New results on q-positivity

Y. García Ramos[1], J.E. Martínez-Legaz[2]*, S. Simons[3]
[1]Centro de Investigación de la Universidad del Pacífico, Lima, Peru
[2]Universitat Autònoma de Barcelona, Spain
[3]University of California, Santa Barbara, USA

Abstract

In this paper we discuss symmetrically self-dual spaces, which are simply real vector spaces with a symmetric bilinear form. Certain subsets of the space will be called q-positive, where q is the quadratic form induced by the original bilinear form. The notion of q-positivity generalizes the classical notion of the monotonicity of a subset of a product of a Banach space and its dual. Maximal q-positivity then generalizes maximal monotonicity. We discuss concepts generalizing the representations of monotone sets by convex functions, as well as the number of maximally q-positive extensions of a q-positive set. We also discuss symmetrically self-dual Banach spaces, in which we add a Banach space structure, giving new characterizations of maximal q-positivity. The paper finishes with two new examples.

1 Introduction

In this paper we discuss symmetrically self-dual spaces, which are simply real vector spaces with a symmetric bilinear form. Certain subsets of the space will be called q-positive, where q is the quadratic form induced by the original bilinear form. The notion of q-positivity generalizes the classical notion of the monotonicity of a subset of a product of a Banach space and its dual. Maximal q-positivity then generalizes maximal monotonicity.
A modern tool in the theory of monotone operators is the representation of monotone sets by convex functions. We extend this tool to the setting of q-positive sets. We discuss the notion of the intrinsic conjugate of a proper convex function on an SSD space. To each nonempty subset of an SSD space, we associate a convex function, which generalizes the function originally introduced by Fitzpatrick for the monotone case in [3]. In the same paper he posed a problem on convex representations of monotone sets, to which we give a partial solution in the more general context of this paper.

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We prove that maximally q-positive convex sets are always affine, thus extending a previous result in the theory of monotone operators [1, 5].
We discuss the number of maximally q-positive extensions of a q-positive set. We show that either there are an infinite number of such extensions or a unique extension, and in the case when this extension is unique we characterize it. As a consequence of this characterization, we obtain a sufficient condition for a monotone set to have a unique maximal monotone extension to the bidual.
We then discuss symmetrically self-dual Banach spaces, in which we add a Banach space structure to the bilinear structure already considered. In the Banach space case, this corresponds to considering monotone subsets of the product of a reflexive Banach space and its dual. We give new characterizations of maximally q-positive sets, and of minimal convex functions bounded below by q.
We give two examples of q-positivity: Lipschitz mappings between Hilbert spaces, and closed sets in a Hilbert space.

2 Preliminaries

We will work in the setting of symmetrically self-dual spaces, a notion introduced in [10]. A *symmetrically self-dual (SSD) space* is a pair $(B, \lfloor\cdot,\cdot\rfloor)$ consisting of a nonzero real vector space B and a symmetric bilinear form $\lfloor\cdot,\cdot\rfloor : B \times B \to \mathbb{R}$. The bilinear form $\lfloor\cdot,\cdot\rfloor$ induces the quadratic form q on B defined by $q(b) = \frac{1}{2}\lfloor b, b\rfloor$. A nonempty set $A \subset B$ is called *q-positive* [10, Definition 19.5] if $b, c \in A \Rightarrow q(b - c) \geq 0$. A set $M \subset B$ is called *maximally* q-positive [10, Definition 20.1] if it is q-positive and not properly contained in any other q-positive set. Equivalently, a q-positive set A is maximally q-positive if every $b \in B$ which is *q-positively related* to A (i.e. $q(b - a) \geq 0$ for every $a \in A$) belongs to A. The set of all elements of B that are q-positively related to A will be denoted by A^π. The closure of A with respect to the (possibly non Hausdorff) weak topology $w(B, B)$ will be denoted by A^w.
Given an arbitrary nonempty set $A \subset B$, the function $\Phi_A : B \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\Phi_A(x) := q(x) - \inf_{a \in A} q(x - a) = \sup_{a \in A}\{\lfloor x, a\rfloor - q(a)\}.$$

This generalizes the *Fitzpatrick function* from the theory of monotone operators. It is easy to see that Φ_A is a proper $w(B, B)$-lsc convex function. If M is maximally q-positive then

$$\Phi_M(b) \geq q(b), \quad \forall\, b \in B, \tag{1}$$

and

$$\Phi_M(b) = q(b) \Leftrightarrow b \in M. \tag{2}$$

A useful characterization of A^π is the following:

$$b \in A^\pi \text{ if and only if } \Phi_A(b) \leq q(b). \tag{3}$$

The set of all proper convex functions $f : B \to \mathbb{R} \cup \{+\infty\}$ satisfying $f \geq q$ on B will be denoted by $\mathcal{PC}_q(B)$ and, if $f \in \mathcal{PC}_q(B)$,

$$\mathcal{P}_q(f) := \{b \in B : f(b) = q(b)\}. \tag{4}$$

We will say that the convex function $f : B \to \mathbb{R} \cup \{+\infty\}$ is a *q-representation* of a nonempty set $A \subset B$ if $f \in \mathcal{PC}_q(B)$ and $\mathcal{P}_q(f) = A$. In particular, if $A \subset B$ admits a q-representation, then it is q-positive [10, Lemma 19.8]. The converse is not true in general, see for example [10, Remark 6.6].

A q-positive set in an SSD space having a $w(B,B)$-lsc q-representation will be called q-representable (q-representability is identical with $\mathcal{S}$-q-positivity as defined in [9, Def. 6.2] in a more restrictive situation). By (1) and (2), every maximally q-positive set is q-representable.

If B is a Banach space, we will denote by $\langle \cdot, \cdot \rangle$ the duality products between B and B^* and between B^* and the bidual space B^{**}, and the norm in B^* will be denoted by $\|\cdot\|$ as well. The topological closure, the interior and the convex hull of a set $A \subset B$ will be denoted respectively by $\overline{A}$, $intA$ and $convA$. The indicator function $\delta_A : B \to \mathbb{R} \cup \{+\infty\}$ of $A \subset B$ is defined by

$$\delta_A(x) := \begin{cases} 0 & \text{if } x \in A \\ +\infty & \text{if } x \notin A \end{cases}.$$

The convex envelope of $f : B \to \mathbb{R} \cup \{+\infty\}$ will be denoted by $conv\, f$.

3 SSD spaces

Following the notation of [6], for a proper convex function $f : B \to \mathbb{R} \cup \{+\infty\}$, we will consider its intrinsic (Fenchel) conjugate $f^{@} : B \to \mathbb{R} \cup \{+\infty\}$ with respect to the pairing $\lfloor \cdot, \cdot \rfloor$:

$$f^{@}(b) := \sup\{\lfloor c, b \rfloor - f(c) : c \in B\}.$$

Proposition 1 ([10, 6]) *Let A be a q-positive subset of an SSD space B. The following statements hold:*
(1) For every $b \in B$, $\Phi_A(b) \leq \Phi_A^{@}(b)$ and $q(b) \leq \Phi_A^{@}(b)$;
(2) For every $a \in A$, $\Phi_A(a) = q(a) = \Phi_A^{@}(a)$;
(3) $\Phi_A^{@}$ is the largest $w(B,B)$-lsc convex function majorized by q on A;
(4) A is q-representable if, and only if, $\mathcal{P}_q(\Phi_A^{@}) \subset A$;
(5) A is q-representable if, and only if, for all $b \in B$ such that, for all $c \in B$, $\lfloor c, b \rfloor \leq \Phi_A(c) + q(b)$, one has $b \in A$.

Proof. *(1)* and *(2)*. Let $a \in A$ and $b \in B$. Since A is q-positive, the infimum $\inf_{a' \in A} q(a - a')$ is attained at $a' = a$; hence we have the first equality in *(2)*. Using this equality, one gets

$$\Phi_A^{@}(b) = \sup_{c \in B}\{\lfloor c, b \rfloor - \Phi_A(c)\} \geq \sup_{a \in A}\{\lfloor a, b \rfloor - \Phi_A(a)\} = \sup_{a \in A}\{\lfloor a, b \rfloor - q(a)\} = \Phi_A(b),$$

which proves the first inequality in *(1)*. In view of this inequality, given that $\Phi_A^@(b) = \sup_{c\in B}\{\lfloor c,b\rfloor - \Phi_A(c)\} \geq \lfloor b,b\rfloor - \Phi_A(b) = 2q\,(b) - \Phi_A(b)$, we have $\Phi_A^@(b) \geq \max\{2q\,(b) - \Phi_A(b), \Phi_A(b)\} = q\,(b) + |q\,(b) - \Phi_A(b)| \geq q\,(b)$, so that the second inequality in *(1)* holds true. From the definition of Φ_A it follows that $\Phi_A(c) \geq \lfloor c,a\rfloor - q(a)$ for every $c \in B$; therefore

$$\Phi_A^@(a) = \sup_{c\in B}\{\lfloor c,a\rfloor - \Phi_A(c)\} \leq q\,(a)\,.$$

From this inequality and the second one in *(1)* we obtain the second equality in *(2)*.
(3). Let f be a $w\,(B,B)$-lsc convex function majorized by q on A. Then, for all $b \in B$,

$$\begin{aligned}
\Phi_A(b) &= \sup_{a\in A}\{\lfloor b,a\rfloor - q\,(a)\} = \sup_{a\in A}\{\lfloor a,b\rfloor - q\,(a)\} \\
&\leq \sup_{a\in A}\{\lfloor a,b\rfloor - f\,(a)\} \leq \sup_{c\in B}\{\lfloor c,b\rfloor - f\,(c)\} = f^@\,(b)\,.
\end{aligned}$$

Thus $\Phi_A \leq f^@$ on B. Consequently $f^{@@} \leq \Phi_A^@$ on B. Since f is $w\,(B,B)$-lsc, from the (non Hausdorff) Fenchel-Moreau theorem [11, Theorem 10.1], $f \leq \Phi_A^@$ on B.
(4). We note from *(1)* and *(2)* that $\Phi_A^@ \in \mathcal{PC}_q(B)$ and $A \subset \mathcal{P}_q(\Phi_A^@)$. It is clear from these observations that if $\mathcal{P}_q(\Phi_A^@) \subset A$ then $\Phi_A^@$ is a $w\,(B,B)$-lsc q-representation of A. Suppose, conversely, that A is q-representable, so that there exists a $w(B;B)$-lsc function $f \in \mathcal{PC}_q(B)$ such that $\mathcal{P}_q(f) = A$. It now follows from *(3)* that $f \leq \Phi_A^@$ on A, and so $\mathcal{P}_q(\Phi_A^@) \subset \mathcal{P}_q(f) = A$.
(5). This statement follows from *(4)*, since the inequality $\lfloor c,b\rfloor \leq \Phi_A(c) + q\,(b)$ holds for all $c \in B$ if, and only if, $b \in \mathcal{P}_q(\Phi_A^@)$. ■

The next results should be compared with [9, Theorems 6.3.(b) and 6.5.(a)].

Corollary 2 *Let A be a q-positive subset of an SSD space B. Then $\mathcal{P}_q(\Phi_A^@)$ is the smallest q-representable superset of A.*

Proof. By Proposition 1.*(2)*, $\mathcal{P}_q(\Phi_A^@)$ is a q-representable superset of A. Let C be a q-representable superset of A. Since $A \subset C$, we have $\Phi_A \leq \Phi_C$ and hence $\Phi_C^@ \leq \Phi_A^@$. Therefore, by Proposition 1.*(4)*, $\mathcal{P}_q(\Phi_A^@) \subset \mathcal{P}_q(\Phi_C^@) \subset C$. ■

Corollary 3 *Let A be a q-positive subset of an SSD space B, and denote by C the smallest q-representable superset of A. Then $\Phi_C = \Phi_A$.*

Proof. Since $A \subset C$, we have $\Phi_A \leq \Phi_C$. On the other hand, by Corollary 2, $C = \mathcal{P}_q(\Phi_A^@)$; hence $\Phi_A^@$ is majorized by q on C. Therefore, by Proposition 1.*(3)*, $\Phi_A^@ \leq \Phi_C^@$. Since Φ_A and Φ_C are $w\,(B,B)$-lsc, from the (non Hausdorff) Fenchel-Moreau theorem [11, Theorem 10.1], $\Phi_C = \Phi_C^{@@} \leq \Phi_A^{@@} = \Phi_A$. We thus have $\Phi_C = \Phi_A$. ■

We continue with a result about the domain of $\Phi_A^@$ which will be necessary in the sequel.

Lemma 4 (about the domain of $\Phi_A^@$) *Let A be a q-positive subset of an SSD space B. Then,*

$$conv A \subset \mathrm{dom}\Phi_A^@ \subset conv^w A.$$

Proof. Since $\Phi_A^@$ coincides with q in A, we have that $A \subset \mathrm{dom}\Phi_A^@$, hence from the convexity of $\Phi_A^@$ it follows that

$$conv A \subset \mathrm{dom}\Phi_A^@.$$

On the other hand, from Proposition 1*(3)* $\Phi_A^@ + \delta_{conv^w A} \leq \Phi_A^@$, because $\Phi_A^@ + \delta_{conv^w A}$ is $w\,(B,B)$-lsc, convex and majorized by q on A. Thus,

$$\mathrm{dom}\Phi_A^@ \subset \mathrm{dom}\left(\Phi_A^@ + \delta_{conv^w A}\right) \subset conv^w A.$$

This finishes the proof. ■

3.1 On a problem posed by Fitzpatrick

Let B be an SSD space and $f : B \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function. The generalized Fenchel-Young inequality establishes that

$$f(a) + f^@(b) \geq \lfloor a, b \rfloor, \quad \forall\, a, b \in B. \tag{5}$$

We define the *q-subdifferential* of f at $a \in B$ by

$$\partial_q f(a) := \left\{ b \in B : f(a) + f^@(b) = \lfloor a, b \rfloor \right\}$$

and the set

$$G_f := \{b \in B : b \in \partial_q f(b)\}.$$

In this Subsection we are interested in identifying sets $A \subset B$ with the property that $G_{\Phi_A} = A$. The problem of characterizing such sets is an abstract version of an open problem on monotone operators posed by Fitzpatrick [3, Problem 5.2].

Proposition 5 *Let B be an SSD space and $f : B \to \mathbb{R}\cup\{+\infty\}$ be a $w\,(B,B)$-lsc proper convex function such that $G_f \neq \emptyset$. Then the set G_f is q-representable.*

Proof. Taking the $w\,(B,B)$-lsc proper convex function $h := \frac{1}{2}(f + f^@)$, we have that

$$G_f = \mathcal{P}_q(h).$$

■

Theorem 6 *Let A be a q-positive subset of an SSD space B. Then*
(1) $A \subset \mathcal{P}_q(\Phi_A^@) \subset G_{\Phi_A} \subset A^\pi \cap conv^w A$;
(2) If A is convex and $w\,(B,B)$-closed,

$$A = G_{\Phi_A};$$

(3) If A is maximally q-positive,

$$A = G_{\Phi_A}.$$

Proof. *(1)*. By Proposition 1*(2)*, we have the first inclusion in *(1)*. Let $b \in \mathcal{P}_q(\Phi_A^{@})$. Since $\Phi_A(b) \leq \Phi_A^{@}(b) = q(b)$, we get

$$2q(b) \leq \Phi_A(b) + \Phi_A^{@}(b) \leq 2q(b).$$

It follows that $b \in G_{\Phi_A}$. This shows that $\mathcal{P}_q(\Phi_A^{@}) \subset G_{\Phi_A}$.Using Proposition 1*(1)*, we infer that for any $a \in G_{\Phi_A}$, $\Phi_A(a) \leq q(a)$, so $G_{\Phi_A} \subset A^{\pi}$. On the other hand, since $G_{\Phi_A} \subset \operatorname{dom}\Phi_A^{@}$, Lemma 4 implies that $G_{\Phi_A} \subset conv^w A$. This proves the last inclusion in *(1)*.
(2). This is immediate from *(1)* since $conv^w A = A$.
(3). This follows directly from Proposition 5 and *(1)*. ■

Proposition 7 *Let A be a nonempty subset of an SSD space B and let D be a $w(B,B)$-closed convex subset of B such that*

$$\Phi_A(b) \geq q(b) \qquad \forall\, b \in D. \tag{6}$$

Suppose that $A^{\pi} \cap D \neq \emptyset$. Then $A^{\pi} \cap D$ is q-representable.

Proof. We take $f = \Phi_A + \delta_D$; this function is $w(B,B)$-lsc, proper (because $A^{\pi} \neq \emptyset$) and convex. Let $b \in B$ be such that $f(b) \leq q(b)$, so

$$\Phi_A(b) \leq q(b) \text{ and } b \in D.$$

This implies that $b \in A^{\pi} \cap D$. From (6) we infer that $f(b) = \Phi_A(b) = q(b)$. It follows that $f \in \mathcal{PC}_q(B)$. It is easy to see that f is a q-representative function for $A^{\pi} \cap D$. ■

Proposition 8 *Let A be a q-positive subset of an SSD space B. If $C = A^{\pi} \cap conv^w A$ is q-positive, then*

$$C = G_{\Phi_C} = C^{\pi} \cap conv^w C.$$

Proof. Clearly $conv^w A \supset C$, from which $conv^w A \supset conv^w C$. Since $C \supset A$, $A^{\pi} \supset C^{\pi}$. Thus $C = A^{\pi} \cap conv^w A \supset C^{\pi} \cap conv^w C$. However, from Theorem 6*(1)*, $C \subset G_{\Phi_C} \subset C^{\pi} \cap conv^w C$. ■

Proposition 9 *Let A be a q-positive subset of an SSD space B. If*

$$\Phi_A(b) \geq q(b) \qquad \forall\, b \in conv^w A, \tag{7}$$

then

$$G_{\Phi_A} = \mathcal{P}_q(\Phi_A^{@}).$$

Proof. It is clear from Theorem 6*(1)* and (7) that, for all $b \in G_{\Phi_A}$, $\Phi_A(b) = q(b)$; thus $\Phi_A^{@}(b) = \lfloor b, b \rfloor - \Phi_A(b) = q(b)$, so $G_{\Phi_A} \subset \mathcal{P}_q(\Phi_A^{@})$. The opposite inclusion also holds, according to Theorem 6*(1)*. ■

Corollary 10 *Let A be a q-positive subset of an SSD space B. If $\Phi_A \in \mathcal{PC}_q(B)$, then*

$$G_{\Phi_A} = \mathcal{P}_q(\Phi_A^@).$$

Proposition 11 *Let A be a q-representable subset of an SSD space B. If $\Phi_A(b) \geq q(b)$ for all $b \in conv^w A$, then*

$$A = G_{\Phi_A}.$$

Proof. Since A is a q-representable set, $A = \mathcal{P}_q(f)$ for some $w(B,B)$-lsc $f \in \mathcal{PC}_q(B)$. By Proposition 1*(3)*, $f \leq \Phi_A^@$; hence, by Corollary **??**, $\mathcal{P}_q(f) \supset \mathcal{P}_q(\Phi_A^@) \supset A = \mathcal{P}_q(f)$, so that $A = \mathcal{P}_q(\Phi_A^@)$. The result follows by applying Proposition 9. ∎

Lemma 12 *Let A be a q-positive subset of an SSD space B. If for some topological vector space Y there exists a $w(B,B)$-continuous linear mapping $f : B \to Y$ satisfying*
(1) $f(A)$ is convex and closed,
(2) $f(x) = 0$ implies $q(x) = 0$,
then

$$\Phi_A(b) \geq q(b) \qquad \forall\, b \in conv^w A. \tag{8}$$

Proof. Since

$$f(A) \subset f\,(conv^w A) \subset \overline{conv} f(A) = f(A),$$

it follows that

$$f\,(conv^w A) = f(A).$$

Let $b \in conv^w A$. Then there exists $a \in A$ such that $f(b) = f(a)$, hence $f(a-b) = 0$. By *2*, $q(a-b) = 0$, and so we obviously have (8). ∎

Corollary 13 *Let $T : X \rightrightarrows X^*$ be a representable monotone operator on a Banach space X. If $DomT$ $(RanT)$ is convex and closed, then*

$$T = G_{\varphi_T}.$$

Proof. Take $f = P_X$ or $f = P_{X^*}$, the projections onto X and X^*, respectively, in Lemma 12 and apply Proposition 11. Notice that when $X \times X^*$ is endowed with the topology $w(X \times X^*, X^* \times X)$, P_X and P_{X^*} are continuous onto X with its weak topology and X^* with the weak* topology, respectively. ∎

3.2 Maximally q-positive convex sets

The following result extends [5, Lemma 1.5] (see also [1, Thm. 4.1]).

Theorem 14 *Let A be a maximally q-positive convex set in an SSD space B. Then A is actually affine.*

Proof. Take $x_0 \in A$. Clearly, the set $A - x_0$ is also maximally q-positive and convex. To prove that A is affine, we will prove that $A - x_0$ is a cone, that is,

$$\lambda (x - x_0) \in A - x_0 \qquad \text{for all } x \in A \text{ and } \lambda \geq 0, \tag{9}$$

and that it is symmetric with respect to the origin, that is,

$$-(x - x_0) \in A - x_0 \qquad \text{for all } x \in A. \tag{10}$$

Let $x \in A$ and $\lambda \geq 0$. If $\lambda \leq 1$, then $\lambda (x - x_0) = \lambda x + (1 - \lambda) x_0 - x_0 \in A - x_0$, since A is convex. If $\lambda \geq 1$, for every $y \in A$ we have $q(\lambda (x - x_0) - (y - x_0)) = \lambda^2 q\left(x - \left(\frac{1}{\lambda}(y - x_0) + x_0\right)\right) \geq 0$, since $\frac{1}{\lambda}(y - x_0) \in A - x_0$. Hence, as $A - x_0$ is maximally q-positive, $\lambda (x - x_0) \in A - x_0$ also in this case. This proves (9). To prove (10), let $x, y \in A$. Then $q(-(x - x_0) - (y - x_0)) = q((x + y - x_0) - x_0) \geq 0$, since $x + y - x_0 \in A$ (as $A - x_0$ is a convex cone) and $x_0 \in A$. Using that $A - x_0$ is maximally q-positive, we conclude that $-(x - x_0) \in A - x_0$, which proves (10). ■

3.3 About the number of maximally q-positive extensions of a q-positive set

Proposition 15 *Let $x_1, x_2 \in B$ be such that*

$$q(x_1 - x_2) \leq 0. \tag{11}$$

Then $\lambda x_1 + (1 - \lambda) x_2 \in \{x_1, x_2\}^{\pi\pi}$ for every $\lambda \in [0, 1]$.

Proof. Let $x \in \{x_1, x_2\}^{\pi}$. Since

$$q(x_1 - x_2) = q((x_1 - x) - (x_2 - x)) = q(x_1 - x) - \lfloor x_1 - x, x_2 - x \rfloor + q(x_2 - x),$$

(11) implies that

$$\lfloor x_1 - x, x_2 - x \rfloor \geq q(x_1 - x) + q(x_2 - x).$$

Then, writing $x_\lambda := \lambda x_1 + (1 - \lambda) x_2$,

$$\begin{aligned}
q(x_\lambda - x) &= q(\lambda (x_1 - x) + (1 - \lambda)(x_2 - x)) \\
&= \lambda^2 q(x_1 - x) + \lambda(1 - \lambda) \lfloor x_1 - x, x_2 - x \rfloor + (1 - \lambda)^2 q(x_2 - x) \\
&\geq \lambda^2 q(x_1 - x) + \lambda(1 - \lambda)(q(x_1 - x) + q(x_2 - x)) \\
&\quad + (1 - \lambda)^2 q(x_2 - x) \\
&= \lambda q(x_1 - x) + (1 - \lambda) q(x_2 - x) \geq 0.
\end{aligned}$$

■

We will use the following lemma:

Lemma 16 *Let $A \subset B$. Then $A^{\pi\pi\pi} = A^{\pi}$.*

Proof. Since q is an even function, from the definition of A^π it follows that $A \subset A^{\pi\pi}$. Replacing A by A^π in this inclusion, we get $A^\pi \subset A^{\pi\pi\pi}$. On the other hand, since the mapping $A \longmapsto A^\pi$ is inclusion reversing, from $A \subset A^{\pi\pi}$ we also obtain $A^{\pi\pi\pi} \subset A^\pi$. ■

Proposition 17 *Let A be a q-positive set. If A has more than one maximally q-positive extension, then it has a continuum of such extensions.*

Proof. Let M_1, M_2 be two different maximally q-positive extensions of A. By the maximality of M_1 and M_2, there exists $x_1 \in M_1$ and $x_2 \in M_2$ such that $q(x_1 - x_2) < 0$. Notice that $\{x_1, x_2\} \subset A^\pi$; hence, using proposition 15 and Lemma 16, we deduce that, for every $\lambda \in [0,1]$, $\lambda x_1 + (1-\lambda)\,x_2 \in \{x_1, x_2\}^{\pi\pi} \subset A^{\pi\pi\pi} = A^\pi$. This shows that, for each $\lambda \in [0,1]$, $A \cup \{x_\lambda\}$, with $x_\lambda := \lambda x_1 + (1-\lambda)\,x_2$, is a q-positive extension of A; since $q(x_{\lambda_1} - x_{\lambda_2}) = q\left((\lambda_1 - \lambda_2)\,(x_1 - x_2)\right) = (\lambda_1 - \lambda_2)^2\, q\,(x_1 - x_2) < 0$ for all $\lambda_1, \lambda_2 \in [0,1]$ with $\lambda_1 \neq \lambda_2$, the result follows using Zorn's Lemma. ■

3.4 Premaximally q-positive sets

Let $(B, \lfloor\cdot,\cdot\rfloor)$ be an SSD space.

Definition 18 *Let P be a q–positive subset of B. We say that P is* premaximally q–positive *if there exists a unique maximally q–positive superset of P. It follows from [9, Lemma 5.4] that this superset is P^π (which is identical with $P^{\pi\pi}$). The same reference also implies that*

$$P \text{ is premaximally } q\text{–positive} \iff P^\pi \text{ is } q\text{–positive.} \tag{12}$$

Lemma 19 *Let P be a q–positive subset of B and*

$$\Phi_P \geq q \text{ on } B. \tag{13}$$

Then P is premaximally q–positive and $P^\pi = \mathcal{P}_q(\Phi_P)$.

Proof. Suppose that M is a maximally q–positive subset of B and $M \supset P$. Let $b \in M$. Since M is q–positive, $b \in M^\pi \subset P^\pi$, thus $\Phi_P(b) \leq q(b)$. Combining this with (13), $\Phi_P(b) = q(b)$, and so $b \in \mathcal{P}_q(\Phi_P)$. Thus we have proved that $M \subset \mathcal{P}_q(\Phi_P)$. It now follows from the maximality of M and the q–positivity of $\mathcal{P}_q(\Phi_P)$ that $P^\pi = \mathcal{P}_q(\Phi_P)$. ■

The next result contains a partial converse to Lemma 19.

Lemma 20 *Let P be a premaximally q–positive subset of B. Then either (13) is true, or $P^\pi = \operatorname{dom}\Phi_P$ and P^π is an affine subset of B.*

Proof. Suppose that (13) is not true. We first show that

$$\operatorname{dom}\Phi_P \text{ is } q\text{–positive.} \tag{14}$$

Since (13) fails, we can first fix $b_0 \in B$ such that $(\Phi_P - q))(b_0) < 0$. Now let $b_1, b_2 \in \operatorname{dom} \Phi_P$. Let $\lambda \in]0,1[$. Then

$$(\Phi_P - q)\left((1-\lambda)b_0 + \lambda b_1\right) \le (1-\lambda)\Phi_P(b_0) + \lambda \Phi_P(b_1) - q\left((1-\lambda)b_0 + \lambda b_1\right). \tag{15}$$

Since $\Phi_P(b_1) \in \mathbb{R}$ and quadratic forms on finite–dimensional spaces are continuous, the right–hand expression in (15) converges to $\Phi_P(b_0) - q(b_0)$ as $\lambda \to 0+$. Now $\Phi_P(b_0) - q(b_0) < 0$ and so, for all sufficiently small $\lambda \in]0,1[$, $(\Phi_P - q)\left((1-\lambda)b_0 + \lambda b_1\right) < 0$, from which $(1-\lambda)b_0 + \lambda b_1 \in P^\pi$. Similarly, for all sufficiently small $\lambda \in]0,1[$, $(1-\lambda)b_0 + \lambda b_2 \in P^\pi$. Thus we can choose $\lambda_0 \in]0,1[$ such that both $(1-\lambda_0)b_0 + \lambda_0 b_1 \in P^\pi$ and $(1-\lambda_0)b_0 + \lambda b_2 \in P^\pi$. Since P^π is q–positive,

$$0 \le q\left([(1-\lambda_0)b_0 + \lambda_0 b_1] - [(1-\lambda_0)b_0 + \lambda b_2]\right) = \lambda_0^2 q(b_1 - b_2).$$

So we have proved that, for all $b_1, b_2 \in \operatorname{dom} \Phi_P$, $q(b_1 - b_2) \ge 0$. This establishes (14). Therefore, since $\operatorname{dom} \Phi_P \supset P$, we have $\operatorname{dom} \Phi_P \subset P^\pi$. On the other hand, if $b \in P^\pi$, then $\Phi_P(b) \le q(p)$, and so $b \in \operatorname{dom} \Phi_P$. This completes the proof that $P^\pi = \operatorname{dom} \Phi_P$. Finally, since $P^\pi (= \operatorname{dom} \Phi_P)$ is convex, Theorem 14 implies that P^π is an affine subset of B. ■

Our next result is a new characterization of premaximally q–positive sets.

Theorem 21 *Let P be a q–positive subset of B. Then P is premaximally q–positive if, and only if, either (13) is true or P^π is an affine subset of B.*

Proof. "Only if" is clear from Lemma 20. If, on the other hand, (13) is true then Lemma 19 implies that P is premaximally q–positive. It remains to prove that if P^π is an affine subset of B then P is premaximally q–positive. So let P^π be an affine subset of B. Suppose that $b_1, b_2 \in P^\pi$, and let $p \in P$. Since P is q–positive, $p \in P^\pi$, and since P^π is affine, $p + b_1 - b_2 \in P^\pi$, from which $q(b_1 - b_2) = q([p + b_1 - b_2] - p) \ge 0$. Thus we have proved that P^π is q–positive. It now follows from (12) that P is premaximally q–positive. ■

Corollary 22 *Let P be an affine q–positive subset of B. Then P is premaximally q–positive if and only if P^π is an affine subset of B.*

Proof. In view of Theorem 21, we only need to prove the "only if" statement. Assume that P is premaximally q–positive. Since the family of affine sets A such that $P \subset A \subset P^\pi$ is inductive, by Zorn's Lemma it has a maximal element M. Let $b \in P^\pi$, $m_1, m_2 \in M$, $p \in P$ and $\lambda, \mu, \nu \in \mathbb{R}$ be such that $\lambda + \mu + \nu = 1$. If $\lambda \ne 0$ then $q\left(\lambda b + \mu m_1 + \nu m_2 - p\right) = \lambda^2 q\left(b - \frac{1}{\lambda}\left(p - \mu m_1 - \nu m_2\right)\right) \ge 0$, since $\frac{1}{\lambda}\left(p - \mu m_1 - \nu m_2\right) \in M \subset P^\pi$ and P^π is q–positive (by [9, Lemma 5.4]). If, on the contrary, $\lambda = 0$ then $q\left(\lambda b + \mu m_1 + \nu m_2 - p\right) = q\left(\mu m_1 + \nu m_2 - p\right) \ge 0$, because in this case $\mu m_1 + \nu m_2 \in M \subset P^\pi$. Therefore $\lambda b + \mu m_1 + \nu m_2 \in P^\pi$. We have thus proved that the affine set generated by $M \cup \{b\}$ is contained in P^π. Hence, by the maximality of M, we have $b \in M$, and we conclude that $P^\pi = M$. ■

Definition 23 *Let E be a nonzero Banach space and A be a nonempty monotone subset of $E \times E^*$. We say that A is of type (NI) if,*

$$\text{for all } (y^*, y^{**}) \in E^* \times E^{**}, \quad \inf_{(a,a^*)\in A} \langle a^* - y^*, \widehat{a} - y^{**} \rangle \leq 0.$$

We define $\iota\colon\ E \times E^* \to E^* \times E^{**}$ by $\iota(x, x^*) = (x^*, \widehat{x})$, where $\widehat{x}$ is the canonical image of x in E^{**}. We say that A is unique *if there exists a unique maximally monotone subset M of $E^* \times E^{**}$ such that $M \supset \iota(A)$. We now write $B := E^* \times E^{**}$ and define $\lfloor \cdot, \cdot \rfloor\colon\ B \times B \to \mathbb{R}$ by $\lfloor (x^*, x^{**}), (y^*, y^{**}) \rfloor := \langle y^*, x^{**} \rangle + \langle x^*, y^{**} \rangle$. $(B, \lfloor \cdot, \cdot \rfloor)$ is an SSD space. Clearly, for all $(y^*, y^{**}) \in E^* \times E^{**}$, $q(y^*, y^{**}) = \langle y^*, y^{**} \rangle$. Now $\iota(A)$ is q–positive, A is of type (NI) exactly when $\Phi_{\iota(A)} \geq q$ on B, and A is unique exactly when $\iota(A)$ is premaximally q–positive. In this case, we write $\iota(A)^\pi$ for the unique maximally monotone subset of $E^* \times E^{**}$ that contains $\iota(A)$.*

Corollary 24(a) appears in [8], and Corollary 24(c) appears in [5, Theorem 1.6].

Corollary 24 *Let E be a nonzero Banach space and A be a nonempty monotone subset of $E \times E^*$.*
(a) If A is of type (NI) then A is unique and $\iota(A)^\pi = \mathcal{P}_q\left(\Phi_{\iota(A)}\right)$.
(b) If $\iota(A)^\pi$ is an affine subset of $E^* \times E^{**}$ then A is unique.
(c) Let A be unique. Then either A is of type (NI), or

$$\iota(A)^\pi = \{(y^*, y^{**}) \in E^* \times E^{**} \colon\ \inf_{(a,a^*)\in A} \langle a^* - y^*, \widehat{a} - y^{**} \rangle > -\infty\} \tag{16}$$

and $\iota(A)^\pi$ is an affine subset of $E^* \times E^{**}$.
(d) Let A be maximally monotone and unique. Then either A is of type (NI), or A is an affine subset of $E \times E^*$ and $A = \operatorname{dom} \varphi_A$, where φ_A is the Fitzpatrick function of A in the usual sense.

Proof. (a), (b) and (c) are immediate from Lemmas 19 and 20 and Theorem 21, and the terminology introduced in Definition 23.
(d). From (c) and the linearity of ι, $\iota^{-1}\left(\iota(A)^\pi\right)$ is an affine subset of $E \times E^*$. Furthermore, it is also easy to see that $\iota^{-1}\left(\iota(A)^\pi\right)$ is a monotone subset of $E \times E^*$. Since $A \subset \iota^{-1}\left(\iota(A)^\pi\right)$, the maximality of A implies that $A = \iota^{-1}\left(\iota(A)^\pi\right)$. Finally, it follows from (16) that $\iota^{-1}\left(i(A)^\pi\right) = \operatorname{dom} \varphi_A$. ∎

3.5 Minimal convex functions bounded below by q

This section extends some results of [7].

Lemma 25 *Let B be an SSD space and $f : B \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function. Then, for every $x, y \in B$ and every $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$, one has*

$$\alpha \max\{f(x), q(x)\} + \beta \max\left\{f^{@}(y), q(y)\right\} \geq q(\alpha x + \beta y).$$

Proof. Using (5) one gets

$$\begin{aligned}
q(\alpha x+\beta y) &= \alpha^2 q(x)+\alpha\beta\lfloor x,y\rfloor+\beta^2 q(y)\\
&\le \alpha^2 q(x)+\alpha\beta\left(f(x)+f^{@}(y)\right)+\beta^2 q(y)\\
&= \alpha\left(\alpha q(x)+\beta f(x)\right)+\beta\left(\alpha f^{@}(y)+\beta q(y)\right)\\
&\le \alpha\max\left\{f(x),q(x)\right\}+\beta\max\left\{f^{@}(y),q(y)\right\}.
\end{aligned}$$

■

Corollary 26 *Let B be an SSD space, $f : B \to \mathbb{R}\cup\{+\infty\}$ be a proper convex function such that $f \geq q$ and $x \in B$. Then there exists a convex function $h : B \to \mathbb{R}\cup\{+\infty\}$ such that*

$$f \geq h \geq q \qquad \textit{and} \qquad \max\left\{f^{@}(x), q(x)\right\} \geq h(x).$$

Proof. Let $h := conv\ \min\left\{f, \delta_{\{x\}} + \max\left\{f^{@}(x), q(x)\right\}\right\}$. Clearly, h is convex, $f \geq h$, and $\max\left\{f^{@}(x), q(x)\right\} \geq h(x)$; so, we only have to prove that $h \geq q$. Let $y \in B$. Since the functions f and $\delta_{\{x\}} + \max\left\{f^{@}(x), q(x)\right\}$ are convex, we have

$$\begin{aligned}
h(y) &= \inf_{\substack{u,v\in B\\ \alpha,\beta\geq 0,\ \alpha+\beta=1\\ \alpha u+\beta v=y}} \left\{\alpha f(u)+\beta\left(\delta_{\{x\}}(v)+\max\left\{f^{@}(x),q(x)\right\}\right)\right\}\\
&= \inf_{\substack{u\in B\\ \alpha,\beta\geq 0,\ \alpha+\beta=1\\ \alpha u+\beta x=y}} \left\{\alpha f(u)+\beta\max\left\{f^{@}(x),q(x)\right\}\right\}\\
&\geq \inf_{\substack{u\in B\\ \alpha,\beta\geq 0,\ \alpha+\beta=1\\ \alpha u+\beta x=y}} q(\alpha u+\beta x)=q(y),
\end{aligned}$$

the above inequality being a consequence of the assumption $f \geq q$ and Lemma 25. We thus have $h \geq q$. ■

Theorem 27 *Let B be an SSD space and $f : B \to \mathbb{R}\cup\{+\infty\}$ be a minimal element of the set of convex functions minorized by q. Then $f^{@} \geq f$.*

Proof. It is easy to see that f is proper. Let $x \in B$ and consider the function h provided by Corollary 26. By the minimality of f, we actually have $h = f$; on the other hand, from (5) it follows that $\frac{1}{2}\left(f(x)+f^{@}(x)\right) \geq \frac{1}{2}\lfloor x,x\rfloor = q(x)$. Therefore $f(x) = h(x) \leq \max\left\{f^{@}(x), q(x)\right\} \leq \max\left\{f^{@}(x), \frac{1}{2}\left(f(x)+f^{@}(x)\right)\right\}$; from these inequalities one easily obtains that $f(x) \leq f^{@}(x)$. ■

Proposition 28 *Let B be an SSD space and $f : B \to \mathbb{R}\cup\{+\infty\}$ be a convex function such that $f \geq q$ and $f^{@} \geq q$. Then*

$$conv\ \min\left\{f, f^{@}\right\} \geq q.$$

Proof. Since f and $f^{@}$ are convex, for every $x \in B$ we have

$$\begin{aligned} conv\ \min\left\{f, f^{@}\right\}(x) &= \inf_{\substack{u,v\in B \\ \alpha,\beta\geq 0,\ \alpha+\beta=1 \\ \alpha u+\beta v=x}} \left\{\alpha f(u) + \beta f^{@}(v)\right\} \\ &\geq \inf_{\substack{u,v\in B \\ \alpha,\beta\geq 0,\ \alpha+\beta=1 \\ \alpha u+\beta v=x}} q(\alpha u + \beta v) = q(x), \end{aligned}$$

the inequality following from the assumptions $f \geq q$ and $f^{@} \geq q$ and Lemma 25. ■

4 SSDB spaces

We say that $(B, \lfloor\cdot,\cdot\rfloor, \|\cdot\|)$ is a symmetrically self-dual Banach (SSDB) space if $(B, \lfloor\cdot,\cdot\rfloor)$ is an SSD space, $(B, \|\cdot\|)$ is a Banach space, the dual B^* is exactly $\{\lfloor\cdot, b\rfloor : b \in B\}$ and the map $i : B \to B^*$ defined by $i(b) = \lfloor\cdot, b\rfloor$ is a surjective isometry. In this case, the quadratic form q is continuous. By [6, Proposition 3] we know that every SSDB space is reflexive as a Banach space. If A is convex in an SSDB space then $A^w = \overline{A}$.
Let B be an SSDB space. In this case, for a proper convex function $f : B \to \mathbb{R} \cup \{+\infty\}$ it is easy to see that $f^{@} = f^* \circ i$, where $f^* : B^* \to \mathbb{R} \cup \{+\infty\}$ is the Banach space conjugate of f. Define $g_0 : B \to \mathbb{R}$ by $g_0(b) := \frac{1}{2}\|b\|^2$. Then for all $b^* \in B^*$, $g_0^*(b^*) = \frac{1}{2}\|b^*\|^2$.

4.1 A characterization of maximally q-positive sets in SSDB spaces

Lemma 29 *The set* $\mathcal{P}_q(g_0) = \{x \in B : g_0(x) = q(x)\}$ *is maximally* q*-positive and the set* $\mathcal{P}_{-q}(g_0) = \{x \in B : g_0(x) = -q(x)\}$ *is maximally* $-q$*-positive.*

Proof. To prove that $\mathcal{P}_q(g_0)$ is maximally q-positive, apply [9, Thm. 4.3(b)] (see also [6, Thm. 2.7]) after observing that $g_0^{@} = g_0^* \circ i = g_0$. Since replacing q by $-q$ changes $\mathcal{P}_q(g_0)$ into $\mathcal{P}_{-q}(g_0)$, it follows that $\mathcal{P}_{-q}(g_0)$ is maximally $-q$-positive too. ■

From now on, to distinguish the function Φ_A of $A \subset B$ corresponding to q from that corresponding to $-q$, we will use the notations $\Phi_{q,A}$ and $\Phi_{-q,A}$, respectively. Notice that $\Phi_{-q,\mathcal{P}_{-q}(g_0)}$ is finite-valued; indeed,

$$\begin{aligned} \Phi_{-q,\mathcal{P}_{-q}(g_0)}(x) &= \sup_{a\in\mathcal{P}_{-q}(g_0)} \{-\lfloor x, a\rfloor + q(a)\} \\ &= \sup_{a\in\mathcal{P}_{-q}(g_0)} \{-\langle x, i(a)\rangle - g_0(a)\} \\ &= \sup_{a\in\mathcal{P}_{-q}(g_0)} \{-\langle x, i(a)\rangle - g_0^*(i(a))\} \leq g_0(x). \end{aligned}$$

Theorem 30 *Let B be an SSDB space and A be a q-positive subset of B, and consider the following statements:*
(1) A is maximally q-positive.
(2) $A+C=B$ for every maximally $-q$-positive set $C\subseteq B$ such that $\Phi_{-q,C}$ is finite-valued.
(3) There exists a set $C\subseteq B$ such that $A+C=B$, and there exists $p\in C$ such that

$$q(z-p)<0 \qquad \forall\, z\in C\setminus\{p\}.$$

Then (1), (2) and (3) are equivalent.

Proof. *(1) $\Longrightarrow$ (2).* Let $x_0\in B$ and $A':=A-\{x_0\}$. We have

$$\Phi_{q,A'}(x)+\Phi_{-q,C}(-x)\geq q(x)-q(-x)=0 \qquad \forall\, x\in C.$$

Hence, as $\Phi_{-q,C}$ is continuous because it is lower semicontinuous and finite-valued, by the Fenchel-Rockafellar duality theorem there exists $y^*\in B^*$ such that

$$\Phi^*_{q,A'}(y^*)+\Phi^*_{-q,C}(y^*)\leq 0.$$

Since, by Proposition 1*(1)*, $\Phi^*_{q,A'}\circ i=\Phi^{@}_{q,A'}\geq\Phi_{q,A'}$ and, correspondingly, $\Phi^*_{-q,C}\circ(-i)=\Phi^{@}_{-q,C}\geq\Phi_{-q,C}$, we thus have

$$\begin{aligned} 0 &\geq \left(\Phi^*_{q,A'}\circ i\right)\left(i^{-1}(y^*)\right)+\left(\Phi^*_{-q,C}\circ(-i)\right)\left(-i^{-1}(y^*)\right)\\ &\geq \Phi_{q,A'}\left(i^{-1}(y^*)\right)+\Phi_{-q,C}\left(-i^{-1}(y^*)\right)\geq q\left(i^{-1}(y^*)\right)-q\left(-i^{-1}(y^*)\right)=0. \end{aligned}$$

Therefore

$$\Phi_{q,A'}\left(i^{-1}(y^*)\right)=q\left(i^{-1}(y^*)\right) \text{ and } \Phi_{-q,C}\left(-i^{-1}(y^*)\right)=-q\left(-i^{-1}(y^*)\right),$$

that is,

$$i^{-1}(y^*)\in A' \text{ and } -i^{-1}(y^*)\in C,$$

which implies that

$$x_0=x_0+i^{-1}(y^*)-i^{-1}(y^*)\in x_0+A'+C=A+C.$$

(2) $\Longrightarrow$ (3). Take $C:=\mathcal{P}_{-q}(g_0)$ (see Lemma 29) and $p:=0$.
(3) $\Longrightarrow$ (1). Let $x\in A^\pi$, and take p as in *(3)*. Since $x+p\in B=A+C$, we have $x+p=y+z$ for some $y\in A$ and $z\in C$. We have $x-y=z-p$; hence, since $x\in A^\pi$ and $y\in A$, we get $0\leq q(x-y)=q(z-p)\leq 0$. Therefore $q(z-p)=0$, which implies $z=p$. Thus from $x+p=y+z$ we obtain $x=y\in A$. This proves that $A^\pi\subset A$, which, together with the fact that A is q-positive, shows that A is maximally q-positive. ■

Corollary 31 *One has*

$$\mathcal{P}_q(g_0)+\mathcal{P}_{-q}(g_0)=B.$$

Proof. Since the set $\mathcal{P}_q(g_0)$ is maximally q-positive by Lemma 29, the result follows from the implication *(1) $\Longrightarrow$ (2)* in the preceding theorem. ■

4.2 Minimal convex functions on SSDB spaces bounded below by q

Theorem 32 *If B is an SSDB space and $f : B \to \mathbb{R} \cup \{+\infty\}$ is a minimal element of the set of convex functions minorized by q then $f = \Phi_M$ for some maximally q-positive set $M \subset B$.*

Proof. We first observe that f is lower semicontinuous; indeed, this is a consequence of its minimality and the fact that its lower semicontinuous closure is convex and minorized by q because q is continuous. By Theorem 27 and [9, Thm. 4.3(b)] (see also [6, Thm. 2.7]), the set $\mathcal{P}_q(f)$ is maximally q-positive, and hence $\Phi_{\mathcal{P}_q(f)} \geq q$. From [6, Thm. 2.2] we deduce that $\Phi_{\mathcal{P}_q(f)} \leq f$, which, by the minimality of f, implies that $\Phi_{\mathcal{P}_q(f)} = f$. ■

5 Examples

5.1 Lipschitz mappings between Hilbert spaces

Let $K > 0$. Let H_1, H_2 be two real Hilbert spaces and let $f : D \subset H_1 \to H_2$ be a K-Lipschitz mapping, i.e.

$$\|f(x_1) - f(y_1)\|_{H_2} \leq K\|x_1 - y_1\|_{H_1}, \qquad \forall\, x_1, y_1 \in D. \tag{17}$$

Remark 33 *It is well known that there exists an extension $\tilde{f} : H_1 \to H_2$ which is K-Lipschitz (see [4, 12]). Let $D \subset H_1$; we will denote by $\mathcal{F}(D)$ the family of K-Lipschitz mappings defined on D and by $\mathcal{F} := \mathcal{F}(H_1)$ the family of K-Lipschitz mappings defined everywhere on H_1.*

Proposition 34 *Let H_1, H_2 be two real Hilbert spaces, let $B = H_1 \times H_2$ and let $\lfloor\cdot,\cdot\rfloor : B \times B \to \mathbb{R}$ be the bilinear form defined by*

$$\lfloor (x_1, x_2), (y_1, y_2) \rfloor = K^2 \langle x_1, y_1 \rangle_{H_1} - \langle x_2, y_2 \rangle_{H_2}. \tag{18}$$

Then
(1) A nonempty set $A \subset B$ is q-positive if and only if there exists $f \in \mathcal{F}(P_{H_1}(A))$ such that $A = graph(f)$;
(2) A set $A \subset B$ is maximally q-positive if and only if there exists $f \in \mathcal{F}$ such that $A = graph(f)$.

Proof. *(1).* If $A = graph(f)$ with $f \in \mathcal{F}(P_{H_1}(A))$, it is easy to see that A is q-positive.
Assume that $A \subset B$ is q-positive. From the definition we have that for all $(x_1, y_1), (x_2, y_2) \in A$,

$$0 \leq q\left((x_1, y_1) - (x_2, y_2)\right) = \frac{1}{2}\left(K^2\|x_1 - x_2\|^2_{H_1} - \|y_1 - y_2\|^2_{H_2}\right).$$

Equivalently,

$$\|y_1 - y_2\|_{H_2} \leq K\|x_1 - x_2\|_{H_1}. \tag{19}$$

For $x \in P_{H_1}(A)$ we define $f(x) = \{y : (x, y) \in A\}$. We will show that f is a K-Lipschitz mapping. If $y_1, y_2 \in f(x)$, from (19) $y_1 = y_2$, so f is single-valued. Now, for $x_1, x_2 \in P_{H_1}(A)$ from (19) we have that

$$\|f(x_1) - f(x_2)\|_{H_2} \leq K\|x_1 - x_2\|_{H_1},$$

which shows that $f \in \mathcal{F}(P_{H_1}(A))$.
(2). Let $A \subset B$ be maximally q-positive. From *(1)*, there exists $f \in \mathcal{F}(P_{H_1}(A))$ such that $A = graph(f)$, and from the Kirszbraun-Valentine extension theorem [4, 12] f has a K-Lipschitz extension $\tilde{f}$ defined everywhere on H_1; since $graph(\tilde{f})$ is also q-positive we must have $f = \tilde{f}$. Now, let $f \in \mathcal{F}$ and $(x, y) \in H_1 \times H_2$ be q-positively related to every point in $graph(f)$. We have that $graph(f) \cup \{(x, y)\}$ is q-positive, so from (1) we easily deduce that $y = f(x)$. This finishes the proof of *(2)*. ■

Clearly, the $w(B, B)$ topology of the SSD space $(B, \lfloor\cdot,\cdot\rfloor)$ coincides with the weak topology of the product Hilbert space $H_1 \times H_2$. Therefore, every q-representable set is closed, so that it corresponds to a K-Lipschitz mapping with closed graph. Notice that, by the Kirszbraun-Valentine extension theorem, a K-Lipschitz mapping between two Hilbert spaces has a closed graph if and only if its domain is closed. The following example shows that not every K-Lipschitz mapping with closed domain has a q-representable graph.

Example 35 *Let $H_1 := \mathbb{R} =: H_2$ and let $f : \{0, 1\} \to H_2$ be the restriction of the identity mapping. Clearly, f is nonexpansive, so we will consider the SSD space corresponding to $K = 1$. Then we will show that the smallest q-representable set containing $graph(f)$ is the graph of the restriction $\widehat{f}$ of the identity to the closed interval $[0, 1]$. Notice that this graph is indeed q-representable, since the lsc function $\delta_{graph(\widehat{f})}$ belongs to $\mathcal{PC}_q(B)$ and one has $graph(\widehat{f}) = \mathcal{P}_q\left(\delta_{graph(\widehat{f})}\right)$. We will see that $graph(\widehat{f}) \subset \mathcal{P}_q(\varphi)$ for every $\varphi \in \mathcal{PC}_q(B)$ such that $graph(f) \subset \mathcal{P}_q(\varphi)$. Indeed, for $t \in [0, 1]$ one has $\varphi(t, t) \leq (1-t)\varphi(0, 0) + t\varphi(1, 1) = (1-t)q(0, 0) + tq(1, 1) = 0 = q(t, t)$; hence $(t, t) \in \mathcal{P}_q(\varphi)$, which proves the announced inclusion.*

Our next two results provide sufficient conditions for q-representability in the SSD space we are considering.

Proposition 36 *Let H_1, H_2, B and $\lfloor\cdot,\cdot\rfloor$ be as in Proposition 34 and let $f : D \subset H_1 \to H_2$ be a K'-Lipschitz mapping, with $0 < K' < K$. If D is nonempty and closed, then $graph(f)$ is q-representable.*

Proof. We will prove that $graph(f)$ coincides with the intersection of all the graphs of K-Lipschitz extensions $\tilde{f}$ of f to the whole of H_1. Since any such graph is maximally q-positive, we have $graph(\tilde{f}) = \mathcal{P}_q\left(\Phi_{graph(\tilde{f})}\right)$; hence that intersection is equal to $\mathcal{P}_q(\varphi)$, where φ denotes the supremum of all the functions $\Phi_{graph(\tilde{f})}$; so the considered intersection is q-representable. As one clearly has $graph(f) \subset \mathcal{P}_q(\varphi)$, we will only prove the opposite inclusion. Let $(x_1, x_2) \in \mathcal{P}_q(\varphi)$. Then $\tilde{f}(x_1) = x_2$ for every $\tilde{f}$, so it will suffice to prove that $x_1 \in D$. Assume, towards a contradiction, that $x_1 \notin D$. By the Kirszbraun-Valentine extension theorem, some $\tilde{f}$ is actually K'-Lipschitz. Take any $y \in H_2 \setminus \{x_2\}$ in the closed ball with center x_2 and radius $(K - K') \inf_{x \in D} \|x - x_1\|_{H_1}$. This number is indeed strictly positive, since D is closed. Let f_y be the extension of f to $D \cup \{x_1\}$ defined by $f_y(x_1) = y$. This mapping is K-Lipschitz, since for every $x \in D$ one has $\|f_y(x) - f_y(x_1)\|_{H_2} = \|f(x) - y\|_{H_2} \leq \|f(x) - x_2\|_{H_2} + \|x_2 - y\|_{H_2} = \left\|\tilde{f}(x) - \tilde{f}(x_1)\right\|_{H_2} + (K - K')\|x - x_1\|_{H_1} \leq K'\|x - x_1\|_{H_1} + (K - K')\|x - x_1\|_{H_1} = K\|x - x_1\|_{H_1}$. Using again the Kirszbraun-Valentine extension theorem, we get the existence of a K-Lipschitz extension $\widetilde{f_y} \in \mathcal{F}$ of f_y. Since $(x_1, x_2) \in \mathcal{P}_q(\varphi) \subset graph(\widetilde{f_y})$, we thus contradict $\widetilde{f_y}(x_1) = f_y(x_1) = y$. ■

Proposition 37 *Let H_1, H_2, B and $\lfloor \cdot, \cdot \rfloor$ be as in Proposition 34 and let $f : D \subset H_1 \to H_2$ be a K-Lipschitz mapping. If D is nonempty, convex, closed and bounded, then $graph(f)$ is q-representable.*

Proof. As in the proof of Proposition 36, it will suffice to show that $graph(f)$ coincides with the intersection of all the graphs of K-Lipschitz extensions $\tilde{f}$ of f to the whole of H_1, and we will do it by proving that for every point (x_1, x_2) in this intersection one necessarily has $x_1 \in D$. If we had $x_1 \notin D$, by the Hilbert projection theorem there would be a closest point $\overline{x}$ to x_1 in D, characterized by the condition $\langle x - \overline{x}, x_1 - \overline{x} \rangle \leq 0$ for all $x \in D$. Let $C := \sup_{x \in D}\{\|x - x_1\| + \|x - \overline{x}\|\}$. Since $x_1 \neq \overline{x}$ and D is nonempty and bounded, $C \in (0, +\infty)$. For every $x \in D$ we have $\|x - x_1\| - \|x - \overline{x}\| = \frac{\|x - x_1\|^2 - \|x - \overline{x}\|^2}{\|x - x_1\| + \|x - \overline{x}\|} = \frac{\|x_1 - \overline{x}\|^2 + 2\langle x - \overline{x}, \overline{x} - x_1 \rangle}{\|x - x_1\| + \|x - \overline{x}\|} \geq \frac{\|x_1 - \overline{x}\|^2}{\|x - x_1\| + \|x - \overline{x}\|} \geq \frac{\|x_1 - \overline{x}\|^2}{C}$. Take $y \in H_2 \setminus \{x_2\}$ in the closed ball with center $f(\overline{x})$ and radius $\frac{K\|x_1 - \overline{x}\|^2}{C}$. Let f_y be the extension of f to $D \cup \{x_1\}$ defined by $f_y(x_1) = y$. This mapping is K-Lipschitz, since for every $x \in D$ one has $\|f_y(x) - f_y(x_1)\|_{H_2} = \|f(x) - y\|_{H_2} \leq \|f(x) - f(\overline{x})\|_{H_2} + \|f(\overline{x}) - y\|_{H_2} \leq K\|x - \overline{x}\|_{H_1} + \|f(\overline{x}) - y\|_{H_2} \leq K\|x - \overline{x}\|_{H_1} + \frac{K\|x_1 - \overline{x}\|^2}{C} \leq K\|x - \overline{x}\|_{H_1} + K(\|x - x_1\| - \|x - \overline{x}\|) = K\|x - x_1\|$. The proof finishes by applying the same reasoning as at the end of the proof of Proposition 36. ■

In this framework, for $A := graph(f)$ the function Φ_A is given by

$$\Phi_A(x_1, x_2) = \frac{1}{2} \sup_{a_1 \in dom f} \{-K^2\|a_1 - x_1\|_{H_1}^2 + \|f(a_1) - x_2\|_{H_2}^2\} + \frac{K^2}{2}\|x_1\|^2 - \frac{1}{2}\|x_2\|^2.$$

It is also evident that $(B, \lfloor \cdot, \cdot \rfloor, \|\cdot\|)$ is an SSDB space if and only if $K = 1$.

5.2 Closed sets in a Hilbert space

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space.and denote by $\|\cdot\|$ the induced norm on H. Clearly, $(H, \langle \cdot, \cdot \rangle, \|\cdot\|)$ is an SSDB space, and the associated quadratic form q is given by $q(x) = \frac{1}{2}\|x\|^2$. Since q is nonnegative, every nonempty set $A \subset H$ is q-positive. We further have:

Proposition 38 *A nonempty set $A \subset H$ is q-representable if and only if it is closed.*

Proof. The "only if" statement being obvious, we will only prove the converse. Define $h : H \to \mathbb{R} \cup \{+\infty\}$ by

$$h(x) = \sup_{y \in H} \left\{ q(y) + \langle y, x - y \rangle + \frac{1}{2} d_A^2(y) \right\},$$

with $d_A(y) := \inf_{a \in A} \|y - a\|$. Clearly, h is convex and lsc.
For all $x \in H$,

$$h(x) \geq q(x) + \langle x, x - x \rangle + \frac{1}{2} d_A^2(x) = q(x) + \frac{1}{2} d_A^2(x) \geq q(x),$$

which implies that $h \geq q$ and $\mathcal{P}_q(h) \subset A$. We will prove that h represents A, that is,

$$A = \mathcal{P}_q(h). \tag{20}$$

To prove the inclusion $\subset$ in (20), let $x \in A$. Then, for all $y \in H$,

$$\begin{aligned} q(y) + \langle y, x - y \rangle + \frac{1}{2} d_A^2(y) &\leq \frac{1}{2}\|y\|^2 + \langle y, x - y \rangle + \frac{1}{2}\|y - x\|^2 = \frac{1}{2}\|x\|^2 \\ &= q(x), \end{aligned}$$

which proves that $h(x) \leq q(x)$. Hence, as $h \geq q$, the inclusion $\subset$ holds in (20). We have thus proved (20), which shows that A is q-representable. ■

Proposition 39 *Let $\emptyset \neq A \subset H$. Then*
(1) $\Phi_A(x) = \frac{1}{2}\|x\|^2 - \frac{1}{2} d_A^2(x)$;
(2) $\Phi_A^{@}(x) = \frac{1}{2}\|x\|^2 + \frac{1}{2} \sup_{b \in H} \{d_A^2(b) - \|x - b\|^2\}$;
(3) $\Phi_A^{@}(x) = \frac{1}{2}\|x\|^2 \Leftrightarrow x \in \overline{A}$;
(4) $G_{\Phi_A} = \left\{x \in H : \sup_{b \in H}\{d_A^2(b) - \|b - x\|^2\} = d_A^2(x)\right\}$

Theorem 40 *Let $\emptyset \neq A \subset H$ be such that $A = G_{\Phi_A}$, and let a_1, $a_2 \in A$ be two different points, $x = \frac{1}{2}(a_1 + a_2)$ and $r = \frac{1}{2}\|a_1 - a_2\|$. Denote by $B_r(x)$ the open ball with center x and radius r. Then,*

$$B_r(x) \cap A \neq \emptyset.$$

Proof. Suppose that

$$A \cap B_r(x) = \emptyset, \tag{21}$$

so, we must have $d_A^2(x) = \|x - a_1\|^2 = \|x - a_2\|^2$.
For $b \in H$, we have

$$\text{either } \langle b - x, x - a_1 \rangle \leq 0 \text{ or } \langle b - x, x - a_2 \rangle \leq 0.$$

If $\langle b-x, x-a_1\rangle \leq 0$,

$$d_A^2(b) - \|b-x\|^2 \leq \|b-a_1\|^2 - \|b-x\|^2 \leq \|x-a_1\|^2 = d_A^2(x).$$

If $\langle b-x, x-a_2\rangle \leq 0$,

$$d_A^2(b) - \|b-x\|^2 \leq \|b-a_2\|^2 - \|b-x\|^2 \leq \|x-a_2\|^2 = d_A^2(x).$$

Thus, we deduce that

$$\sup_{b\in H}\{d_A^2(b) - \|b-x\|^2\} = d_A^2(x),$$

hence by Proposition 39*(4)* $x \in G_{\Phi_A} = A$, which is a contradiction with (21). ∎

Corollary 41 *Let $H = \mathbb{R}$ and $\emptyset \neq A \subset \mathbb{R}$. Then,*

$$A = G_{\Phi_A} \text{ if and only if } A \text{ is closed and convex.}$$

Proof. ($\Longrightarrow$) Since $A = G_{\Phi_A}$, A is closed. Assume that A is not convex, so there exists $a_1, a_2 \in A$ such that $]a_1, a_2[\, \cap A = \emptyset$, hence

$$A \cap B_r(x) = \emptyset, \text{ with } x = \frac{1}{2}(a_1 + a_2) \text{ and } r = \frac{1}{2}|a_1 - a_2|,$$

which contradicts Theorem 40. Thus A is convex.
($\Longleftarrow$) Since A is closed, it is q-positive; hence we can apply Theorem 6*(2)*. ∎

We will show with a simple example that, leaving aside the case $B = \mathbb{R}$, in general $A = G_{\Phi_A}$ does not imply that A is convex.

Example 42 *Let $H = \mathbb{R}^2$, and let $A = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 x_2 = 0\right\}$. We will show that $A = G_{\Phi_A}$. Let $x = (x_1, x_2) \in \mathbb{R}^2 \setminus A$. Then*

$$d_A(x) = \min\{|x_1|, |x_2|\}.$$

If $\lambda \in \mathbb{R}$, let $f(\lambda) := d_A^2(\lambda x) - \|\lambda x - x\|^2 = \lambda^2 d_A^2(x) - (\lambda-1)^2\|x\|^2$. Then $f'(1) = 2d_A^2(x) > 0$ and so, if λ is slightly greater than 1, $f(\lambda) > f(1)$, that is to say, $d_A^2(\lambda x) - \|\lambda x - x\|^2 > d_A^2(x)$. Hence we have

$$\sup_{y\in H}\left\{d_A^2(y) - \|y-x\|^2\right\} > d_A^2(x);$$

thus, by Proposition 39(4), $x \notin G_{\Phi_A}$. We deduce that $A = G_{\Phi_A}$, and clearly A is not convex.

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