Practice Problems: Partial Fraction Decomposition

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The following are solutions to the Partial Fraction practice problems posted on November 9.

For the following problems, just find the partial fraction decomposition (no need to integrate).

1. $\dfrac{3x}{2x^2 - x - 1}$

   Solution: Factor the denominator: $2x^2 - x - 1 = (2x+1)(x-1)$.

   $$\frac{3x}{(2x+1)(x-1)} = \frac{A}{2x+1} + \frac{B}{x-1}$$

   Clear the denominators by multiplying the denominator on the left side on both sides:

   $$3x = A(x-1) + B(2x+1)$$

   Combine like terms:

   $$3x = (A+2B)x - A + B$$

   Then we get the system of equations: $\begin{cases} A + 2B = 3 \\ -A + B = 0 \end{cases}$

   Solving this system we see that $A = 1, B = 1$. Then

   $$\frac{3x}{(2x+1)(x-1)} = \frac{1}{2x+1} + \frac{1}{x-1}$$

   □

2. $\dfrac{5x+7}{x^3 + 2x^2 - x - 2}$

   Solution: Factor the denominator: $x^3 + 2x^2 - x - 2 = (x^2-1)(x+2) = (x-1)(x+1)(x+2)$.

   $$\frac{5x+7}{(x-1)(x+1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+2}$$

   Clear the denominators:

   $$5x + 7 = A(x+1)(x+2) + B(x-1)(x+2) + C(x-1)(x+1)$$

   Combine like terms:

   $$5x + 7 = (A+B+C)x^2 + (3A+B)x + 2A - 2B - C$$

Then we get the system of equations: $\begin{cases} A+B+C=0 \\ 3A+B=5 \\ 2A-2B-C=7 \end{cases}$

Solving this system we see that $A=2, B=-1, C=-1$. Then

$$\frac{5x+7}{(x-1)(x+1)(x+2)} = \frac{2}{x-1} - \frac{1}{x+1} - \frac{1}{x+2}$$

□

3. $\dfrac{x^2+1}{x(x-1)^3}$

Solution: The denominator is already factored.

$$\frac{x^2+1}{x(x-1)^3} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}$$

Clear the denominators:

$$x^2+1 = A(x-1)^3 + Bx(x-1)^2 + Cx(x-1) + Dx$$

Combine like terms:

$$x^2+1 = (A+B)x^3 + (-3A-2B+C)x^2 + (3A+B-C+D)x - A$$

Then we get the system of equations: $\begin{cases} A+B=0 \\ -3A-2B+C=1 \\ 3A+B-C+D=0 \\ -A=1 \end{cases}$

Solving this system we see that $A=-1, B=1, C=0, D=2$. Then

$$\frac{x^2+1}{x(x-1)^3} = \frac{-1}{x} + \frac{1}{x-1} + \frac{2}{(x-1)^3}$$

□

4. $\dfrac{2x^2-x+4}{x^3+4x}$

Solution: Factor the denominator: $x^3+4x = x(x^2+4)$.

$$\frac{2x^2-x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

Clear the denominators:

$$2x^2-x+4 = A(x^2+4) + (Bx+C)x$$

Combine like terms:

$$2x^2 - x + 4 = (A + B)x^2 + Cx + 4A$$

Then we get the system of equations: $\begin{cases} A + B = 2 \\ C = -1 \\ 4A = 4 \end{cases}$

Solving this system we see that $A = 1, B = 1, C = -1$. Then

$$\frac{2x^2 - x + 4}{x(x^2 + 4)} = \frac{1}{x} + \frac{x - 1}{x^2 + 4}$$

□

5. $\dfrac{x^5 - 3x^2 + 12x - 1}{x^3(x^2 + x + 1)(x^2 + 2)^3}$ (Just write out the form, do not find the constants here.)

Solution: The denominator is already factored. We just need to write out the form:

$$\frac{x^5 - 3x^2 + 12x - 1}{x^3(x^2 + x + 1)(x^2 + 2)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx + E}{x^2 + x + 1} + \frac{Fx + G}{x^2 + 2} + \frac{Hx + I}{(x^2 + 2)^2} + \frac{Jx + K}{(x^2 + 2)^3}$$

□

6. $\dfrac{2x^4 + 4x^3 - 2x^2 + x + 7}{x^3 + 2x^2 - x - 2}$

Solution: We first need to do long division (since the top has degree larger than the bottom degree):

$$\frac{2x^4 + 4x^3 - 2x^2 + x + 7}{x^3 + 2x^2 - x - 2} = 2x + \frac{5x + 7}{x^3 + 2x^2 - x - 2}$$

The right rational function is the same as the one in Problem #2, so

$$\frac{2x^4 + 4x^3 - 2x^2 + x + 7}{x^3 + 2x^2 - x - 2} = 2x + \frac{2}{x - 1} - \frac{1}{x + 1} - \frac{1}{x + 2}$$

□

7. $\dfrac{12}{x^2 - 9}$

Solution: Factor the denominator: $x^2 - 9 = (x - 3)(x + 3)$.

$$\frac{12}{(x - 3)(x + 3)} = \frac{A}{x - 3} + \frac{B}{x + 3}$$

Clear the denominators:

$$12 = A(x + 3) + B(x - 3)$$

Combine like terms:

$$12 = (A + B)x + 3A - 3B$$

Then we get the system of equations: $\begin{cases} A+B=0 \\ 3A-3B=12 \end{cases}$

Solving this system we see that $A=2, B=-2$. Then

$$\frac{12}{(x-3)(x+3)} = \frac{2}{x-3} - \frac{2}{x+3}$$

□

8. $\dfrac{7x-3}{x^3+2x^2-3x}$

Solution: Factor the denominator: $x^3+2x^2-3x = x(x^2+2x-3) = x(x+3)(x-1)$.

$$\frac{7x-3}{x(x+3)(x-1)} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-1}$$

Clear the denominators:

$$7x-3 = A(x+3)(x-1) + Bx(x-1) + Cx(x+3)$$

Combine like terms:

$$7x-3 = (A+B+C)x^2 + (2A-B+3C)x - 3A$$

Then we get the system of equations: $\begin{cases} A+B+C=0 \\ 2A-B+3C=7 \\ -3A=-3 \end{cases}$

Solving this system we see that $A=1, B=-2, C=1$. Then

$$\frac{7x-3}{x(x+3)(x-1)} = \frac{1}{x} - \frac{2}{x+3} + \frac{1}{x-1}$$

□

9. $\dfrac{x-4}{(2x-5)^2}$

Solution: The denominator is already factored.

$$\frac{x-4}{(2x-5)^2} = \frac{A}{2x-5} + \frac{B}{(2x-5)^2}$$

Clear the denominators:

$$x-4 = A(2x-5) + B$$

Combine like terms:

$$x-4 = 2Ax - 5A + B$$

Then we get the system of equations: $\begin{cases} 2A = 1 \\ -5A + B = -4 \end{cases}$

Solving this system we see that $A = \frac{1}{2}, B = -\frac{3}{2}$. Then

$$\frac{x-4}{(2x-5)^2} = \frac{\frac{1}{2}}{2x-5} - \frac{\frac{3}{2}}{(2x-5)^2}$$

□

10. $\dfrac{4x^2 - x - 2}{x^4 + 2x^3}$

Solution: Factor the denominator: $x^4 + 2x^3 = x^3(x+2)$.

$$\frac{4x^2 - x - 2}{x^3(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+2}$$

Clear the denominators:

$$4x^2 - x - 2 = Ax^2(x+2) + Bx(x+2) + C(x+2) + Dx^3$$

Combine like terms:

$$4x^2 - x - 2 = (A+D)x^3 + (2A+B)x^2 + (2B+C)x + 2C$$

Then we get the system of equations: $\begin{cases} A + D = 0 \\ 2A + B = 4 \\ 2B + C = -1 \\ 2C = -2 \end{cases}$

Solving this system we see that $A = 2, B = 0, C = -1, D = -2$. Then

$$\frac{4x^2 - x - 2}{x^3(x+2)} = \frac{2}{x} - \frac{1}{x^3} - \frac{2}{x+2}$$

□

11. $\dfrac{x-3}{x^3 + 3x}$

Solution: Factor the denominator: $x^3 + 3x = x(x^2+3)$.

$$\frac{x-3}{x(x^2+3)} = \frac{A}{x} + \frac{Bx+C}{x^2+3}$$

Clear the denominators:

$$x - 3 = A(x^2+3) + (Bx+C)x$$

Combine like terms:

$$x - 3 = (A+B)x^2 + Cx + 3A$$

Then we get the system of equations: $\begin{cases} A + B = 0 \\ C = 1 \\ 3A = -3 \end{cases}$

Solving this system we see that $A = -1, B = 1, C = 1$. Then

$$\frac{x-3}{x(x^2+3)} = -\frac{1}{x} + \frac{x+1}{x^2+3}$$

□

12. $\dfrac{2x^3 + 7x + 5}{(x^2+x+2)(x^2+1)}$

Solution: We cannot factor the denominator any further.

$$\frac{2x^3 + 7x + 5}{(x^2+x+2)(x^2+1)} = \frac{Ax+B}{x^2+x+2} + \frac{Cx+D}{x^2+1}$$

Clear the denominators:

$$2x^3 + 7x + 5 = (Ax+B)(x^2+1) + (Cx+D)(x^2+x+2)$$

Combine like terms:

$$2x^3 + 7x + 5 = (A+C)x^3 + (B+C+D)x^2 + (A+2C+D)x + B + 2D$$

Then we get the system of equations: $\begin{cases} A + C = 2 \\ B + C + D = 0 \\ A + 2C + D = 7 \\ B + 2D = 5 \end{cases}$

Solving this system we see that $A = 2, B = -5, C = 0, D = 5$. Then

$$\frac{2x^3 + 7x + 5}{(x^2+x+2)(x^2+1)} = \frac{2x-5}{x^2+x+2} + \frac{5}{x^2+1}$$

□

13. $\dfrac{x^4 + x^3 + x^2 - x + 1}{x(x^2+1)^2}$

Solution: We cannot factor the denominator any further.

$$\frac{x^4 + x^3 + x^2 - x + 1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

Clear the denominators:

$$x^4 + x^3 + x^2 - x + 1 = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x$$

Combine like terms:

$$x^4 + x^3 + x^2 - x + 1 = (A + B)x^4 + Cx^3 + (2A + B + D)x^2 + (C + E)x + A$$

Then we get the system of equations: $\begin{cases} A + B = 1 \\ C = 1 \\ 2A + B + D = 1 \\ C + E = -1 \\ A = 1 \end{cases}$

Solving this system we see that $A = 1, B = 0, C = 1, D = -1, E = -2$. Then

$$\frac{x^4 + x^3 + x^2 - x + 1}{x(x^2 + 1)^2} = \frac{1}{x} + \frac{1}{x^2 + 1} + \frac{-x - 2}{(x^2 + 1)^2}$$

□