Math 6B Notes

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Green's Theorem

We say that a closed curve is **positively oriented** if it is oriented counterclockwise. It is negatively oriented if it is oriented clockwise.

Theorem (Green's Theorem)**.** *Let C be a positively oriented, piecewise smooth, simple closed curve in the plane and let D be the region bounded by C. If $\mathbf{F}(x, y) = (P(x, y), Q(x, y))$ with P and Q having continuous partial derivatives on an open region that contains D, then*

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

In other words, Green's Theorem allows us to change from a complicated line integral over a curve C to a less complicated double integral over the region bounded by C.

The Operator ∇

We define the vector differential operator ∇ (called "del") as

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$$

The **gradient** of a function f is given by

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

The **curl** of a function $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is defined as

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F}.$$

Written out more explicitly:

$$\begin{aligned}
\text{curl } \mathbf{F} = \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\
&= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix} \mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{vmatrix} \mathbf{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} \mathbf{k} \\
&= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}
\end{aligned}$$

The **divergence** of function $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is defined as

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Notice that the curl of a function is a vector and the divergence of a function is a scalar (just a number).

Surface Integrals

If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$, then the surface integral of $\mathbf{F}$ over S is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS.$$

If S is given by a vector function $\mathbf{r}(u, v)$, then the above integral can be written as

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA$$

where D is the parameter domain (the projection of S onto a plane).

If we are given S is the function $z = g(x, y)$, we can define the parametrization of S to be

$$\mathbf{r}(x, y) = (x, y, g(x, y)).$$

Then the surface integral of $\mathbf{F}$ over S is given by

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) dA$$

where D is the projection of $z = g(x, y)$ onto the xy plane.

Stokes' Theorem

Theorem (Stokes' Theorem)**.** *Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}.$$

In other words, the line integral around the boundary curve of a surface S of the tangential component of $\mathbf{F}$ is equal to the surface integral of the normal component of the curl of $\mathbf{F}$.

Note: This looks very similar to Green's Theorem! In fact, Green's Theorem is a special case of Stokes' Theorem; it is when $\mathbf{F}$ is restricted to the xy plane. Stokes' Theorem can then be thought of as the higher-dimensional version of Green's Theorem.

Divergence Theorem

Theorem (Divergence Theorem)**.** *Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then*

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div } \mathbf{F} dV.$$

In other words, the divergence theorem states that the flux of $\mathbf{F}$ across the boundary surface of E is equal to the triple integral of the divergence of $\mathbf{F}$ over E. This theorem allows us to change from a double integral to a triple integral.

Sequences

A **sequence** $\{a_n\}$ is a function $f(n)$ on the natural numbers $\mathbb{N}$; i.e. it is a list of numbers

$$a_1, a_2, \ldots.$$

We say that our sequence **converges** to a limit L if the terms of a_n eventually get really close to L; that is,

$$\lim_{n\to\infty} a_n = L.$$

If a sequence doesn't converge it **diverges**.

We can use some of the following theorems to help determine when a sequence converges or diverges.

Theorem. *If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$, then $\lim_{n\to\infty} a_n = L$.*

The above theorem is useful for when we need to use L'Hôpital's Rule to evaluate a limit.

Theorem (Squeeze Theorem)**.** *If $a_n \leq b_n \leq c_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.*

The above theorem is useful whenever we have sine or cosine terms in our sequence.

Theorem. *If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.*

Use this theorem whenever you have an alternating sequence that appears to converge.

Theorem. *If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then*

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L).$$

Use this theorem when you need to move the limit inside a function.

Series and the Divergence Test

The **infinite series** (or **series**) $\sum_{n=1}^{\infty} a_n$ is the sum of the terms of the sequence a_n:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \dots.$$

We define the **partial sum** of a series to be $s_k = \sum_{n=1}^{k} a_n$. If $\lim_{k\to\infty} s_k$ exists, then the series $\sum_{k=1}^{\infty} a_k$ is **convergent**. If the limit of partial sums doesn't converge, the series **diverges**.

The **geometric series**

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots$$

is convergent if $|r| < 1$ and it converges to

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \quad |r| < 1.$$

If $|r| \geq 1$, the geometric series is divergent.

Theorem. *If the series* $\sum_{n=1}^{\infty}$ *is convergent, then* $\lim_{n\to\infty} = 0$.

CAUTION/WARNING/DANGER AHEAD: The converse of this theorem is false. If $\lim_{n\to\infty} a_n = 0$, it is not necessarily true that $\sum_{n=1}^{\infty} a_n$ converges. An example of this is the **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

We see that $\lim_{n\to\infty} \frac{1}{n} = 0$, but the harmonic series is actually divergent.

The above theorem can be rewritten the following way, known as the Divergence Test (or Test for Divergence:

Theorem (Divergence Test)**.** *If* $\lim_{n\to\infty} a_n \neq 0$, *then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

Example. Consider the series $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{2n^2+1}{n^2+1}\right)$. Since

$$\lim_{n\to\infty} \ln\left(\frac{2n^2+1}{n^2+1}\right) = \ln\left(\lim_{n\to\infty} \frac{2n^2+1}{n^2+1}\right) = \ln 2 \neq 0,$$

the series is divergent by the Divergence Test. □

If $\lim_{n\to\infty} a_n \neq 0$ then we can conclude from the Divergence Test above that the series $\sum_{n=1}^{\infty} a_n$ diverges. If $\lim_{n\to\infty} a_n = 0$, the series $\sum_{n=1}^{\infty} a_n$ may or may not converge (as seen as with the harmonic series above). So how do we determine when a series converges or diverges? The series tests below will help us determine the convergence of a series.

The Integral Test, p-Series Test

Theorem (Integral Test)**.** *Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$.*

(i) If $\int_1^\infty f(x)dx$ is convergent, then $\sum_{n=1}^\infty a_n$ is convergent.

(ii) If $\int_1^\infty f(x)dx$ is divergent, then $\sum_{n=1}^\infty a_n$ is divergent.

Recall that a function is decreasing if $f' < 0$.

Example. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$. The corresponding function to our series is $f(x) = \dfrac{1}{x^2+1}$. $f(x)$ is continuous, positive, and decreasing on the interval $[1, \infty)$. Since

$$\int_1^\infty f(x)dx = \int_1^\infty \frac{1}{x^2+1}dx = \arctan x\Big|_1^\infty = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4},$$

the integral $\int_1^\infty f(x)$ converges, hence by the Integral Test the series also converges. □

The following is a consequence of the Integral Test:

Theorem (p-Series Test)**.** *The p-series $\sum_{p=1}^\infty \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.*

The Comparison Test

Theorem (The Comparison Test)**.** *Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.*

(i) If $a_n \leq b_n$ and $\sum b_n$ is convergent, then $\sum a_n$ is also convergent.

(ii) If $a_n \geq b_n$ and $\sum a_n$ is divergent, then $\sum b_n$ is also divergent.

Example. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2+1}$. Since $\cos^2 n \leq 1$, we have

$$\frac{\cos^2 n}{n^2+1} \leq \frac{1}{n^2+1}.$$

In the previous example we showed that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges, thus by the Comparison Test our series also converges. □

This example shows that the Comparison Test is useful when we have a bounded function (like sin, cosin, arctan) in our series.

Example. Consider the series $\sum_{n=1}^{\infty} \frac{1}{2n-3}$. Since $2n-3 < 2n$, then

$$\frac{1}{2n-3} > \frac{1}{2n}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, thus by the Comparison test our series also diverges. □

Example. Consider the series $\sum_{n=1}^{\infty} \frac{1}{2n+3}$. Since $n > 1$, $3n > 3$ and so $2n+3 < 2n+3n$, therefore

$$\frac{1}{2n+3} > \frac{1}{2n+3n}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{2n+3n} = \sum_{n=1}^{\infty} \frac{1}{5n} = \frac{1}{5}\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, thus by the Comparison Test our series also converges. □

These two previous examples illustrate something important: If we have something that looks like a p-series, it is wise to compare it with a p-series.

The Alternating Series Test

Theorem (The Alternating Series Test)**.** *The alternating series* $\sum_{n=1}^{\infty}(-1)^n a_n$ *converges if the following are true:*

(i) $a_n \geq a_{n+1}$ *(a_n is a decreasing sequence)*

(ii) $\lim_{n\to\infty} a_n \to 0$.

Example. Consider the series $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^{3/4}}$. Since $\cos n\pi = 1$ for $n = 2, 4, 6, ...$, $\cos n\pi = -1$ for $n = 1, 3, 5, ...$, we can rewrite out series as $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/4}}$. Let $a_n = \frac{1}{n^{3/4}}$. We need to show that a_n is a decreasing sequence. There are two ways to do this: (i) show $a_n \geq a_{n+1}$ is a true statement, or (ii) for $a_n = f(n)$, show the function $f(x)$ is decreasing, i.e. $f' < 0$. We will show both ways.

$a_n \geq a_{n+1}$ implies

$$\frac{1}{n^{3/4}} \geq \frac{1}{(n+1)^{3/4}}.$$

We need to rearrange this statement until we come up with something true. Cross multiply, raise each side to the fourth power, then take the cubed root:

$$(n+1)^{3/4} \geq n^{3/4} \quad \Rightarrow \quad (n+1)^3 \geq n^3 \quad \Rightarrow \quad n+1 \geq n,$$

which is clearly a true statement. This proves that a_n is decreasing.
Let's use the derivative now to show that a_n is decreasing. Let $f(x) = \frac{1}{x^{3/4}} = x^{-3/4}$. Then

$$f'(x) = -\frac{3}{4}x^{-7/4}.$$

Since we are working on the domain $[1, \infty)$, $x^{-7/4}$ will always be positive, therefore $-\frac{3}{4}x^{-7/4} < 0$. Thus $f' < 0$, or f is decreasing.

Now we just need to show $\lim_{n\to\infty} a_n = 0$:

$$\lim_{n\to\infty} \frac{1}{n^{3/4}} = 0.$$

Since a_n is decreasing and its limit is zero, by the Alternating Series Test our series converges. □

Absolute Convergence, Ratio Test, and Root Test

A series $\sum a_n$ is **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent. We say $\sum a_n$ is conditionally convergent if it is convergent but $\sum |a_n|$ is divergent.

Example. The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is absolutely convergent since

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent by the p-Series Test.

The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/4}}$ is conditionally convergent since it is convergent by the Alternating Series Test (see previous example), but

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^{3/4}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{3/4}}$$

is divergent by the p-Series Test. □

Theorem. *If a series $\sum a_n$ is absolutely convergent, it is convergent.*

Theorem (The Ratio Test)**.** *Consider the series $\sum_{n=1}^{\infty} a_n$ and the limit*

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L.$$

(i) If $L < 1$ then the series is absolutely convergent (and therefore convergent by the previous theorem.)

(ii) If $L > 1$ then the series is divergent.

(iii) If L = 1, the test is inconclusive.

Example. Consider the series $\sum_{n=1}^{\infty}(-1)^n\frac{n^3}{3^n}$. Let $a_n = (-1)^n\frac{n^3}{3^n}$. Then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\frac{(n+1)^3}{3^{n+1}}}{(-1)^n\frac{n^3}{3^n}}\right| = \lim_{n\to\infty}\frac{(n+1)^3}{3^{n+1}}\cdot\frac{3^n}{n^3} = \lim_{n\to\infty}\frac{1}{3}\left(\frac{n+1}{n}\right)^3 = \lim_{n\to\infty}\frac{1}{3}\left(1+\frac{1}{n}\right)^3 = \frac{1}{3}.$$

Since the limit converged and was less than 1, our series is absolutely convergent by the Ratio Test, thus it is convergent. □

Theorem. *Consider the series* $\sum_{n=1}^{n} a_n$ *and the limit*

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = L.$$

(i) If L < 1, the series is absolutely convergent (and therefore convergent by the theorem above.)

(ii) If L > 1, the series is divergent.

(iii) If L = 1, the test is inconclusive.

Example. Consider the series $\sum_{n=1}^{\infty}\left(\frac{2n+3}{3n+2}\right)^n$. Let $a_n = \left(\frac{2n+3}{3n+2}\right)^n$. Then

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{2n+3}{3n+2}\right)^n} = \lim_{n\to\infty}\frac{2n+3}{3n+2} = \frac{2}{3}.$$

Since the limit converged and was less than 1, our series is absolutely convergent by the Root Test, thus it is convergent. □

Power Series

A **power series** about a is given by

$$\sum_{n=0}^{\infty} c_n(x-a)^n.$$

To find the radius of convergence R of a power series, we can use the Ratio test. We just need to find when $\lim_{n\to\infty}\left|\frac{a_n+1}{a_n}\right| < 1$ for $a_n = c_n(x-a)^n$. To find the interval of convergence, we need to evaluate $|x-a| < R$ at the endpoints.

What functions can be represented as power series? We start by construction series from the geometric series. Recall that a geometric series is given by

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad |r| < 1.$$

We can use this formula to find the power series representation of $\frac{1}{1-x}$:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + ..., \quad |x| < 1.$$

We can also use this representation to derive other similar functions, like $f(x) = \frac{1}{1+x^2}$, $f(x) = \frac{3}{1-x^3}$, etc. We can also use differentiation and integration to derive other power series:

Theorem. *If $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ with radius of convergence R, then*

(i) $\frac{d}{dx}\left(\sum c_n(x-a)^n\right) = \sum \frac{d}{dx}(c_n(x-a)^n)$

(ii) $\int \left(\sum c_n(x-a)^n\right) dx = \sum \left(\int c_n(x-a)^n dx\right)$

Note: be careful with the numbering of your indices when it comes to integration and differentiation.

Taylor Series

Theorem. *If f has a power series expansion about a given by*

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, \quad |x-a| < R$$

then its coefficients are given by $c_n = \frac{f^{(n)}(a)}{n!}$.

The **Taylor series** representation of $f(x)$ about a is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + ...$$

The **Maclaurin series** representation of $f(x)$ is the Taylor series of $f(x)$ about $a = 0$. In particular,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x^1 + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + ...$$

The following are some common Maclaurin series:

$$
\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \text{for all } x \\
\sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad \text{for all } x \\
\cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad \text{for all } x \\
(1+x)^k &= \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + ..., \quad |x| < 1 \text{ (Binomial series)} \\
\ln(1-x) &= -\sum_{n=1}^{\infty} \frac{x^n}{n}, \quad |x| < 1 \\
\tan^{-1}(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad |x| < 1
\end{aligned}
$$

As seen in the previous section, we can integrate and differentiate Taylor series. We can also use Taylor series to evaluate limits.

We expect Taylor series expansions of polynomials to be finite. If we have a finite Taylor expansion, it has radius of convergence $R = \infty$.

Fourier Series

A function $f(x)$ is called **periodic** if there exists a positive number p, called the **period**, such that $f(x+p) = f(x)$.

If $f(x)$ is a periodic function with period 2π and can be represented by a trigonometric series, then

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is called the **Fourier series** of $f(x)$ with the Fourier coefficients

$$
\begin{aligned}
a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx \\
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \quad n \in \mathbb{N} \\
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx.
\end{aligned}
$$

Note that the above is also true if you are on the interval $[0, 2\pi]$.

We also have the following orthogonality properties:

$$\begin{aligned}
&\int_{-\pi}^{\pi} \cos nx \cos mx dx = 0, \quad && \int_{0}^{\pi} \cos nx \cos mx dx = 0 \quad \text{if } n \neq m \\
&\int_{-\pi}^{\pi} \sin nx \sin mx dx = 0, \quad && \int_{0}^{\pi} \sin nx \sin mx dx = 0 \quad \text{if } n \neq m \\
&\int_{-\pi}^{\pi} \sin nx \cos mx dx = 0 \quad \text{for all } n, m
\end{aligned}$$

We can also find the Fourier series of a $2L$ periodic function on an interval $[-L, L]$. The formulas are similar to above:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \left(\frac{n\pi x}{L}\right) \right)$$

where

$$\begin{aligned}
a_0 &= \frac{1}{2L} \int_{-L}^{L} f(x) dx \\
a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n \in \mathbb{N} \\
b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx.
\end{aligned}$$

We also orthogonality properties on arbitrary periods:

$$\begin{aligned}
&\int_{-L}^{L} \cos\left(\frac{n\pi}{L}x\right) \cos\left(\frac{m\pi}{L}x\right) dx = 0, \quad && \int_{0}^{L} \cos\left(\frac{n\pi}{L}x\right) \cos\left(\frac{m\pi}{L}x\right) dx = 0 \quad \text{if } n \neq m \\
&\int_{-L}^{L} \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) dx = 0, \quad && \int_{0}^{L} \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) dx = 0 \quad \text{if } n \neq m \\
&\int_{-L}^{L} \sin\left(\frac{n\pi}{L}x\right) \cos\left(\frac{m\pi}{L}x\right) dx = 0 \quad \text{for all } n, m
\end{aligned}$$

A function is **even** if $f(-x) = f(x)$ for all x, **odd** if $f(-x) = -f(x)$. An even function is symmetric about the y-axis, an odd function is symmetric about the origin. We also have multiplicative properties of even and odd functions:

$$\text{even} \cdot \text{even} = \text{even}, \quad \text{odd} \cdot \text{even} = \text{odd}, \quad \text{odd} \cdot \text{odd} = \text{even}$$

Knowing whether a function is even or odd can help us in our path to finding Fourier series as seen in the Theorem below.

Theorem. *Suppose f is a $2L$-periodic function and has a Fourier series representation. Then*

(i) f is even if and only if $b_n = 0$ for all n, i.e.

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{1}{L}\int_0^L f(x)dx$$
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx.$$

(ii) f is odd if and only if $a_n = 0$ for all n, i.e.

$$f(x) = \sum_{n=1}^{\infty} b_n\left(\frac{n\pi x}{L}\right)$$

where

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

In other words, even functions only have cosine expansions and odd functions only have sine expansions.

Finding the **half range expansions** of a function means to compute the even expansion (cosine expansion) and the odd expansion (sine expansion) using the equations in the theorem above.

The Fourier Transform

The **Fourier** transform of a function f is given by

$$\mathscr{F}\{f(x)\} = \hat{f}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx.$$

Some properties of the Fourier transform:

- $\mathscr{F}\{af + bg\} = a\mathscr{F}\{f\} + b\mathscr{F}\{g\}$ for a, b constants, f, g functions
- If $f(x)$ is continuous, $f(x) \to 0$ as $x \to \pm\infty$, f' is integrable, then

$$\mathscr{F}\{f'(x)\} = iw\mathscr{F}\{f(x)\}.$$

Partial Differential Equations

A **partial differential equation (PDE)** is an equation involving one or more partial derivatives of an unknown function. The **order** of the highest derivative is the order of the PDE.

The one dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

is a second order PDE. The one dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

is also second order.

To solve an ODE we only needed initial conditions. To solve PDEs we need initial conditions *and* boundary conditions.

The Wave Equation

Consider the wave equation problem

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le L,\ t \ge 0 \\ u(0,t) = 0, &\quad u(L,t) = 0 \ \text{(Boundary conditions)} \\ u(x,0) = f(x), &\quad u_t(x,0) = g(x) \ \text{(Initial conditions)} \end{aligned}$$

The general solution to the wave equation with the above conditions is

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L} x\right) \left(b_n \cos \lambda_n t + b_n^* \sin \lambda_n t\right)$$

where

$$\begin{aligned} b_n &= \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx \\ b_n^* &= \frac{2}{cn\pi} \int_0^L g(x) \sin\left(\frac{n\pi}{L} x\right) dx \\ \lambda_n &= \frac{cn\pi}{L}, \quad n = 1, 2, ... \end{aligned}$$

The Wave Equation - D'Alembert's Method

Consider the wave equation problem

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty,\ t \ge 0 \\ u(0,t) = 0, &\quad u(L,t) = 0 \ \text{(Boundary conditions)} \\ u(x,0) = f(x), &\quad u_t(x,0) = g(x) \ \text{(Initial conditions)} \end{aligned}$$

The general solution to the wave equation with the above conditions is

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds.$$

The Heat Equation

Consider the heat equation problem

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 \leq x \leq L, \ t \geq 0$$
$$u(0,t) = 0, \quad u(L,t) = 0 \ \ \text{(Boundary conditions)}$$
$$u(x,0) = f(x) \ \ \text{(Initial condition)}$$

The general solution to the heat equation with the above conditions is

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n^2 t} \sin\left(\frac{n\pi}{L}x\right)$$

where

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx$$
$$\lambda_n = \frac{cn\pi}{L}, \quad n = 1, 2, ...$$