\gamma}, \hat{z})$, and the result follows by passing to the limit. □

LEMMA 14.5. *Let $z \in H$, $\{x_\gamma^{**}\}$ be a net of elements of E^{**}, $x^{**} \in E^{**}$ and $x_\gamma^{**} \to x^{**}$ in $\mathcal{T}_{\mathcal{CLB}}(E^{**})$. Then $(x_\gamma^{**}, \hat{z}) \to (x^{**}, \hat{z})$ in $\mathcal{T}_{\mathcal{CLB}}((E \times H)^{**})$.*

Proof. Let h be an arbitrary element of $\mathcal{CLB}(E \times H)$, and define $g \in \mathcal{CLB}(E)$ as in Lemma 14.4. It follows from the definition of $\mathcal{T}_{\mathcal{CLB}}(E^{**})$ that $g^{**}(x_\gamma^{**}) \to g^{**}(x^{**})$ hence, using Lemma 14.4, $h^{**}(x_\gamma^{**}, \hat{z}) \to h^{**}(x^{**}, \hat{z})$. The result now follows from the definition of $\mathcal{T}_{\mathcal{CLB}}((E \times H)^{**})$. □

THEOREM 14.6. *Let $\{(x_\gamma^{**}, z_\gamma)\}$ be a net of elements of $E^{**} \times H$ and $(x^{**}, z) \in E^{**} \times H$. Then*

$$\begin{aligned}&(x_\gamma^{**}, \widehat{z_\gamma}) \to (x^{**}, \hat{z}) \text{ in } \mathcal{T}_{\mathcal{CLB}}((E \times H)^{**})\\ &\quad \Leftrightarrow (x_\gamma^{**}, z_\gamma) \to (x^{**}, z) \text{ in } \mathcal{T}_{\mathcal{CLB}}(E^{**}) \times \mathcal{T}_{\| \|}(H).\end{aligned}$$

Proof. (⇒) follows from Lemma 14.2 and Theorem 3.4(b), while (⇐) follows from Lemmas 14.5 and 14.3(⇐). □

Corollary 14.7 will be used in Lemma 15.1 below.

COROLLARY 14.7. *Let E and F be Banach spaces. Suppose that $\{(x_\gamma^{**}, y_\gamma^*)\}$ is a net of elements of $E^{**} \times F^*$, $(x^{**}, y^*) \in E^{**} \times F^*$, $\{(x_\gamma^*, y_\gamma)\}$ is a net of elements of $E^* \times F$ and $(x^*, y) \in E^* \times F$. Then*:

$$\begin{aligned}&(x_\gamma^{**}, \widehat{y_\gamma^*}) \to (x^{**}, \widehat{y^*}) \text{ in } \mathcal{T}_{\mathcal{CLB}}((E \times F^*)^{**}) \quad \text{and}\\ &(x_\gamma^*, y_\gamma) \to (x^*, y) \text{ in } \mathcal{T}_{\| \|}(E^* \times F)\\ &\quad \Leftrightarrow (x_\gamma^{**}, \widehat{y_\gamma}) \to (x^{**}, \hat{y}) \text{ in } \mathcal{T}_{\mathcal{CLB}}((E \times F)^{**}) \quad \text{and}\\ &(x_\gamma^*, y_\gamma^*) \to (x^*, y^*) \text{ in } \mathcal{T}_{\| \|}(E^* \times F^*).\end{aligned}$$

Proof. It is clear from two applications of Theorem 14.6, one with $H := F$ and the other with $H := F^*$, that each of the statements above is equivalent to:

$$x_\gamma^{**} \to x^{**} \text{ in } \mathcal{T}_{\mathcal{CLB}}(E^{**}) \quad \text{and}$$
$$(x_\gamma^*, y_\gamma, y_\gamma^*) \to (x^*, y, y^*) \text{ in } \mathcal{T}_{\|\ \|}(E^* \times F \times F^*). \qquad \square$$

15. The Multifunction Associated with a Saddle-Function

Let E and F be Banach spaces. A function $k\colon\ E \times F \mapsto [-\infty, \infty]$ is said to be a *saddle-function* if, for all $x \in E$, the function $k_x := k(x, \cdot)$ is convex on F and, for all $y \in F$, the function $-k^y := -k(\cdot, y)$ is convex on E. If k is a saddle-function, we write

$$\operatorname{dom} k := \{(x, y) \in E \times F\colon\ k_x(F) \subset \mathbb{R} \cup \{\infty\} \text{ and } -k^y(E) \subset \mathbb{R} \cup \{\infty\}\}.$$

We suppose that $\operatorname{dom} k \neq \emptyset$. We define the multifunction $\sigma k\colon\ E \times F \mapsto (E \times F)^*$ by

$$((x, y), (x^*, y^*)) \in G(\sigma k) \Leftrightarrow$$
$$(x, y) \in \operatorname{dom} k,\ x^* \in \partial(-k^y)(x) \text{ and } y^* \in \partial(k_x)(y).$$

Rockafellar proved in [19, Theorem 3, p. 248] that if F is reflexive and k is "closed" in a sense made specific there then σk is maximal monotone. It is also noted on p. 249 of [19] that if all the functions k_x for $x \in E$ and all the functions $-k^y$ for $y \in F$ are lower semicontinuous then k is closed. We will show in Theorem 15.2 below that, in the situation of [19, Theorem 3], σk is in fact maximal monotone of type (ED), so the inequality-splitting conclusions of Theorem 8.6 are valid for $T := \sigma k$. We first need a preliminary lemma on "partially inverting" a multifunction.

LEMMA 15.1. *Let F be reflexive and $S\colon\ E \times F^* \mapsto 2^{(E\times F^*)^*}$ be maximal monotone of type (ED). Define $P\colon\ E \times F \mapsto 2^{(E\times F)^*}$ by declaring that $(x^*, y^*) \in P(x, y)$ exactly when $(x^*, \hat{y}) \in S(x, y^*)$. Then P is maximal monotone of type (ED).*

Proof. We leave to the reader the proof that P is maximal monotone. Now suppose that $((x^{**}, \hat{y}), (x^*, y^*)) \in G(\overline{P})$. Then, by direct computation, $((x^{**}, \widehat{y^*}), (x^*, \hat{y})) \in G(\overline{S})$. Since S is of type (ED), there exists a net $\{(s_\gamma, t_\gamma^*), (s_\gamma^*, \widehat{t_\gamma})\}$ of elements of $G(S)$ such that

$$\widehat{(s_\gamma, t_\gamma^*)} \to (x^{**}, \widehat{y^*}) \text{ in } \mathcal{T}_{\mathcal{CLB}}((E \times F^*)^{**}) \quad \text{and}$$
$$(s_\gamma^*, \widehat{t_\gamma}) \to (x^*, \hat{y}) \text{ in } \mathcal{T}_{\|\ \|}((E \times F^*)^*),$$

which is easily seen to be equivalent to

$$(\widehat{s_\gamma}, \widehat{t_\gamma^*}) \to (x^{**}, \widehat{y^*}) \text{ in } \mathcal{T}_{\mathcal{CLB}}((E \times F^*)^{**}) \quad \text{and}$$
$$(s_\gamma^*, t_\gamma) \to (x^*, y) \text{ in } \mathcal{T}_{\|\ \|}(E^* \times F).$$

Using Corollary 14.7, this last is equivalent to

$$(\widehat{s_\gamma}, \widehat{t_\gamma}) \to (x^{**}, \hat{y}) \text{ in } \mathcal{T}_{\mathcal{CLB}}((E \times F)^{**}) \quad \text{and}$$
$$(s_\gamma^*, t_\gamma^*) \to (x^*, y^*) \text{ in } \mathcal{T}_{\|\ \|}(E^* \times F^*).$$

Now, for all γ, $((s_\gamma, t_\gamma^*), (s_\gamma^*, \widehat{t_\gamma})) \in G(S)$, from which $((s_\gamma, t_\gamma), (s_\gamma^*, t_\gamma^*)) \in G(P)$. Since $(\widehat{s_\gamma}, \widehat{t_\gamma}) = \widehat{(s_\gamma, t_\gamma)}$, we have proved that P is of type (ED), as required. □

THEOREM 15.2. *Let F be reflexive and k be a closed saddle-function such that* $\operatorname{dom} k \neq \emptyset$. *Then σk is maximal monotone of type* (*ED*).

Proof. It is shown by Rockafellar on p. 248 of [19] that there exists $f \in \mathcal{PCLSC}(E \times F^*)$ such that

$$(x^*, y^*) \in \sigma k(x, y) \Leftrightarrow (x^*, \hat{y}) \in \partial f(x, y^*).$$

The result now follows from Theorem 12.6(b) and Lemma 15.1. □

It is not so clear what happens if F is not assumed to be reflexive. Rockafellar proved in [19, Theorem 2, pp. 245–247] that if k is finite-valued and separately continuous then σk is maximal monotone. However, we do not know the answer to the following problem:

PROBLEM 15.3. *Let k be finite-valued and separately continuous. Is σk necessarily of type* (*D*)?

Acknowledgements

The author would like to thank Professor Michel Théra for sending him a preprint of [16], which was a considerable source of inspiration. He would also like to thank Professor Robert Phelps for reading through an earlier version of this paper, and making a number of suggestions which have improved the exposition considerably.

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