Practice Problems: Surface Areas

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Solutions to the practice problems posted on November 30.

Find the surface area obtained by rotating the curve about the specified axis.

1. $x = 1 + 2y^2, 1 \leq y \leq 2$, about the x-axis

   Solution: We are rotating about the x-axis so we want to use the formula $S = \int 2\pi y ds$. Since $1 \leq y \leq 2$, then we want to use $ds = \sqrt{1 + (\frac{dx}{dy})^2}dy$.

   Find $\frac{dx}{dy}$: $\frac{dx}{dy} = 4y$

   Find ds: $ds = \sqrt{1 + (\frac{dx}{dy})^2}dy = \sqrt{1 + (4y)^2}dy = \sqrt{1 + 16y^2}dy$

   Find S: $S = \int 2\pi y ds = \int_1^2 2\pi y\sqrt{1 + 16y^2}dy$

   Let $u = 1 + 16y^2$, then $du = 32y$. When $y = 1, u = 17$; when $y = 2, u = 65$:

   $$S = \int_1^2 2\pi y\sqrt{1 + 16y^2}dy = \frac{2\pi}{32}\int_{17}^{65}\sqrt{u}\ du = \frac{\pi}{16}\cdot\frac{2}{3}u^{3/2}\Big|_{17}^{65} = \frac{\pi}{24}(65\sqrt{65} - 17\sqrt{17})$$

   □

2. $y = 1 - x^2, 0 \leq x \leq 1$, about the y-axis

   Solution: We are rotating about the y-axis so we want to use the formula $S = \int 2\pi x ds$. Since $0 \leq x \leq 1$, then we want to use $ds = \sqrt{1 + (\frac{dy}{dx})^2}dx$.

   Find $\frac{dy}{dx}$: $\frac{dy}{dx} = -2x$

   Find ds: $ds = \sqrt{1 + (\frac{dy}{dx})^2}dx = \sqrt{1 + 4x^2}dx$

   Find S: $S = \int 2\pi x ds = \int_0^1 2\pi x\sqrt{1 + 4x^2}dx$

   Let $u = 1 + 4x^2$, then $du = 8xdx$. When $x = 0, u = 1$; when $x = 1, u = 5$:

   $$S = \int_0^1 2\pi x\sqrt{1 + 4x^2}dx = \frac{2\pi}{8}\int_1^5\sqrt{u}\ du = \frac{\pi}{4}\cdot\frac{2}{3}u^{3/2}\Big|_1^5 = \frac{\pi}{6}(5\sqrt{5} - 1)$$

   □

3. $9x = y^2 + 18, 2 \leq x \leq 6$, about the x-axis

   Solution: We are rotating about the x-axis so we want to use the formula $S = \int 2\pi y ds$. Since $2 \leq x \leq 6$, then we want to use $ds = \sqrt{1 + (\frac{dy}{dx})^2}dx$.

First, solve for y in the above equation (there will be a $\pm$ solution, just pick the positive one): $y = \sqrt{9x - 18} = 3\sqrt{x-2}$

Find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{3}{2\sqrt{x-2}}$$

Find ds:

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \sqrt{1 + \frac{9}{4(x-2)}} dx = \sqrt{\frac{4(x-2)+9}{4(x-2)}} dx = \sqrt{\frac{4x+1}{4(x-2)}} dx = \frac{1}{2}\sqrt{\frac{4x+1}{x-2}} dx$$

Find S:

$$S = \int 2\pi y ds = \int_2^6 2\pi \cdot 3\sqrt{x-2} \cdot \frac{1}{2}\sqrt{\frac{4x+1}{x-2}} dx = 3\pi \int_2^6 \sqrt{4x+1}\ dx$$

Let $u = 4x+1$, then $du = 4dx$. When $x = 2, u = 9$; when $x = 6, u = 25$:

$$= \frac{3\pi}{4}\int_9^{25} \sqrt{u}\ du = \frac{3\pi}{4} \cdot \frac{2}{3} u^{3/2}\Big|_9^{25} = 49\pi$$

□

4. $x = \sqrt{a^2 - y^2}, 0 \leq y \leq a/2$, about the y-axis

Solution: We are rotating about the y-axis so we want to use the formula $S = \int 2\pi x ds$. Since $0 \leq y \leq a/2$, then we want to use $ds = \sqrt{1 + (\frac{dx}{dy})^2} dy$.

Find $\frac{dx}{dy}$:

$$\frac{dx}{dy} = \frac{1}{2\sqrt{a^2 - y^2}} \cdot (-2y) = \frac{-y}{\sqrt{a^2 - y^2}}$$

Find ds:

$$ds = \sqrt{1 + (\frac{dx}{dy})^2} dy = \sqrt{1 + \frac{y^2}{a^2 - y^2}} dy = \sqrt{\frac{a^2 - y^2 + y^2}{a^2 - y^2}} dy = \sqrt{\frac{a^2}{a^2 - y^2}} dy = \frac{a}{\sqrt{a^2 - y^2}} dy$$

Find S:

$$S = \int 2\pi x ds = \int_0^{a/2} 2\pi\sqrt{a^2 - y^2}\frac{a}{\sqrt{a^2 - y^2}} dy = 2\pi a \int_0^{a/2} dy = 2\pi a y\Big|_0^{a/2} = \pi a^2$$

□

5. $y = \sqrt{1+e^x}, 0 \le x \le 1$, about the x-axis

Solution: We are rotating about the x-axis so we want to use the formula $S = \int 2\pi y ds$. Since $0 \le x \le 1$, then we want to use $ds = \sqrt{1 + (\frac{dy}{dx})^2} dx$.

Find $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{e^x}{2\sqrt{1+e^x}}$$

Find ds:

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \sqrt{1 + \frac{e^{2x}}{4(1+e^x)}} dx = \sqrt{\frac{4+4e^x+e^{2x}}{4(1+e^x)}} dx = \sqrt{\frac{(2+e^x)^2}{4(1+e^x)}} dx = \frac{2+e^x}{2\sqrt{1+e^x}} dx$$

Find S:

$$S = \int 2\pi y ds = \int_0^1 2\pi\sqrt{1+e^x}\frac{2+e^x}{2\sqrt{1+e^x}} dx = \pi \int_0^1 (2+e^x) dx = \pi(2x+e^x)\Big|_0^1 = (e+1)\pi$$

□