

EULER, THE SYMMETRIC GROUP AND THE RIEMANN ZETA FUNCTION

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1. INTRODUCTION

Let π be a permutation in the symmetric group S_n . An *ascent* is an occurrence of $\pi(j) < \pi(j+1)$ for $1 \leq j \leq n-1$. For example, the permutation 24513 has 3 ascents. The *Eulerian number* $\langle n \rangle_k$ is defined to be the number of permutations in S_n with exactly k ascents. (The Eulerian numbers are not to be confused with the Euler numbers E_n .) Some of the elementary facts about them [2, chapter 6.2] are the recursion

$$\langle n \rangle_k = (k+1) \langle n-1 \rangle_k + (n-k) \langle n-1 \rangle_{k-1}$$

with boundary conditions

$$\langle n \rangle_0 = 1, \quad \langle n \rangle_{n-1} = 1,$$

and the symmetry

$$\langle n \rangle_k = \langle n \rangle_{n-1-k}.$$

We have an obvious identity

$$\sum_k \langle n \rangle_k = n!.$$

The point of this paper is a surprising identity for alternating sums of Eulerian numbers.

Theorem. Let $\zeta(s)$ be the Riemann zeta function defined by $\sum_{m=1}^{\infty} m^{-s}$ for $\operatorname{Re}(s) > 1$, analytically continued to $\mathbb{C} \setminus \{1\}$. For integer $n \geq 1$ we have

$$\zeta(-n) = \frac{\sum_{k=1}^n (-1)^k \langle n \rangle_k}{2^{n+1}(1-2^{n+1})}.$$

Of course, $\zeta(-n)$ can be expressed in closed form in terms of the Bernoulli numbers by

$$\zeta(-n) = -\frac{B_{n+1}}{n+1},$$

so the theorem is also an identity relating Eulerian numbers to Bernoulli numbers. However, the proof is direct.

2. EULER OPERATOR

Let $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ denote the *Stirling number of the second kind*. It is the number of ways to partition a set of n elements into k nonempty subsets. These, too are connected to Bernoulli numbers by the identity (6.99) of [2] with $k = 1$

$$B_m = \sum_{j \geq 0} \left\{ \begin{smallmatrix} m \\ j \end{smallmatrix} \right\} \frac{j!(-1)^j}{j+1}.$$

Let \mathcal{D} be the derivative operator $\frac{d}{dz}$ and \mathcal{E} be the Euler operator $z \frac{d}{dz}$. They are related by

$$(1) \quad \mathcal{E}^n = \sum_k \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} z^k \mathcal{D}^k$$

which is proved by induction [2, Exercise 6.13].

Lemma. If we apply \mathcal{E}^n to $f(z) = 1/(1+z)$ we get

$$(2) \quad \mathcal{E}^n \frac{1}{1+z} = \frac{\sum_j (-1)^{j+1} \left\langle \begin{smallmatrix} n \\ j \end{smallmatrix} \right\rangle z^{j+1}}{(1+z)^{n+1}}$$

Proof. By (1) we have

$$\mathcal{E}^n \frac{1}{1+z} = \sum_k \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} z^k \frac{(-1)^k k!}{(1+z)^{k+1}}.$$

Put every term over the common denominator $(1+z)^{n+1}$; then the numerator is

$$\begin{aligned} & \sum_k \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} z^k (-1)^k k! (1+z)^{n-k} \\ &= \sum_k \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} (-1)^k k! \sum_j \binom{n-k}{j} z^{n-j} \\ &= \sum_j (-1)^{n-j} \left(\sum_k \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} (-1)^{n-j-k} k! \binom{n-k}{j} \right) z^{n-j} \end{aligned}$$

by [2, (6.40)] the term in parenthesis simplifies to give

$$= \sum_j (-1)^{n-j} \langle n \rangle_j z^{n-j} = \sum_j (-1)^{n-j} \left\langle n \atop n-1-j \right\rangle z^{n-j}$$

by the symmetry property. Change j to $n-1-j$ to get the numerator as

$$\sum_j (-1)^{j+1} \langle n \rangle_j z^{j+1}.$$

□

Lemma. We have the identity

$$(3) \quad \langle n \rangle_j = \sum_{k=0}^j \binom{n+1}{k} (j+1-k)^n (-1)^k.$$

This is (6.38) in [2], without proof supplied.

Proof. Expand $1/(1+z)$ as a power series and apply \mathcal{E}^n term by term.

$$\begin{aligned} \frac{1}{1+z} &= \sum_{m=0}^{\infty} (-1)^m z^m \\ \mathcal{E}^n \frac{1}{1+z} &= \sum_{m=0}^{\infty} (-1)^m m^n z^m. \end{aligned}$$

Multiply both sides of (2) by $(1+z)^{n+1}$ to get

$$\begin{aligned} \sum_j (-1)^{j+1} \langle n \rangle_j z^{j+1} &= (1+z)^{n+1} \sum_{m=0}^{\infty} (-1)^m m^n z^m \\ &= \sum_k \binom{n+1}{k} z^k \sum_{m=0}^{\infty} (-1)^m m^n z^m \\ &= \sum_l \left(\sum_{k=0}^l \binom{n+1}{k} (-1)^{l-k} (l-k)^n \right) z^l \end{aligned}$$

Comparing coefficients of z^{j+1} gives

$$\begin{aligned} (-1)^{j+1} \langle n \rangle_j &= \sum_{k=0}^{j+1} \binom{n+1}{k} (-1)^{j+1-k} (j+1-k)^n \\ &= \sum_{k=0}^j \binom{n+1}{k} (-1)^{j+1-k} (j+1-k)^n \end{aligned}$$

□

3. ABEL SUMMATION

The connection to the Riemann zeta function is, of course, through Abel summation. Let $f(z)$ be a function, continuous at $z = 1$ and with power series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

absolutely convergent for $|z| < 1$. Then we define [3] the following series to be convergent in the sense of Abel summation,

$$\sum_{n=0}^{\infty} a_n \stackrel{A}{=} f(1).$$

For example, we have $\sum_n (-1)^n z^n = 1/(1+z)$ for $|z| < 1$, and the right side is continuous at $z = 1$ so

$$\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + 1 - 1 \dots \stackrel{A}{=} \frac{1}{2}.$$

This has nothing to do with limits of partial sums; it is a new definition. (It is probably worth mentioning also that some people use the term ‘Abel summation’ to mean instead the discrete analog of integration by parts; i.e summation by parts.) It is easy to go astray here; observe that

$$\frac{1}{1+x+x^2} = \frac{1-x}{1-x^3} = \sum_{n=0}^{\infty} (x^{3n} - x^{3n+1})$$

so, for example,

$$1 - 1 + 0 + 1 - 1 + 0 + 1 - 1 + 0 \dots \stackrel{A}{=} \frac{1}{3}.$$

Introducing zeros into the sum changes the value.

For $\text{Re}(s) > 1$ we have absolutely convergent series

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}, \quad \phi(s) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m^s},$$

We have

$$\phi(s) = (1 - 2^{1-s})\zeta(s),$$

and the alternating series $\phi(s)$ converges conditionally for $0 < s < 1$.

Euler used Abel summation to compute values of $\phi(s)$ and then the Riemann zeta function at negative integers; surprisingly, it gives the correct answer (see [5, §8.4].) That is, it agrees with the values obtained by Riemann’s analytic continuation of $\zeta(s)$. In fact, Euler

conjectured the functional equation for $\zeta(s)$ correctly, based on this calculation.

Euler's idea was that $\phi(s)$ is summable, in the sense of Abel summation, for $s = -n$ a negative integer. With $f(z) = 1/(1+z)$ we have

$$\begin{aligned}\phi(-n) &= \sum_{m=1}^{\infty} (-1)^{m-1} m^n \stackrel{A}{=} -\mathcal{E}^n f(1), \\ \zeta(-n) &= (1 - 2^{n+1})^{-1} \phi(-n) \stackrel{A}{=} -(1 - 2^{n+1})^{-1} \mathcal{E}^n f(1).\end{aligned}$$

By (2) we get that

$$\zeta(-n) = \frac{\sum_j (-1)^j \left\langle \begin{matrix} n \\ j \end{matrix} \right\rangle}{2^{n+1}(1 - 2^{n+1})}.$$

4. RANDOM MATRIX THEORY

Euler certainly could have proved this theorem, although there is no evidence he actually did. It provides a connection between the combinatorics of the symmetric group (the Eulerian numbers $\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle$) and the trivial zeros of the Riemann zeta function at the negative even integers $-n = -2j$.

Conjecturally, there is an indirect connection between the statistics of the nontrivial zeros ρ of $\zeta(s)$ satisfying $0 < \text{Re}(\rho) < 1$, and the combinatorics of the symmetric group, via random matrix theory. Assume the Riemann hypothesis, that the zeros are on the critical line $\rho = 1/2 + i\gamma$. The (suitably normalized) spacings of the gaps between the zeros are conjectured to satisfy what is called the GUE distribution, the probability distribution for the eigenvalues of random unitary matrices. The numerical evidence for this is impressive [4]. Meanwhile, for a permutation π of S_n and $1 \leq i_1 \leq \dots \leq i_k \leq n$, we say that $\pi(i_1), \dots, \pi(i_k)$ is an *increasing subsequence* if

$$\pi(i_1) < \pi(i_2) < \dots < \pi(i_k).$$

Let $\ell_n(\pi)$ be the length of the longest increasing subsequence, Rains and Odlyzko proved (see [1] for an exposition) that

$$P(\ell_n \leq \ell) = \frac{1}{n!} \int_{U_\ell} |\text{Tr} M|^{2n} dM,$$

where dM is Haar measure on the unitary group U_ℓ . And suitably normalized and re-scaled, the length ℓ_n of the longest increasing subsequence of a random permutation behaves statistically like largest

eigenvalue of a GUE matrix, according to a theorem of Baik, Deift, and Johansson described in detail in [1].

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