Factoring Cubic Polynomials

March 3, 2016

A cubic polynomial is of the form

$$p(x) = a_3x^3 + a_2x^2 + a_1x + a_0.$$

The Fundamental Theorem of Algebra guarantees that if a_0, a_1, a_2, a_3 are all real numbers, then we can factor my polynomial into the form

$$p(x) = a_3(x - b_1)(x^2 + b_2c + b_3).$$

In other words, I can always factor my cubic polynomial into the product of a first degree polynomial and a second degree polynomial. Sometimes we can factor even further into the form

$$p(x) = a_3(x - c_1)(x - c_2)(x - c_3),$$

where c_1, c_2, c_3 are real numbers, but this is not always the case.

But how do we find such a factorization? The following are two methods which I recommend.

Factoring by Grouping

This is by far the nicest method of the two, but it only works in some cases. Consider the polynomial

$$p(x) = x^3 - 4x^2 + 3x - 12.$$

We group the first two terms and the last two terms together:

$$p(x) = (x^3 - 4x^2) + (3x - 12)$$

and then we pull out the common factors:

$$p(x) = x^2(x - 4) + 3(x - 4).$$

Notice now that these two terms now have $x - 4$ in common with each other; factor it out:

$$p(x) = (x - 4)(x^2 + 3).$$

$x^2 + 3$ is an irreducible quadratic, so it cannot factor into real terms. However, we can use the quadratic formula to solve for the roots.

Factoring Using the Rational Root Theorem

This method works as long as the coefficients a_0, a_1, a_2, a_3 are all rational numbers. The Rational Root Theorem says that the possible roots of a polynomial are the factors of the last term divided by the factors of the first term. In our case, since we are factoring the cubic polynomial above, the possible roots are

$$\pm\frac{\text{factors of } a_0}{\text{factors of } a_3}.$$

Example. List the possible roots of the following polynomials.

1. $p(x) = 4x^2 + 8x - 5x + 10$ The factors of 10 are $\pm 1, 2, 5, 10$, and the factors of 4 are $1, 2, 4$. Therefore the possible zeros of $p(x)$ are

$$\pm \frac{1, 2, 5, 10}{1, 2, 4} = \pm 1, \frac{1}{2}, \frac{1}{4}, 2, 5, \frac{5}{2}, \frac{5}{4}, 10.$$

2. $p(x) = -x^3 + 9x^2 - 13x + 18$ The factors of 18 are $\pm 1, 2, 3, 6, 9$ and the factors of -1 are ± 1. Therefore the possible zeros of $p(x)$ are

$$\pm \frac{1, 2, 3, 6, 9}{\pm 1} = \pm 1, 2, 3, 6, 9.$$

□

Now, let's put the theorem to action. Consider the polynomial

$$p(x) = x^3 + 5x^2 - 2x - 24.$$

We cannot factor this polynomial by grouping, so we turn to the Rational Root Theorem. The factors of -24 are $\pm 1, 2, 3, 4, 6, 8, 12, 24$, and the possible factors of 1 are ± 1. Therefore the possible zeros of $p(x)$ are $\pm 1, 2, 3, 4, 6, 8, 12, 24$. We will then plug each of these numbers into $p(x)$ until we find a root such that $p = 0$. Let's start with 1:

$$p(1) = 1 + 5 - 2 - 24 \neq 0,$$

and so 1 is not a zero. Let's try -1:

$$p(-1) = -1 + 5 + 2 - 24 \neq 0,$$

and so -1 is not a zero. Let's try 2:

$$p(2) = 8 + 20 - 4 - 24 = 0,$$

and so 2 is a zero. This means that $x - 2$ is a factor of our polynomial. Now we use long division to divide $p(x)$ by $x - 2$ to find out what the other possible zeros could be:

$$\frac{x^3 + 5x^2 - 2x - 24}{x - 2} = x^2 + 7x + 12.$$

This means that $p(x) = (x - 2)(x^2 + 7x + 12)$. The last term factors nicely:

$$p(x) = (x - 2)(x + 3)(x + 4),$$

and we are done.