Practice Problems: Riemann Sums

Written by Victoria Kala
vtkala@math.ucsb.edu
December 6, 2014

Solutions to the practice problems posted on November 30.

Evaluate the following Riemann sums by turning them into integrals.

1. $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left(8\left(1+\frac{i}{n}\right)^3 + 3\left(1+\frac{i}{n}\right)^2 \right)$ (Hint: Interval is $[1, 2]$)

   Solution: Need to find Δx and x_i:

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$$

$$x_i = a + i\Delta x = 1 + \frac{i}{n}$$

   Now we want to plug these into our Riemann Sum:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left(8\left(1+\frac{i}{n}\right)^3 + 3\left(1+\frac{i}{n}\right)^2 \right) = \lim_{n\to\infty} \Delta x \sum_{i=1}^{n} \left(8x_i^3 + 3x_i^2\right) = \int_1^2 (8x^3 + 3x^2)dx$$

$$= 2x^4 + x^3 \Big|_1^2 = 37$$

   □

2. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi^2 i}{n^2} \cos^2\left(\frac{\pi i}{n}\right)$ (Hint: Interval is $[0, \pi]$)

   Solution: Need to find Δx and x_i:

$$\Delta x = \frac{b-a}{n} = \frac{\pi - 0}{n} = \frac{\pi}{n}$$

$$x_i = a + i\Delta x = 0 + \frac{i\pi}{n} = \frac{i\pi}{n}$$

   Now we want to plug these into our Riemann Sum:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi^2 i}{n^2} \cos^2\left(\frac{\pi i}{n}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{n} \cdot \frac{\pi i}{n} \cos^2\left(\frac{\pi i}{n}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x \cdot x_i \cos^2 x_i$$

$$= \int_0^\pi x \cos^2 x \, dx = \frac{1}{2} \int_0^\pi x(1 + \cos 2x)dx$$

Use integration by parts on the integral, tabular makes it easy:

$$= \frac{1}{2}\left(\frac{x^2}{2} + \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x\right)\Big|_0^{\pi} = \frac{\pi^2}{4}$$

□

3. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\frac{i}{n^2}}{\left(\frac{2i}{n}+1\right)^3}$ (Hint: Interval is $[0,1]$)

Solution: Need to find Δx and x_i:

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$
$$x_i = a + i\Delta x = 0 + \frac{i}{n} = \frac{i}{n}$$

Now we want to plug these into our Riemann Sum:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\frac{i}{n^2}}{\left(\frac{2i}{n}+1\right)^3} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\frac{i}{n}\frac{1}{n}}{\left(2\frac{i}{n}+1\right)^3} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{x_i \Delta x}{\left(2x_i+1\right)^3} = \int_0^1 \frac{x}{(2x+1)^3}dx$$

Need to use a u-substitution to solve this. Let $u = 2x+1$, then $du = 2dx$ and $x = \frac{1}{2}(u-1)$. When $x = 0, u = 1$; when $x = 1, u = 3$:

$$= \frac{1}{2}\int_1^3 \frac{\frac{1}{2}(u-1)}{u^3}du = \frac{1}{4}\int_1^3 \frac{u-1}{u^3}du = \frac{1}{4}\int_1^3 \left(\frac{1}{u^2} - \frac{1}{u^3}\right)du = \frac{1}{4}\left(-\frac{1}{u} + \frac{1}{2u^2}\right)\Big|_1^3 = \frac{1}{18}$$

□

4. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{96i+20n}{8in+n^2}$ (Hint: Interval is $[0,4]$)

Solution: Need to find Δx and x_i:

$$\Delta x = \frac{b-a}{n} = \frac{4-0}{n} = \frac{4}{n}$$
$$x_i = a + i\Delta x = 0 + \frac{i4}{n} = \frac{4i}{n}$$

Now we want to plug these into our Riemann Sum:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{96i+20n}{8in+n^2} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n}\cdot\frac{24i+5n}{8i+n} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n}\cdot\frac{\frac{24i+5n}{n}}{\frac{8i+n}{n}} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n}\cdot\frac{24\frac{i}{n}+5}{8\frac{i}{n}+1}$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n}\cdot\frac{6\frac{4i}{n}+5}{2\frac{4i}{n}+1} = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x\cdot\frac{6x_i+5}{2x_i+1} = \int_0^4 \frac{6x+5}{2x+1}dx = \int_0^4 \left(3 + \frac{2}{2x+1}\right)dx$$
$$= 3x + \ln|2x+1|\Big|_0^4 = 12 + \ln 9$$

□