Stronger maximal monotonicity properties of linear operators

by
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0. Introduction

Let E be a real Banach space with dual E^* and $S\colon\ E \mapsto 2^{E^*}$. Write

$$G(S) := \{(x, x^*)\colon\ x \in E,\ x^* \in Sx\}.$$

S is said to be *monotone* if

$$(x, x^*) \text{ and } (s, s^*) \in G(S) \quad \Longrightarrow \quad \langle x - s, x^* - s^* \rangle \ge 0.$$

S is said to be *maximal monotone* if S is monotone and: if $x \in E$, $x^* \in E^*$ and

$$\text{for all } (s, s^*) \in G(S), \quad \langle x - s, x^* - s^* \rangle \ge 0$$

then

$$(x, x^*) \in G(S).$$

It is the purpose of this paper to discuss four subfamilies of the maximal monotone operators, in particular in relation to the (single–valued) linear operators. This is a good place to point out that if we want to assume that a linear operator is bounded, we will state that fact explicitly.

We now introduce the first subfamily of the maximal monotone operators that we wish to discuss.

Definition 0.1. Let E be a real Banach space with dual E^* and $S\colon\ E \mapsto 2^{E^*}$ be monotone. We say that S is *strongly maximal monotone* if S is monotone and: if K is a nonempty weakly compact convex subset of E, $x^* \in E^*$ and

$$\text{for all } (s, s^*) \in G(S), \quad \text{there exists } x \in K \text{ such that} \quad \langle x - s, x^* - s^* \rangle \ge 0$$

then

$$\text{there exists } x \in K \text{ such that} \quad (x, x^*) \in G(S)$$

and, further, if L is a nonempty weak* compact convex subset of E^*, $x \in E$ and

$$\text{for all } (s, s^*) \in G(S), \quad \text{there exists } x^* \in L \text{ such that} \quad \langle x - s, x^* - s^* \rangle \ge 0$$

then

$$\text{there exists } x^* \in L \text{ such that} \quad (x, x^*) \in G(S).$$

It is clear (by taking K or L to be a singleton) that every strongly maximal monotone multifunction is maximal monotone.

Here is our motivation for introducing strongly maximal monotone operators. A well known result of Rockafellar (see [6]) asserts that *if $f\colon\ E \mapsto \mathbb{R} \cup \{\infty\}$ is proper, convex and lower semicontinuous then the associated subdifferential mapping $\partial f\colon E \mapsto 2^{E^*}$ is maximal monotone*. It was proved in [9] Theorems 6.1 and 6.2, p. 1386 and, in a different way in [11], Theorem 32.5, p. 128 that, in fact, *∂f is strongly maximal monotone*. This leads naturally to the following problem:

Problem 0.2. Find a maximal monotone multifunction $S\colon E \mapsto 2^{E^*}$ that is not strongly maximal monotone. (As we have already observed, S cannot be a subdifferential. We shall show in Theorem 1.1 that S cannot be a monotone linear operator either.) This problem is open even if $E = \mathbb{R}^2$.

By way of introduction to the other subclasses of the maximal monotone operators that we wish to discuss, we mention the following result that was proved in [11], Corollary 10.4, p. 36: *Let E be reflexive and $S\colon E \mapsto 2^{E^*}$ be monotone. Then S is maximal monotone if, and only if, for all $(x, x^*) \in E \times E^* \setminus G(S)$, there exists $(w, w^*) \in G(S)$ such that*

$$w \neq x,\ w^* \neq x^* \quad \text{and} \quad \langle w - x, w^* - x^* \rangle = -\|w - x\|\|w^* - x^*\|. \tag{0.2.1}$$

We now formalize the above result into a definition.

Definition 0.3. Let E be a real Banach space with dual E^* and $S\colon E \mapsto 2^{E^*}$ be monotone. We say that S is *of type (NA)* if, for all $(x, x^*) \in E \times E^* \setminus G(S)$, there exists $(w, w^*) \in G(S)$ satisfying (0.2.1). (NA) stands for "negative alignment".

It obviously follows from the result mentioned above that if E is reflexive then every maximal monotone operator on E is of type (NA). The following simple example shows that there is no hope of getting this result if E is not reflexive.

Example 0.4. Let E be a nonreflexive Banach space. Let $Sx := \{0\}$ $(x \in E)$. From James's theorem, there exists $x^* \in E^*$ that does not attain its norm on the unit ball of E. Then S is maximal monotone and $(0, x^*) \in E \times E^* \setminus G(S)$, but there does not exist $(w, w^*) \in G(S)$ satisfying (0.2.1): since $(w, w^*) \in G(S) \Longrightarrow w^* = 0$, (0.2.1) would imply that $\langle w, x^* \rangle = \|w\|\|x^*\|$. Setting $b := w/\|w\|$, we would have $\|b\| = 1$ and $\langle b, x^* \rangle = \|x^*\|$, contradicting our choice of x^*.

The above considerations lead us to the following weakening of Definition 0.3:

Definition 0.5. Let E be a real Banach space with dual E^* and $S\colon E \mapsto 2^{E^*}$ be monotone. We say that S is *of type (ANA)* if, whenever $(x, x^*) \in E \times E^* \setminus G(S)$ then, for all $n \geq 1$, there exists $(w_n, w_n^*) \in G(S)$ such that $w_n \neq x$, $w_n^* \neq x^*$ and

$$\frac{\langle w_n - x, w_n^* - x^* \rangle}{\|w_n - x\|\|w_n^* - x^*\|} \to -1 \quad \text{as } n \to \infty.$$

(ANA) stands for "almost negative alignment". If S is of type (ANA) then S is maximal monotone. Further, it is clear that every monotone multifunction of type (NA) is of type (ANA). The converse of this statement is false — the operator S defined in Example 0.4 is not of type (NA) but, from Theorem 2.1 below, it is of type (ANA).

Our motivation for introducing multifunctions of type (ANA) again stems from the properties of subdifferentials, since it was proved in [10], Theorem 13, p. 229 and Theorem 26, p. 237 that *if $f\colon E \mapsto \mathbb{R} \cup \{\infty\}$ is proper, convex and lower semicontinuous then the associated subdifferential mapping $\partial f\colon E \mapsto 2^{E^*}$ is maximal monotone of type (ANA).* This leads naturally to the following problem:

Problem 0.6. Find a maximal monotone multifunction $S\colon\ E \mapsto 2^{E^*}$ that is not of type (ANA). (As we have already observed, E must be nonreflexive and S cannot be a subdifferential.) If such an example can be found, find a subspace D of E and a linear maximal monotone $T\colon D \mapsto E^*$ that is not of type (ANA). (We shall show in Theorems 2.1 and 2.3 that T would have to be unbounded and not surjective.)

We shall use the following minimax theorem, which follows from a result of Fan (see [1]). (See also [3] and [8] for simple generalizations of Fan's result.)

Theorem 0.7. *Let A be a nonempty convex subset of a vector space, B be a nonempty convex subset of a vector space and B also be a compact Hausdorff topological space. Let $h : A \times B \mapsto \mathbb{R}$ be convex on A, and concave and upper semicontinuous on B. Then*

$$\inf_A \max_B h = \max_B \inf_A h.$$

1. The strong maximality of linear maximal monotone maps

It is clear from (b) and (c) of Theorem 1.1 below that every linear maximal monotone operator is strongly maximal monotone.

Theorem 1.1. *Let D be a subspace of E and $T\colon D \mapsto E^*$ be linear and maximal monotone. Then*
(a) The function from D into $\mathbb{R}$ defined by $s \mapsto \langle s, Ts\rangle$ is convex.
(b) If K is a nonempty weakly compact convex subset of E, $x^* \in E^*$ and

$$\text{for all } s \in D, \quad \text{there exists } x \in K \text{ such that} \quad \langle x - s, x^* - Ts\rangle \geq 0 \tag{1.1.1}$$

then

$$\text{there exists } x \in K \text{ such that} \quad (x, x^*) \in G(T). \tag{1.1.2}$$

(c) If L is a nonempty weak compact convex subset of E^*, $x \in E$ and*

$$\text{for all } s \in D, \quad \text{there exists } x^* \in L \text{ such that} \quad \langle x - s, x^* - Ts\rangle \geq 0 \tag{1.1.3}$$

then

$$\text{there exists } x^* \in L \text{ such that} \quad (x, x^*) \in G(T). \tag{1.1.4}$$

Proof. (a) follows from the observation that if s, $t \in D$, λ, $\mu > 0$ and $\lambda + \mu = 1$ then

$$\lambda\langle s, Ts\rangle + \mu\langle t, Tt\rangle - \langle \lambda s + \mu t, T(\lambda s + \mu t)\rangle = \lambda\mu\langle s - t, Ts - Tt\rangle \geq 0.$$

(b) Define $h\colon\ D \times K \mapsto \mathbb{R}$ by

$$h(s, y) := \langle y - s, x^* - Ts\rangle = \langle y, x^*\rangle - \langle y, Ts\rangle - \langle s, x^*\rangle + \langle s, Ts\rangle.$$

From (a), h is convex on D. h is also affine and weakly continuous on K. From (1.1.1), $\inf_D \max_K h \geq 0$ hence, from Theorem 0.7, $\max_K \inf_D h \geq 0$, that is to say,

$$\text{there exists } x \in K \text{ such that,} \quad \text{for all } s \in D, \quad \langle x - s, x^* - Ts \rangle \geq 0.$$

Since T is maximal monotone, it follows from this last inequality that $(x, x^*) \in G(T)$, which gives (1.1.2).

(c) Define h: $D \times L \mapsto \mathbb{R}$ by

$$h(s, y^*) := \langle x - s, y^* - Ts \rangle = \langle x, y^* \rangle - \langle x, Ts \rangle - \langle s, y^* \rangle + \langle s, Ts \rangle.$$

From (a), h is convex on D. h is also affine and weak* continuous on L. From (1.1.3), $\inf_D \max_L h \geq 0$ hence, from Theorem 0.7, $\max_L \inf_D h \geq 0$, that is to say,

$$\text{there exists } x^* \in L \text{ such that,} \quad \text{for all } s \in D, \quad \langle x - s, x^* - Ts \rangle \geq 0.$$

Since T is maximal monotone, it follows from this last inequality that $(x, x^*) \in G(T)$, which gives (1.1.4). ▌

2. Bounded and surjective monotone linear maps

Theorem 2.1. *Let T: $E \mapsto E^*$ be linear and monotone. Then T is maximal monotone of type (ANA).*

Proof. We first note from the local boundedness theorem for monotone multifunctions (see, for instance, [4] Theorem 2.28, p. 28) that T is bounded. Suppose that $(x, x^*) \in E \times E^* \setminus G(T)$. Then $Tx \neq x^*$. For all $n \geq 1$, we can find $z_n \in E$ such that $\|z_n\| = 1$ and

$$\langle z_n, Tx - x^* \rangle \to -\|Tx - x^*\| \quad \text{as } n \to \infty. \tag{2.1.1}$$

For all $n \geq 1$, let $w_n := x + z_n/n \neq x$. Then $\|Tw_n - Tx\| = \|Tz_n\|/n \leq \|T\|/n$ hence

$$\|Tw_n - Tx\| \to 0 \quad \text{and} \quad \|Tw_n - x^*\| \to \|Tx - x^*\| \neq 0 \quad \text{as } n \to \infty. \tag{2.1.2}$$

Now, for all sufficiently large $n \geq 1$, $Tw_n \neq x^*$ and we have the inequality

$$\frac{|\langle w_n - x, Tw_n - Tx \rangle|}{\|w_n - x\| \|Tw_n - x^*\|} \leq \frac{\|Tw_n - Tx\|}{\|Tw_n - x^*\|}.$$

Combining this with (2.1.2), we obtain that

$$\frac{\langle w_n - x, Tw_n - Tx \rangle}{\|w_n - x\| \|Tw_n - x^*\|} \to 0 \quad \text{as } n \to \infty. \tag{2.1.3}$$

On the other hand, from (2.1.1) and (2.1.2),

$$\frac{\langle w_n - x, Tx - x^* \rangle}{\|w_n - x\| \|Tw_n - x^*\|} = \frac{\langle z_n/n, Tx - x^* \rangle}{\|z_n/n\| \|Tw_n - x^*\|} \to \frac{-\|Tx - x^*\|}{\|Tx - x^*\|} = -1 \quad \text{as } n \to \infty.$$

Adding this to (2.1.3), we obtain that

$$\frac{\langle w_n - x, Tw_n - x^*\rangle}{\|w_n - x\|\|Tw_n - x^*\|} \to -1 \quad \text{as } n \to \infty.$$

This completes the proof that T is of type (ANA). As we have already observed, this implies that T is maximal monotone. ▮

As we stated in Problem 0.6, we would like to have an example of a linear maximal monotone operator that is not of type (ANA). It is clear from Theorem 2.1 that any such example must be unbounded. One common way of constructing unbounded linear operators is to use differentiation. In Example 2.2 below, we discuss a simple case of the differentiation technique (which, unfortunately, does not provide the example that we want).

Example 2.2. Let Lip[0, 1] be the set of all real Lipschitz functions on [0, 1]. If $f \in \mathrm{Lip}[0,1]$ then f' exists almost everywhere on [0, 1], and its absolute value is bounded (by the Lipschitz constant of f). Let

$$D := \{[f]\colon\ f \in \mathrm{Lip}[0,1],\ f(0) = 0\},$$

where $[f]$ is the equivalence class of f in $\mathrm{L}^1[0,1]$. If $x \in D$ then there exists a unique element f_x of Lip[0, 1] such that $f_x(0) = 0$ and $[f_x] = x$. Define $T\colon\ D \mapsto \mathrm{L}^\infty[0,1]$ by

$$Tx := [f_x'] \quad (x \in D),$$

where $[f_x']$ is the equivalence class of f_x' in $\mathrm{L}^\infty[0,1]$. T is clearly linear. Further, if $x \in D$ then

$$\langle x, Tx\rangle = \int f_x f_x' = \frac{1}{2} f_x(1)^2 \ge 0,$$

from which T is monotone. Despite the fact that T is not continuous, it is nevertheless maximal monotone of type (ANA). This follows from Theorem 2.3 below and the fact that $R(T) = \mathrm{L}^\infty[0,1] = E^*$. ($R(T)$ is the "range" of T, i.e., $T(D)$.) So, as we have already observed, T will not provide us with the example that we want. We refer the reader to [5], Example 4.2 for many other properties of T.

Theorem 2.3(a) extends [5], Proposition 3.1(j). We will actually prove a result stronger than Theorem 2.3(b) in Corollary 3.7. We introduce the following notation: if $x \in E$, $\widehat{x}$ stands for the canonical image of x in the bidual E^{**} of E; further, we write

$$N(T) := \{x \in D\colon\ Tx = 0\}.$$

Theorem 2.3. *Let D be a subspace of E and $T\colon D \mapsto E^*$ be linear and monotone and $R(T) = E^*$. Then*
(a) T^{-1} exists as a single–valued monotone linear operator from E^* into E.
(b) T is maximal monotone of type (ANA).

Proof. Define $S\colon E^* \mapsto 2^{E^{**}}$ by $G(S) := \{(Tx, \widehat{x})\colon\ x \in D\}$. S is monotone and, since $R(T) = E^*$,

$$x^* \in E^* \quad \Longrightarrow \quad Sx^* \neq \emptyset.$$

It follows from the local boundedness theorem (again, see [4] Theorem 2.28, p. 28) that there exist $\delta > 0$ and $M \geq 0$ such that

$$(x^*, x^{**}) \in G(S) \text{ and } \|x^*\| < \delta \quad \Longrightarrow \quad \|x^{**}\| \leq M. \tag{2.3.1}$$

We now prove that

$$N(T) = \{0\}, \tag{2.3.2}$$

from which (a) follows easily. So suppose that $x \in N(T)$. Let $n \geq 1$. Then $nx \in N(T)$ and so

$$(0, n\widehat{x}) = (T(nx), \widehat{nx}) \in G(S).$$

Since $\|0\| \leq \delta$, it follows from (2.3.1) that $\|n\widehat{x}\| \leq M$, i.e., $\|x\| \leq M/n$. Letting $n \to \infty$, we obtain $x = 0$, which establishes (2.3.2) and completes the proof of (a).
(b) Define the monotone linear map $V\colon E^* \mapsto E^{**}$ by $Vx^* := \widehat{T^{-1}x^*}$. It is clear from Theorem 2.1 that V is of type (ANA), from which it follows easily that T is of type (ANA) also. As we have already observed, this implies that T is maximal monotone. ∎

3. Ultramaximal monotone linear operators

Since Theorem 2.3 tells us that a linear solution to Problem 0.6 cannot be surjective, we now consider a modification of the differentiation technique already discussed in Example 2.2 that yields a non–surjective operator.

Example 3.1. Let

$$D := \{[f]\colon\ f \in \mathrm{Lip}[0,1],\ f(0) = f(1)\},$$

where $[f]$ is the equivalence class of f in $\mathrm{L}^1[0,1]$. If $x \in D$ then there exists a unique element f_x of $\mathrm{Lip}[0,1]$ such that $f_x(0) = f_x(1)$ and $[f_x] = x$. Define $T\colon\ D \mapsto \mathrm{L}^\infty[0,1]$ by

$$Tx := [f_x'] \quad (x \in D),$$

where $[f_x']$ is the equivalence class of f_x' in $\mathrm{L}^\infty[0,1]$. T is clearly linear. Further, if $x \in D$ then

$$\langle x, Tx\rangle = \int f_x f_x' = \frac{1}{2} f_x(1)^2 - \frac{1}{2} f_x(0)^2 = 0,$$

from which T is skew, and so certainly monotone. (This example is taken from [5], Example 6.2.) Since T is unbounded and not surjective, it is a reasonable candidate to be a linear maximal monotone operator that is not of type (ANA). However, we will see that T does not provide the example that we want. On the contrary, it will follow from Corollary 3.9 that T is even of type (NA). It will be convenient to introduce some additional notation in order to establish this.

Definition 3.2. Let D be a subspace of E and $T: D \mapsto E^*$ be linear and monotone. We say that T is *ultramaximal monotone* if the following is true: if $(v^*, v^{**}) \in E^* \times E^{**}$ and

$$s \in D \quad \Longrightarrow \quad \langle Ts - v^*, \widehat{s} - v^{**} \rangle \geq 0 \tag{3.2.1}$$

then

$$\text{there exists } w \in D \text{ such that } (v^*, v^{**}) = (Tw, \widehat{w}). \tag{3.2.2}$$

(This is equivalent to the statement that the multifunction S introduced in the proof of Theorem 2.3(a) is maximal monotone.) It was proved in [5], Theorem 6.4 that if T is an ultramaximal monotone linear map then T is maximal monotone of type (D) and locally maximal monotone, and T^* is monotone. Also, an example of a bounded monotone linear operator T was given in [5], Example 6.5 such that T is not ultramaximal monotone but all the other conditions mentioned above are satisfied.

Here, our main result about ultramaximal linear operators is Theorem 3.5. Before embarking on that, we give two preliminary results.

Lemma 3.3. *Let $x \in E$ and $x^* \in E^*$.*
(a) Then

$$\|x\|^2 + \|x^*\|^2 + 2\langle x, x^* \rangle \geq 0. \tag{3.3.1}$$

(b) If we have equality in (3.3.1) *then $\|x\| = \|x^*\|$ and so $\langle x, x^* \rangle = -\|x\|\|x^*\|$.*

Proof. $\|x\|^2 + \|x^*\|^2 + 2\langle x, x^* \rangle \geq \|x\|^2 + \|x^*\|^2 - 2\|x\|\|x^*\| = (\|x\| - \|x^*\|)^2$. ▌

Theorem 3.4. *Let D be a nonempty convex subset of a vector space, F be a Banach space, f: $D \mapsto \mathbb{R}$ be convex and g: $D \mapsto F$ be affine. Then* (3.4.1) $\Longleftrightarrow$ (3.4.2).

$$a \in D \quad \Longrightarrow \quad f(a) + \|g(a)\|^2 \geq 0. \tag{3.4.1}$$

$$\left.\begin{array}{l} \textit{There exists } y^* \in F^* \textit{ such that} \\ a \in D \quad \Longrightarrow \quad f(a) - 2\langle g(a), y^* \rangle \geq \|y^*\|^2. \end{array}\right\} \tag{3.4.2}$$

Proof. See [11], Theorem 7.2, p. 27 or [12], Theorem 3. ▌

We now come to our main result about ultramaximal monotone linear operators.

Theorem 3.5. Let D be a subspace of E and $T: D \mapsto E^*$ be linear and ultramaximal monotone. Then T is of type (NA).

Proof. Let $(x, x^*) \in E \times E^* \setminus G(T)$. Write $F := E \times E^*$ with $\|(x, x^*)\| := \sqrt{\|x\|^2 + \|x^*\|^2}$, and define f: $D \mapsto \mathbb{R}$ and g: $D \mapsto F$ by

$$f(s) := 2\langle s - x, Ts - x^* \rangle \quad \text{and} \quad g(s) := (s - x, Ts - x^*) \quad (s \in D).$$

It follows from Theorem 1.1(a) that f is convex, and g is obviously affine. Thus, from Lemma 3.3(a), for all $s \in D$,

$$f(s) + \|g(s)\|^2 = \|s - x\|^2 + 2\langle s - x, Ts - x^* \rangle + \|Ts - x^*\|^2 \geq 0.$$

Since any element y^* of F^* can be written in the form (z^*, z^{**}) for some $(z^*, z^{**}) \in E^* \times E^{**}$ with $\|y^*\| = \sqrt{\|z^*\|^2 + \|z^{**}\|^2}$, it follows from Theorem 3.4 that there exists $(z^*, z^{**}) \in E^* \times E^{**}$ such that

$$s \in D \quad \Longrightarrow \quad 2\langle s - x, Ts - x^*\rangle - 2\langle s - x, z^*\rangle - 2\langle Ts - x^*, z^{**}\rangle \geq \|z^*\|^2 + \|z^{**}\|^2.$$

Adding $2\langle z^*, z^{**}\rangle$ to both sides, we obtain:

$$s \in D \quad \Longrightarrow \quad 2\langle Ts - x^* - z^*, \widehat{s} - \widehat{x} - z^{**}\rangle \geq \|z^*\|^2 + 2\langle z^*, z^{**}\rangle + \|z^{**}\|^2. \tag{3.5.1}$$

Thus, from Lemma 3.3(a),

$$s \in D \quad \Longrightarrow \quad \langle Ts - x^* - z^*, \widehat{s} - \widehat{x} - z^{**}\rangle \geq 0.$$

Since T is ultramaximal monotone, there exists $w \in D$ such that

$$(Tw, \widehat{w}) = (x^* + z^*, \widehat{x} + z^{**}). \tag{3.5.2}$$

Substituting $s = w$ in (3.5.1), we obtain $\|z^*\|^2 + 2\langle z^*, z^{**}\rangle + \|z^{**}\|^2 \leq 0$ and so, from Lemma 3.3(b),

$$\|z^*\| = \|z^{**}\| \quad \text{and} \quad \langle z^*, z^{**}\rangle = -\|z^*\|\|z^{**}\|. \tag{3.5.3}$$

We have from (3.5.2) that $z^* = Tw - x^*$ and $z^{**} = \widehat{w} - \widehat{x}$. Consequently, we derive from (3.5.3) that $\|Tw - x^*\| = \|\widehat{w} - \widehat{x}\|$ and $\langle Tw - x^*, \widehat{w} - \widehat{x}\rangle = -\|Tw - x^*\|\|\widehat{w} - \widehat{x}\|$, that is to say

$$\|w - x\| = \|Tw - x^*\| \quad \text{and} \quad \langle w - x, Tw - x^*\rangle = -\|w - x\|\|Tw - x^*\|.$$

Since $(x, x^*) \notin G(T)$, at least one of $\|w - x\|$ and $\|Tw - x^*\|$ is nonzero. It then follows from the above that both are nonzero. This completes the proof that T is of type (NA). ▌

We next give a simple sufficient condition for T to be ultramaximal monotone and hence, by virtue of the above theorem, of type (NA) also. If $F \subset E$, we write

$$F^\perp := \{x^* \in E^*\colon \text{ for all } x \in F,\ \langle x, x^*\rangle = 0\}.$$

Consequently, if $F \subset E^*$, then

$$F^\perp := \{x^{**} \in E^{**}\colon \text{ for all } x^* \in F,\ \langle x^*, x^{**}\rangle = 0\}.$$

Theorem 3.6. *Let D be a subspace of E, $T\colon D \mapsto E^*$ be linear and monotone, $N(T)^\perp \subset R(T)$ and $R(T)^\perp \subset \widehat{N(T)}$. Then T is ultramaximal monotone, and hence of type (NA).*

Proof. We shall suppose that $(v^*, v^{**}) \in E^* \times E^{**}$ satisfies (3.2.1), and we shall show that (3.2.2) is satisfied. This proof is in two parts. In the first part, we shall show that

$$v^* \in N(T)^\perp. \tag{3.6.1}$$

To this end, let $s \in N(T)$. Let λ be an arbitrary element of $\mathbb{R}$. Then $\lambda s \in N(T) \subset D$ hence, from (3.2.1),

$$\langle T(\lambda s) - v^*, \widehat{\lambda s} - v^{**} \rangle \geq 0,$$

that is to say, $\lambda\langle s, v^* \rangle \leq \langle v^*, v^{**} \rangle$. Since this hold for all λ, it follows that $\langle s, v^* \rangle = 0$, which gives (3.6.1). By hypothesis, $v^* \in R(T)$, so there exists $v \in D$ such that

$$Tv = v^*. \tag{3.6.2}$$

In the second part of the proof, we establish that

$$\widehat{v} - v^{**} \in R(T)^{\perp}. \tag{3.6.3}$$

To this end, let $u^* \in R(T)$. Fix $u \in D$ so that

$$Tu = u^*. \tag{3.6.4}$$

Again, let λ be an arbitrary element of $\mathbb{R}$. From (3.2.1) with $s := v + \lambda u$,

$$\langle T(v + \lambda u) - v^*, \widehat{v} + \lambda \widehat{u} - v^{**} \rangle \geq 0.$$

Using (3.6.2) and (3.6.4), this simplifies to

$$\lambda^2 \langle u, u^* \rangle + \lambda \langle u^*, \widehat{v} - v^{**} \rangle \geq 0.$$

Since this holds for all λ, it follows that $\langle u^*, \widehat{v} - v^{**} \rangle = 0$, which gives (3.6.3). By hypothesis, $\widehat{v} - v^{**} \in \widehat{N(T)}$, hence there exists $w \in E$ such that $\widehat{w} = v^{**}$ and $v - w \in N(T)$. It now follows that $w \in D$ and, from (3.6.2), that $Tw = Tv = v^*$. Thus we have established (3.2.2), which completes the proof that T is ultramaximal monotone. It now follows from Theorem 3.5 that T is also of type (NA). ▌

Our next result strengthens Theorem 2.3(b). (See also [5], Theorem 6.4((a)$\Longrightarrow$(b)).) It implies that the operator T of Example 2.2 is actually of type (NA).

Corollary 3.7. *Let D be a subspace of E and $T: D \mapsto E^*$ be linear and monotone and $R(T) = E^*$. Then T is ultramaximal monotone, and hence of type (NA).*

Proof. This is immediate from Theorem 3.6 since $R(T)^{\perp} = \{0\}$. ▌

If $F \subset E^*$, we write

$$F_{\perp} := \{x \in E\colon \text{ for all } x^* \in F,\ \langle x, x^* \rangle = 0\}.$$

The set $F_{\perp}$ should not be confused with the set $F^{\perp}$ already defined.

Corollary 3.8. *Let D be a subspace of E, $T\colon D \mapsto E^*$ be linear and monotone, $\widehat{N(T)}$ be weak* closed in E^{**}, and $N(T)^\perp \subset R(T)$. Then T is ultramaximal monotone and of type (NA).*

Proof. Since $N(T)^\perp \subset R(T)$, $R(T)^\perp \subset N(T)^{\perp\perp}$. However, since $\widehat{N(T)}$ is weak* closed in E^{**},

$$N(T)^{\perp\perp} = \left(\widehat{N(T)}_\perp\right)^\perp = \widehat{N(T)},$$

and the result follows from Theorems 3.5 and 3.6. ▌

Since any finite dimensional subspace of a Hausdorff topological vector space is closed, the following result follows immediately from Corollary 3.8:

Corollary 3.9. *Let D be a subspace of E, $T\colon D \mapsto E^*$ be linear and monotone, $N(T)$ be finite dimensional, and $N(T)^\perp \subset R(T)$. Then T is ultramaximal monotone and of type (NA).*

If T is as in Example 3.1 then $N(T)$ is the subspace of $\mathrm{L}^1[0,1]$ consisting of the equivalence classes of the constant functions. Thus $N(T)$ has dimension one, and it is easily seen that $N(T)^\perp \subset R(T)$ also. Consequently, it follows from Corollary 3.9 that T is ultramaximal monotone and of type (NA).

Remark 3.10. It is worth noting that certain Banach spaces cannot support continuous linear ultramaximal monotone operators. We first observe that if $T\colon E \mapsto E^*$ is continuous, linear and ultramaximal monotone then

$$v^{**} \in E^{**} \text{ and } T^{**}v^{**} \in \widehat{E^*} \quad \Longrightarrow \quad v^{**} \in \widehat{E}. \tag{3.10.1}$$

To this end, suppose that $v^{**} \in E^{**}$ and $T^{**}v^{**} = \widehat{v^*}$ for some $v^* \in E^*$. Let s be an arbitrary element of E. Then

$$\langle Ts, \widehat{s} - v^{**}\rangle = \langle s, T^*(\widehat{s} - v^{**})\rangle = \langle T^*(\widehat{s} - v^{**}), \widehat{s}\rangle$$

and

$$\langle v^*, \widehat{s} - v^{**}\rangle = \langle \widehat{s} - v^{**}, \widehat{v^*}\rangle = \langle \widehat{s} - v^{**}, T^{**}v^{**}\rangle = \langle T^*(\widehat{s} - v^{**}), v^{**}\rangle.$$

Thus, by subtraction,

$$\langle Ts - v^*, \widehat{s} - v^{**}\rangle = \langle T^*(\widehat{s} - v^{**}), \widehat{s} - v^{**}\rangle,$$

and so, from the monotonicity of T^* mentioned in Definition 3.2, $\langle Ts - v^*, \widehat{s} - v^{**}\rangle \geq 0$. Since T is ultramaximal monotone, there exists $w \in E$ such that $(Tw, \widehat{w}) = (v^*, v^{**})$. In particular, $v^{**} \in \widehat{E}$, which gives (3.10.1). Since (3.10.1) is satisfied, we say that T is *Tauberian* — see Wilansky, [13], p. 175. It follows from [13], Theorem 11–4–2, p. 174–175 that if E is not reflexive then the closure in E^* of the image under T of the unit ball of E is not weakly compact in E^* hence, in the notation of Saab–Saab, [7], Definition 6, p. 378, E does not have "property (w)". Spaces with this property are discussed in [7], p.

378–380 and Proposition 47, p. 386. In particular, E cannot be of the form $c_0(\Gamma)$ or $C(\Omega)$ (Ω compact Hausdorff). See also the discussion in Bauschke, [2], p. 167–169.

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