Comparison Geometry for Ricci Curvature

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Chapter 1

Basic Tools and Characterizations of Ricci Curvature Lower Bound

The most basic tool in studying manifolds with Ricci curvature bound is the Bochner formula, which measures the non-commutativity of the covariant derivative and the connection Laplacian. Applying the Bochner formula to distance functions we get important tools like mean curvature and Laplacian comparison theorems, volume comparison theorem. Each of these tools can be used to give a characterization of the Ricci curvature lower bound. These tools have many applications, see next two chapters.

1.1 Bochner's formula

For a smooth function u on a Riemannian manifold (M^n, g), the gradient of u is the vector field ∇u such that $\langle \nabla u, X\rangle = X(u)$ for all vector fields X on M. The Hessian of u is the symmetric bilinear form

$$\mathrm{Hess}\,(u)(X,Y) = XY(u) - \nabla_X Y(u) = \langle \nabla_X \nabla u, Y\rangle,$$

and the Laplacian is the trace $\Delta u = \mathrm{tr}(\mathrm{Hess}\, u)$. For a bilinear form A, we denote $|A|^2 = \mathrm{tr}(AA^t)$.

The Bochner formula for functions is

Theorem 1.1.1 (Bochner's Formula) *For a smooth function u on a Riemannian manifold (M^n, g),*

$$\frac{1}{2}\Delta|\nabla u|^2 = |\mathrm{Hess}\, u|^2 + \langle \nabla u, \nabla(\Delta u)\rangle + \mathrm{Ric}(\nabla u, \nabla u). \tag{1.1}$$

Proof: We can derive the formula by using local geodesic frame and commuting the derivatives. Fix $x \in M$, let $\{e_i\}$ be an orthonormal frame in a neighborhood of x such that, at x, $\nabla_{e_i} e_j(x) = 0$ for all i, j. At x,

$$\begin{aligned}
\frac{1}{2}\Delta|\nabla u|^2 &= \frac{1}{2}\sum_i e_i e_i \langle \nabla u, \nabla u\rangle \\
&= \sum_i e_i \langle \nabla_{e_i} \nabla u, \nabla u\rangle = \sum_i e_i \operatorname{Hess} u(e_i, \nabla u) \\
&= \sum_i e_i \operatorname{Hess} u(\nabla u, e_i) = \sum_i e_i \langle \nabla_{\nabla u} \nabla u, e_i\rangle \\
&= \sum_i \langle \nabla_{e_i} \nabla_{\nabla u} \nabla u, e_i\rangle \\
&= \sum_i \left[\langle \nabla_{\nabla u} \nabla_{e_i} \nabla u, e_i\rangle + \langle \nabla_{[e_i, \nabla u]} \nabla u, e_i\rangle + \langle R(e_i, \nabla u)\nabla u, e_i\rangle\right]
\end{aligned} \tag{1.2}$$

Now at x,

$$\begin{aligned}
\sum_i \langle \nabla_{\nabla u} \nabla_{e_i} \nabla u, e_i\rangle &= \sum_i \left[\nabla u \langle \nabla_{e_i} \nabla u, e_i\rangle - \langle \nabla_{e_i} \nabla u, \nabla_{\nabla u} e_i\rangle\right] \\
&= \nabla u(\Delta u) = \langle \nabla u, \nabla(\Delta u)\rangle,
\end{aligned} \tag{1.3}$$

and

$$\begin{aligned}
\sum_i \langle \nabla_{[e_i, \nabla u]} \nabla u, e_i\rangle &= \sum_i \operatorname{Hess} u([e_i, \nabla u], e_i) \\
&= \sum_i \operatorname{Hess} u(e_i, \nabla_{e_i} \nabla u) \\
&= \sum_i \langle \nabla_{e_i} \nabla u, \nabla_{e_i} \nabla u\rangle = |\operatorname{Hess} u|^2.
\end{aligned} \tag{1.4}$$

Combining (1.2), (1.3) and (1.4) gives (1.1). ■

Applying the Cauchy-Schwarz inequality $|\operatorname{Hess} u|^2 \geq \frac{(\Delta u)^2}{n}$ to (1.1) we obtain the following inequality

$$\frac{1}{2}\Delta|\nabla u|^2 \geq \frac{(\Delta u)^2}{n} + \langle \nabla u, \nabla(\Delta u)\rangle + \operatorname{Ric}(\nabla u, \nabla u). \tag{1.5}$$

If $\operatorname{Ric} \geq (n-1)H$, then

$$\frac{1}{2}\Delta|\nabla u|^2 \geq \frac{(\Delta u)^2}{n} + \langle \nabla u, \nabla(\Delta u)\rangle + (n-1)H|\nabla u|^2. \tag{1.6}$$

This inequality characterizes a lower bound on Ricci curvature for Riemannian manifolds. Namely, if (1.6) holds for all functions $u \in C^3(M)$, then $\operatorname{Ric} \geq (n-1)H$. This can be seen as follows. Given any $x_0 \in M$ and $v_0 \in T_{x_0}M$, let u be a C^3 function such that $\nabla u(x_0) = v_0$ and $\operatorname{Hess} u(x_0) = \lambda_0 I_n$. Then from (1.1) and (1.6), we have $\operatorname{Ric}(v_0, v_0) \geq (n-1)H|v_0|^2$, so $\operatorname{Ric} \geq (n-1)H$.

The Bochner formula simplifies whenever $|\nabla u|$ or Δu are simply. Hence it is natural to apply it to the distance functions, harmonic functions, and the eigenfunctions among others, getting many applications. The formula has a more general version (Weitzenböck type) for vector fields (1-forms).

1.2 Mean Curvature and Local Laplacian Comparison

Here we apply the Bochner formula to distance functions. We call $\rho : U \to \mathbb{R}$, where $U \subset M^n$ is open, is a distance function if $|\nabla \rho| \equiv 1$ on U.

Example 1.2.1 *Let $A \subset M$ be a submanifold, then $\rho(x) = d(x, A) = \inf\{d(x, y) | y \in A\}$ is a distance function on some open set $U \subset M$. When $A = q$ is a point, the distance function $r(x) = d(q, x)$ is smooth on $M \setminus \{q, C_q\}$, where C_q is the cut locus of q. When A is a hypersurface, $\rho(x)$ is smooth outside the focal points of A.*

For a smooth distance function $\rho(x)$, $\operatorname{Hess}\rho$ is the covariant derivative of the normal direction $\partial_r = \nabla\rho$. Hence $\operatorname{Hess}\rho = II$, the second fundamental form of the level sets $\rho^{-1}(r)$, and $\Delta\rho = m$, the mean curvature. For $r(x) = d(q, x)$, $m(r, \theta) \sim \frac{n-1}{r}$ as $r \to 0$; for $\rho(x) = d(x, A)$, where A is a hypersurface, $m(y, 0) = m_A$, the mean curvature of A, for $y \in A$.

Putting $u(x) = \rho(x)$ in (1.1), we obtain the Riccati equation along a radial geodesic,

$$0 = |II|^2 + m' + \operatorname{Ric}(\partial_r, \partial_r). \tag{2.1}$$

By the Cauchy-Schwarz inequality,

$$|II|^2 \geq \frac{m^2}{n-1}.$$

Thus we have the Riccati inequality

$$m' \leq -\frac{m^2}{n-1} - \operatorname{Ric}(\partial_r, \partial_r). \tag{2.2}$$

If $\operatorname{Ric}_{M^n} \geq (n-1)H$, then

$$m' \leq -\frac{m^2}{n-1} - (n-1)H. \tag{2.3}$$

From now on, unless specified otherwise, we assume $m = \Delta r$, the mean curvature of geodesic spheres. Let M_H^n denote the complete simply connected space of constant curvature H and m_H (or m_H^n when dimension is needed) the mean curvature of its geodesics sphere, then

$$m_H' = -\frac{m_H^2}{n-1} - (n-1)H. \tag{2.4}$$

Let $\mathrm{sn}_H(r)$ be the solution to

$$\mathrm{sn}_H'' + H\mathrm{sn}_H = 0$$

such that $\mathrm{sn}_H(0) = 0$ and $\mathrm{sn}_H'(0) = 1$, i.e. sn_H are the coefficients of the Jacobi fields of the model spaces M_H^n:

$$\mathrm{sn}_H(r) = \begin{cases} \frac{1}{\sqrt{H}} \sin \sqrt{H} r & H > 0 \\ r & H = 0 \\ \frac{1}{\sqrt{|H|}} \sinh \sqrt{|H|} r & H < 0 \end{cases}.$$

Then

$$m_H = (n-1)\frac{\mathrm{sn}_H'}{\mathrm{sn}_H}. \tag{2.5}$$

As $r \to 0$, $m_H \sim \frac{n-1}{r}$. The mean curvature comparison is

Theorem 1.2.2 (Mean Curvature Comparison) *If* $\mathrm{Ric}_{M^n} \geq (n-1)H$, *then along any minimal geodesic segment from* q*,*

$$m(r) \leq m_H(r). \tag{2.6}$$

Moreover, equality holds if and only if all radial sectional curvatures are equal to H*.*

Since $\lim_{r\to 0}(m - m_H) = 0$, this follows from the Riccati equation comparison. However, a direct proof using only the Riccati inequalities (2.3), (2.4) does not seem to be in the literature. From (2.3) and (2.4) we have

$$(m - m_H)' \leq -\frac{1}{n-1}(m^2 - m_H^2). \tag{2.7}$$

Here we present three somewhat different proofs. The first proof uses the continuity method, the second solving linear ODE, the third by considering $\mathrm{sn}_H^2(m - m_H)$ directly. The last two proofs are motived from generalizations of the mean curvature comparison to weaker Ricci curvature lower bounds [34, **?**], allowing natural extensions, see Chapters **??**.
Proof I: Let $m_+^H = (m - m_H)_+ = \max\{m - m_H, 0\}$, amount of mean curvature comparison failed. By (2.7)

$$(m_+^H)' \leq -\frac{1}{n-1}(m + m_H)m_+^H.$$

If $m + m_H \geq 0$, then $(m_+^H)' \leq 0$. When r is small, m is close to m_H, so $m + m_H > 0$. Therefore $m_+^H = 0$ for all r small. Let r_0 be the biggest number such that $m_+^H(r) = 0$ on $[0, r_0]$ and $m_+^H > 0$ on $(r_0, r_0 + \epsilon_0]$ for some $\epsilon_0 > 0$. We have $r_0 > 0$. Claim: $r_0 =$ the maximum of r, where m, m_H are defined on $(0, r]$. Otherwise, we have on $(r_0, r_0 + \epsilon_0]$

$$\frac{(m_+^H)'}{m_+^H} \leq -\frac{1}{n-1}(m + m_H) \tag{2.8}$$

and m, m_H are bounded. Integrate (2.8) from $r_0+\epsilon$ to $r_0+\epsilon_0$ (where $0<\epsilon<\epsilon_0$) gives

$$\ln \frac{m_+^H(r_0+\epsilon_0)}{m_+^H(r_0+\epsilon)} \le \int_{r_0+\epsilon}^{r_0+\epsilon_0} -\frac{1}{n-1}(m+m_H)dr.$$

The right hand side is bounded by $C\epsilon_0$ since m, m_H are bounded on $(r_0, r_0+\epsilon_0]$. Therefore $m_+^H(r_0+\epsilon_0) \le m_+^H(r_0+\epsilon)e^{C\epsilon_0}$. Now let $\epsilon \to 0$ we get $m_+^H(r_0+\epsilon_0) \le 0$, which is a contradiction.

Proof II: We only need to work on the interval where $m - m_H \ge 0$. On this interval $-(m^2 - m_H^2) = -m_+^H(m - m_H + 2m_H) = -m_+^H(m_+^H + 2m_H)$. Thus (2.7) gives

$$(m_+^H)' \le -\frac{(m_+^H)^2}{n-1} - 2\frac{m_+^H \cdot m_H}{n-1} \le -2\frac{m_+^H \cdot m_H}{n-1} = -2\frac{sn_H'}{sn_H}m_+^H.$$

Hence $(sn_H^2 m_+^H)' \le 0$. Since $sn_H^2(0)m_+^H(0) = 0$, we have $sn_H^2 m_+^H \le 0$ and $m_+^H \le 0$. Namely $m \le m_H$.

Proof III: We have

$$\begin{aligned} \left(\mathrm{sn}_H^2(m-m_H)\right)' &= 2\mathrm{sn}_H'\mathrm{sn}_H(m-m_H) + sn_H^2(m-m_H)' \\ &\le \frac{2}{n-1}\mathrm{sn}_H^2 m_H(m-m_H) - \frac{1}{n-1}\mathrm{sn}_H^2(m^2-m_H^2) \\ &= -\frac{\mathrm{sn}_H^2}{n-1}(m-m_H)^2 \le 0 \end{aligned}$$

Here in the 2nd line we have used (2.7) and (2.5).

Since $\lim_{r\to 0} \mathrm{sn}_H^2(m-m_H) = 0$, integrating from 0 to r yields

$$\mathrm{sn}_H^2(r)(m(r) - m_H(r)) \le 0,$$

which gives (3.3.4).

When equality occurs, the Cauchy- Schwarz inequality is an equality, which means $II = \frac{sn_H'}{sn_H} I_{n-1}$ along the minimal geodesic. Therefore all radial sectional curvatures are equal to H. ■

Recall that $m = \Delta r$. From (3.3.4), we get the local Laplacian comparison for distance functions

$$\Delta r \le \Delta_H r, \quad \text{for all } x \in M \setminus \{q, C_q\}. \tag{2.9}$$

The local Laplacian comparison immediately gives us Myers' theorem [**?**], a diameter comparison.

Theorem 1.2.3 (Myers, 1941) *If* $\mathrm{Ric}_M \ge (n-1)H > 0$, *then* $\mathrm{diam}(M) \le \pi/\sqrt{H}$.

Proof: If $\mathrm{diam}(M) > \pi/\sqrt{H}$, let $q, q' \in M$ such that $d(q,q') = \pi/\sqrt{H} + \epsilon$ for some $\epsilon > 0$, and γ be a minimal geodesic connecting q, q' with $\gamma(0) = q, \gamma(\pi/\sqrt{H}+\epsilon) = q'$. Then $\gamma(t) \notin C_q$ for all $0 < t \le \pi/\sqrt{H}$. Let $r(x) = d(q,x)$, then r is smooth at $\gamma(\pi/\sqrt{H})$, therefore Δr is well defined at $\gamma(\pi/\sqrt{H})$. By (2.9) $\Delta r \le \Delta_H r$ at all $\gamma(t)$ with $0 < t < \pi/\sqrt{H}$. Now $\lim_{r\to\pi/\sqrt{H}} \Delta_H r = -\infty$ so Δr is not defined at $\gamma(\pi/\sqrt{H})$. This is a contradiction. ■

Equation (2.4) also holds when $m_H = \Delta d(x, A_H)$, where $A_H \subset M^n_H$ is a hypersurface. Therefore the proof of Theorem 1.2.2 carries over, and we have a comparison of the mean curvature of level sets of $d(x,A)$ and $d(x,A_H)$ when A and A_H are hypersurfaces with $m_A \le m_{A_H}$ and $\mathrm{Ric}_M \ge (n-1)H$. Equation (2.4) doesn't hold if A_H is a submanifold which is not a point or hypersurface, therefore one needs stronger curvature assumption to do comparison [**?**].

1.3 Global Laplacian Comparison

The Laplacian comparison (2.9) holds globally in various weak senses and the standard PDE theory carries over. As a result the Laplacian comparison is very powerful, see next Chapter for some crucial applications.

First we prove an important property about cut locus.

Lemma 1.3.1 *For each $q \in M$, the cut locus C_q has measure zero.*

One can show C_q has measure zero by observing that the region inside the cut locus is star-shaped [**?**, Page 112]. The author comes up with the following argument in proving that Perelman's l-cut locus [31] has measure zero.

since the $\mathcal{L}$-exponential map is smooth and the l-distance function is locally Lipschitz. **Proof:** Recall that if $x \in C_q$, then either x is a (first) conjugate point of q or there are two distinct minimal geodesics connecting q and x [**?**], so $x \in$\{conjugate locus of q\} $\cup$ \{the set where r is not differentiable\}. The conjugate locus of q consists of the critical values of $\exp_q$. Since $\exp_q$ is smooth, by Sard's theorem, the conjugate locus has measure zero. The set where r is not differentiable has measure zero since r is Lipschitz. Therefore the cut locus C_q has measure zero. ■

First we review the definitions (for simplicity we only do so for the Laplacian) and study the relationship between these different weak senses.

For a continuous function f on $M, q \in M$, a function f_q defined in a neighborhood U of q, is an upper barrier of f at q if f_q is $C^2(U)$ and

$$f_q(q) = f(q), \quad f_q(x) \ge f(x) \ (x \in U). \tag{3.10}$$

Definition 1.3.2 *For a continuous function f on M, we say $\Delta f(q) \le c$ in the barrier sense (f is a barrier subsolution to the equation $\Delta f = c$ at q), if for all $\epsilon > 0$, there exists an upper barrier $f_{q,\epsilon}$ such that $\Delta f_{q,\epsilon}(q) \le c + \epsilon$.*

This notion was defined by Calabi [3] back in 1958 (he used the terminology "weak sense" rather than "barrier sense"). A weaker version is in the sense of viscosity, introduced by Crandall and Lions in [9].

Definition 1.3.3 *For a continuous function f on M, we say $\Delta f(q) \leq c$ in the viscosity sense (f is a viscosity subsolution of $\Delta f = c$ at q), if $\Delta\phi(q) \leq c$ whenever $\phi \in C^2(U)$ and $(f-\phi)(q) = \inf_U(f-\phi)$, where U is a neighborhood of q.*

Clearly barrier subsolutions are viscosity subsolutions.

Another very useful notion is subsolution in the sense of distributions.

Definition 1.3.4 *For continuous functions f, h on an open domain $\Omega \subset M$, we say $\Delta f \leq h$ in the distribution sense (f is a distribution subsolution of $\Delta f = h$) on Ω, if $\int_\Omega f\Delta\phi \leq \int_\Omega h\phi$ for all $\phi \geq 0$ in $C_0^\infty(\Omega)$.*

By [19] if f is a viscosity subsolution of $\Delta f = h$ on Ω, then it is also a distribution subsolution and vice verse, see also [26], [**?**, Theorem 3.2.11].

For geometric applications, the barrier and distribution sense are very useful and the barrier sense is often easy to check. Viscosity gives a bridge between them. As observed by Calabi [3] one can easily construct upper barriers for the distance function.

Lemma 1.3.5 *If γ is minimal from p to q, then for all $\epsilon > 0$, the function $r_{q,\epsilon}(x) = \epsilon + d(x, \gamma(\epsilon))$, is an upper barrier for the distance function $r(x) = d(p,x)$ at q.*

Since $r_{q,\epsilon}$ trivially satisfies (3.10) the lemma follows by observing that it is smooth in a neighborhood of q.

Upper barriers for Perelman's l-distance function can be constructed very similarly.

Therefore the Laplacian comparison (2.9) holds globally in all the weak senses above. Cheeger-Gromoll (unaware of Calabi's work at the time) had proved the Laplacian comparison in the distribution sense directly by observing the very useful fact that near the cut locus ∇r points towards the cut locus [**?**], see also [5]. (However it is not clear if this fact holds for Perelman's l-distance function.)

One reason why these weak subsolutions are so useful is that they still satisfy the following classical Hopf strong maximum principle, see [3], also e.g. [5] for the barrier sense, see [27, 20] for the distribution and viscosity senses, also [**?**, Theorem 3.2.11] in the Euclidean case.

Theorem 1.3.6 (Strong Maximum Principle) *If on a connected open set, $\Omega \subset M^n$, the function f has an interior minimum and $\Delta f \leq 0$ in any of the weak senses above, then f is constant on Ω.*

These weak solutions also enjoy the regularity (e.g. if f is a weak sub and sup solution of $\Delta f = 0$, then f is smooth), see e.g. [**?**].

The Laplacian comparison also works for radial functions (functions composed with the distance function). In geodesic polar coordinate, we have

$$\Delta f = \tilde{\Delta} f + m(r,\theta)\frac{\partial}{\partial r}f + \frac{\partial^2 f}{\partial r^2}, \tag{3.11}$$

where $\tilde{\Delta}$ is the induced Laplacian on the sphere and $m(r, \theta)$ is the mean curvature of the geodesic sphere in the inner normal direction. Therefore

Theorem 1.3.7 (Global Laplacian Comparison) *If* $\mathrm{Ric}_{M^n} \geq (n-1)H$, *in all the weak senses above, we have*

$$\Delta f(r) \leq \Delta_H f(r) \quad (\textit{if } f' \geq 0), \tag{3.12}$$
$$\Delta f(r) \geq \Delta_H f(r) \quad (\textit{if } f' \leq 0). \tag{3.13}$$

1.4 Volume Comparison

1.4.1 Volume of Riemannian Manifold

How do we compute the volume of Riemannian manifold? Recall that for a subset $U \subset \mathbb{R}^n$, we define

$$\mathrm{Vol}(U) = \int_U 1\, dvol = \int_U 1\, dx_1 \cdots dx_n,$$

where $x_1, \cdots, x_n$ are the standard coordinate. One can compute it with different coordinates by using the change of variable formula.

Lemma 1.4.1 (Change of Variables Formula) *Suppose* $U, V \subset \mathbb{R}^n$ *and that* $\psi : V \to U$ *is a diffeomorphism. Suppose* $\psi(x) = y$. *Then*

$$\int_U dvol = \int_U 1\, dy_1 \cdots dy_n = \int_V |Jac(\psi)|\, dx_1 \cdots dx_n.$$

For a general Riemannian manifold M^n, let

$$\psi_\alpha : U_\alpha \to \mathbb{R}^n$$

be a chart and set $E_{ip} = (\psi_\alpha^{-1})_*(\frac{\partial}{\partial x_i})$. In general, the E_{ip}'s are not orthonormal. Let $\{e_k\}$ be an orthonormal basis of T_pM. Then

$$E_{ip} = \sum_{k=1}^n a_{ik} e_k.$$

The volume of the parallelepiped spanned by $\{E_{ip}\}$ is $|\det(a_{ik})|$. Now

$$g_{ij} = \sum_{k=1}^n a_{ik} a_{jk},$$

so $\det(g_{ij}) = (\det(a_{ij}))^2$. Thus

$$\mathrm{Vol}(U_\alpha) = \int_{\psi(U_\alpha)} \sqrt{|\det(g_{ij})|} \circ (\psi_\alpha^{-1})\, dx_1 \cdots dx_n.$$

By the change of variables formula in $\mathbb{R}^n$ (Lemma 1.4.1), this volume is well defined, namely it is independent of local coordinate charts.

Definition 1.4.2 (Volume Form) *Any term*

$$dvol = \sqrt{|\det(g_{ij})|} \circ (\psi_\alpha^{-1})\, dx_1 \cdots dx_n$$

is called a volume density element, or volume form, on M*.*

Now we can compute the volume of M by partition of unity,

$$\mathrm{Vol}(M) = \int_M 1\, dvol = \sum_\alpha \int_{\psi(U_\alpha)} f_\alpha\, dvol,$$

where $\{U_\alpha\}$ are coordinate charts covering M, $\{f_\alpha\}$ is a partition of unity subordinate to $\{U_\alpha\}$.

Since partitions of unity are not practically effective, we look for charts that cover all but a measure zero set. Since the cut locus has measure zero, the best is the exponential coordinate. For $q \in M^n$, let $D_q \subset T_qM$ be the segment disk. Then

$$\exp_q : D_q \to M \setminus C_q$$

is a diffeomorphism. We can either use Euclidean coordinates or polar coordinates on D_q. For balls it is convenient to use polar coordinate. From the diffeomorphism

$$\exp_q : D_q \setminus \{0\} \to M \setminus (C_q \cup \{q\}),$$

set

$$E_i = (\exp_q)_*(\frac{\partial}{\partial \theta_i})$$

and

$$E_n = (\exp_q)_*(\frac{\partial}{\partial r}).$$

To compute the g_{ij}'s, we want explicit expressions for E_i and E_n. Since $\exp_q$ is a radial isometry, $g_{nn} = 1$ and $g_{ni} = 0$ for $1 \le i < n$. Let $J_i(r, \theta)$ be the Jacobi field with $J_i(0) = 0$ and $J_i{}'(0) = \frac{\partial}{\partial \theta_i}$. Then

$$E_i(\exp_q(r, \theta)) = J_i(r, \theta).$$

If we write J_i and $\frac{\partial}{\partial r}$ in terms of an orthonormal basis $\{e_k\}$, we have $J_i = \sum_{k=1}^n a_{ik} e_k$. Thus

$$\sqrt{\det(g_{ij})(r, \theta)} = |\det(a_{ik})| = ||J_1 \wedge \cdots \wedge J_{n-1} \wedge \frac{\partial}{\partial r}||.$$

Let

$$\mathcal{A}(r, \theta) = ||J_1 \wedge \cdots \wedge J_{n-1} \wedge \frac{\partial}{\partial r}||, \tag{4.1}$$

the volume density, or volume element, of M is

$$dvol = \mathcal{A}(r, \theta)\, dr d\theta_{n-1}.$$

Example 1.4.3 *$\mathbb{R}^n$ has Jacobi equation $J'' = R(T, J)T$. Thus, if $J(0) = 0$ and $J'(0) = \frac{\partial}{\partial\theta_i}$,*

$$J(r) = r\frac{\partial}{\partial\theta_i}.$$

Hence the volume element is

$$dvol = r^{n-1}\, dr d\theta_{n-1}.$$

Example 1.4.4 *S^n has $J_i(r) = \sin(r)\frac{\partial}{\partial\theta_i}$. Hence*

$$dvol = \sin^{n-1}(r)\, dr d\theta_{n-1}.$$

Example 1.4.5 *$\mathbb{H}^n$ has $J_i(r) = \sinh(r)\frac{\partial}{\partial\theta_i}$. Hence*

$$dvol = \sinh^{n-1}(r)\, dr d\theta_{n-1}.$$

Example 1.4.6 *We can compute the volume of unit disk in $\mathbb{R}^n$.*

$$\omega_n = \int_{S^{n-1}} \int_0^1 r^{n-1}\, dr d\theta_{n-1} = \frac{1}{n}\int_{S^{n-1}} d\theta_{n-1},$$

noting that

$$\int_{S^{n-1}} d\theta_{n-1} = \frac{2(\pi)^{n/2}}{\Gamma(n/2)}.$$

In general, since J_i has the Taylor expansion

$$J_i(r) = r\frac{\partial}{\partial\theta_i} + \frac{r^3}{3!}R(\partial_r, \frac{\partial}{\partial\theta_i})\partial_r + \cdots,$$

plug this into (4.1), we have the following Taylor expansion for $\mathcal{A}$,

$$\mathcal{A}(r,\theta) = r^{n-1} - \frac{r^{n+1}}{6}\mathrm{Ric}(\partial_r, \partial_r) + \cdots. \tag{4.2}$$

1.4.2 Comparison of Volume Elements

Theorem 1.4.7 *Suppose M^n has $\mathrm{Ric}_M \geq (n-1)H$. Let $dvol = \mathcal{A}(r,\theta)\, dr d\theta_{n-1}$ be the volume element of M in geodesic polar coordinate at q and let $dvol_H = \mathcal{A}_H(r,\theta)\, dr d\theta_{n-1}$ be the volume element of the model space M^n_H. Then*

$$\frac{\mathcal{A}(r,\theta)}{\mathcal{A}_H(r)} \text{ is nonincreasing along any minimal geodesic segment from } q. \tag{4.3}$$

This follows from the following lemma and the mean curvature comparison.

Lemma 1.4.8 *The relative rate of change of the volume element is given by the mean curvature,*

$$\frac{\mathcal{A}'}{\mathcal{A}}(r,\theta) = m(r,\theta). \tag{4.4}$$

Proof: Let γ be a unit speed geodesic with $\gamma(0) = q$, $J_i(r)$ be the Jacobi field along γ with $J_i(0) = 0$ and $J_i'(0) = \frac{\partial}{\partial\theta_i}$ for $i = 1 \cdots n-1$ and $J_n'(0) = \gamma'(0)$ where $\{\frac{\partial}{\partial\theta_1}, \frac{\partial}{\partial\theta_2}, \cdots, \gamma'(0)\}$ is an orthonormal basis of T_qM. Recall

$$\frac{\mathcal{A}'(r,\theta)}{\mathcal{A}(r,\theta)} = \frac{||J_1 \wedge \cdots \wedge J_n||'}{||J_1 \wedge \cdots \wedge J_n||}.$$

For any $r = r_0$ such that $\gamma|_{[0,r_0+\epsilon)}$ is minimal, let $\{\bar{J}_i(r_0)\}$ be an orthonormal basis of $T_{\gamma(r_0)}M$ with $\bar{J}_n(r_0) = \gamma'(r_0)$. Since we are inside the cut locus, there are no conjugate points. Therefore, $\{J_i(r_0)\}$ is also a basis of $T_{\gamma(r_0)}M$. So we can write

$$\bar{J}_i(r_0) = \sum_{k=1}^{n} b_{ik} J_k(r_0).$$

For all $0 \le r < r_0 + \epsilon$, define

$$\bar{J}_i(r) = \sum_{k=1}^{n} b_{ik} J_k(r).$$

Then $\{\bar{J}_i\}$ are Jacobi fields along γ which is an orthonormal basis at $\gamma(r_0)$. Since

$$||\bar{J}_1 \wedge \cdots \wedge \bar{J}_n|| = \det(b_{ij})||J_1 \wedge \cdots \wedge J_n||$$

for all $r \in [0, r_0 + \epsilon)$, and $b_{ij}s$ are constant,

$$\frac{||J_1 \wedge \cdots \wedge J_n||'}{||J_1 \wedge \cdots \wedge J_n||}(r) = \frac{||\bar{J}_1 \wedge \cdots \wedge \bar{J}_n||'}{||\bar{J}_1 \wedge \cdots \wedge \bar{J}_n||}(r).$$

At r_0, $||\bar{J}_1 \wedge \cdots \wedge \bar{J}_n||(r_0) = 1$. Therefore

$$\begin{aligned} \frac{\mathcal{A}'(r,\theta)}{\mathcal{A}(r,\theta)}(r_0) &= ||\bar{J}_1 \wedge \cdots \wedge \bar{J}_n||'(r_0) \\ &= \sum_{k=1}^{n} ||\bar{J}_1 \wedge \cdots \wedge \bar{J}_k' \wedge \cdots \wedge \bar{J}_n||. \end{aligned} \tag{4.5}$$

Since $\{\bar{J}_i(r_0)\}$ is an orthonormal basis of $T_{\gamma(r_0)}M$ we have that

$$\bar{J}_k'(r_0) = \sum_{l=1}^{n} \langle \bar{J}_k'(r_0), \bar{J}_l(r_0)\rangle \bar{J}_l(r_0).$$

Plug this into (4.5), we get

$$\frac{\mathcal{A}'(r,\theta)}{\mathcal{A}(r,\theta)}(r_0) = \sum_{k=1}^{n} \langle \bar{J}_k'(r_0), \bar{J}_k(r_0)\rangle = \sum_{k=1}^{n-1} \langle \nabla_{\bar{J}_k}\gamma', \bar{J}_k\rangle(r_0) = m(r_0, \gamma'(0)).$$

■

Proof of Theorem 1.4.7: By (4.1),

$$\left(\frac{\mathcal{A}(r,\theta)}{\mathcal{A}_H(r)}\right)' = \frac{m\mathcal{A}\mathcal{A}_H - \mathcal{A}m_H\mathcal{A}_H}{\mathcal{A}_H^2} = (m - m_H)\frac{\mathcal{A}(r,\theta)}{\mathcal{A}_H(r)}.$$

The mean curvature comparison (3.3.4) gives $m - m_H \leq 0$, therefore $\frac{\mathcal{A}(r,\theta)}{\mathcal{A}_H(r)}$ is nonincreasing in r. ∎

(4.1) and (4.2) give the following Taylor expansion for the mean curvature,

$$m(r,\theta) = \frac{n-1}{r} - \frac{r}{3}\mathrm{Ric}(\partial_r, \partial_r) + \cdots. \tag{4.6}$$

1.4.3 Volume Comparison

Integrating (4.3) along the sphere directions, and then the radial direction gives the relative volume comparison of geodesic spheres and balls. Let $\Gamma_r = \{\theta \in S^{n-1} |$ the normal geodesic γ with $\gamma(0) = x, \gamma'(0) = \theta$ has $d(\gamma(0), \gamma(r)) = r\}$. Then the volume of the geodesic sphere, $S(x,r) = \{y \in M | d(x,y) = r\}$, $A(x,r) = \int_{\Gamma_r} \mathcal{A}(r,\theta) d\theta_{n-1}$. Extend $\mathcal{A}(r,\theta)$ by zero to all S^{n-1}, we have $A(x,r) = \int_{S^{n-1}} \mathcal{A}(r,\theta)d\theta_{n-1}$. Let $A_H(r)$ be the volume of the geodesic sphere in the model space. If $\mathcal{A}(r_0,\theta) = 0$, then $\mathcal{A}(r_0,\theta) = 0$ for all $r \geq r_0$, so (4.3) also holds in the extended region.

Theorem 1.4.9 (Bishop-Gromov's Relative Volume Comparison) *Suppose M^n has $\mathrm{Ric}_M \geq (n-1)H$. Then*

$$\frac{A(x,r))}{A_H(r))} \text{ and } \frac{\mathrm{Vol}\,(B(x,r))}{\mathrm{Vol}_H(B(r))} \text{ are nonincreasing in } r. \tag{4.7}$$

In particular,

$$\mathrm{Vol}\,(B(x,r)) \leq \mathrm{Vol}_H(B(r)) \quad \text{for all } \; r > 0, \tag{4.8}$$

$$\frac{\mathrm{Vol}\,(B(x,r))}{\mathrm{Vol}\,(B(x,R))} \geq \frac{\mathrm{Vol}_H(B(r))}{\mathrm{Vol}_H(B(R))} \quad \text{for all } \; 0 < r \leq R, \tag{4.9}$$

and equality holds if and only if $B(x,r)$ is isometric to $B_H(r)$.

Proof:

$$\frac{d}{dr}\left(\frac{A(x,r)}{A_H(r)}\right) = \frac{1}{\mathrm{Vol}S^{n-1}}\int_{S^{n-1}} \frac{d}{dr}\left(\frac{\mathcal{A}(r,\theta)}{\mathcal{A}_H(r)}\right) d\theta_{n-1} \leq 0.$$

The monotonicity of the ratio of volume of balls follows from this and the lemma below since $\mathrm{Vol}(B(x,r)) = \int_0^r A(x,t)dt$. ∎

Lemma 1.4.10 *If $f(t)/g(t)$ is nonincreasing in t, with $g(t) > 0$, then*

$$H(r,R) = \frac{\int_r^R f(t)dt}{\int_r^R g(t)dt}$$

is nonincreasing in r and R.

Proof: We have

$$\frac{\partial}{\partial r}H(r,R) = \frac{-f(r)\int_r^R g(t)dt + g(r)\int_r^R f(r)dr}{\left(\int_r^R g(t)dt\right)^2}.$$

Now

$$\frac{f(t)}{g(t)} \leq \frac{f(r)}{g(r)}$$

implies

$$g(r)f(t) \leq f(r)g(t),$$

so

$$\int_r^R g(r)f(t)dt \leq \int_r^R f(r)g(t)dt.$$

Thus $\frac{\partial}{\partial r}H(r,R) \leq 0$. Similarly we have $\frac{\partial}{\partial R}H(r,R) \leq 0$. ■

Instead of integrating (4.3) along the whole unit sphere and/or radial direction, we can integrate along any sector of S^{n-1} and/or segment of the radial direction. Fix $x \in M^n$, for any measurable set $B \subset M$, connect every point of $y \in B$ to x with a minimal geodesic γ_y such that $\gamma_y(0) = x, \gamma_y(1) = y$. For $t \in [0,1]$, let $B_t = \{\gamma_y(t) | y \in B\}$. Since cut locus C_x has measure zero, B_t is uniquely determined up to a modification on a null measure set. (4.3) gives

$$\mathrm{Vol}(B_t) \geq t\int_B \frac{\mathcal{A}_H(td(x,y))}{\mathcal{A}_H(d(x,y))}dvol_y. \tag{4.10}$$

Chapter 2

Comparison for Integral Ricci Curvature

2.1 Integral Curvature: an Overview

What's integral curvature? A natural integral curvature is the L^p-norm of the curvature tensor. For a compact Riemannian manifold M^n, $x \in M$, let $\sigma(x) = \max_{v,w,\in T_xM} |K(v,w)|$, where $K(v,w)$ is the sectional curvature of the plane spanned by v, w. The L^p-norm of the curvature tensor is

$$\|Rm\|_p = \left(\int_M \sigma(x)^p dvol \right)^{1/p}.$$

When the metric g scales by λ^2, the sectional curvature scales by λ^{-2}, volume by λ^n, so $\|Rm\|_p$ scales by $\lambda^{\frac{n}{p}-2}$. Therefore when $p = \frac{n}{2}$, $\|Rm\|_p$ is scale invariant, while for $p < \frac{n}{2}$, one can make $\|Rm\|_p$ small just by choosing λ small, not a very restrictive condition. Sometime the normalized norm,

$$\overline{\|Rm\|}_p = \left(\frac{1}{\mathrm{Vol} M} \int_M \sigma(x)^p dvol \right)^{1/p},$$

which scales like curvature, is more appropriate. When M is noncompact, one can define the integral over a ball (see below).

Bounds on these integral curvatures are extensions of two sided pointwise curvature bounds to integral. What about one sided curvature bound? Or integral curvature lower bound? Here we specify for Ricci curvature. Given $H \in \mathbb{R}$, we can measure the amount of Ricci curvature lying below $(n-1)H$ in L^p norm.

For each $x \in M^n$ let $\rho(x)$ denote the smallest eigenvalue for the Ricci tensor $\mathrm{Ric} : T_xM \to T_xM$, and $\mathrm{Ric}_-^H(x) = ((n-1)H - \rho(x))_+ = \max\{0, (n-1)H - \rho(x)\}$, amount of Ricci curvature below $(n-1)H$. Let

$$\|\mathrm{Ric}_-^H\|_p(R) = \sup_{x\in M}\left(\int_{B(x,R)} (\mathrm{Ric}_-^H)^p \, dvol\right)^{\frac{1}{p}}. \tag{1.1}$$

Then $\|\mathrm{Ric}_-^H\|_p$ measures the amount of Ricci curvature lying below $(n-1)H$ in the L^p sense. Clearly $\|\mathrm{Ric}_-^H\|_p(R) = 0$ iff $\mathrm{Ric}_M \geq (n-1)H$.

Similarly we can define Ricci curvature integral upper bound, or for sectional curvature integral lower bound $\|K_-^H\|_p$. When $H = 0$, we will omit the superscript, e.g. denote $\|\mathrm{Ric}_-^0\|_p$ by $\|\mathrm{Ric}_-\|_p$.

Why do we study integral curvature? Many geometric problems lead to integral curvatures, for example, the isospectral problems, geometric variational problems and extremal metrics, and Chern-Weil's formula for characteristic numbers. Since integral curvature bound is much weaker than pointwise curvature bound, one naturally asks what geometric and topological results can be extended to integral curvature.

In general one can not extend results from pointwise curvature bounds to integral curvature bounds. This can be illustrated by an example by D. Yang [44].

Recall a very important result in Riemannian geometry is Cheeger's finiteness theorem [**?**]. Namely the class of manifolds M^n with

$$|K_M| \leq H, \ \mathrm{Vol}_M \geq v, \ \mathrm{diam}_M \leq D$$

has only finite many diffeomorphism types. A key estimate is Cheeger's estimate on the length of the shortest closed geodesics. This is not true if $|K_M| \leq H$ is replaced by $\|Rm\|_p$ is bounded. In fact we have [44]

Example 2.1.1 (D. Yang 1992) *For all $p \geq \frac{n}{2}$ there are manifolds M_k^n such that*

$$\|Rm\|_p \leq H, \ \mathrm{Vol} \geq v, \ \mathrm{diam} \leq D$$

but $b_2(M_k) \to \infty$ as $k \to \infty$.

Hence some smallness is needed. For $p \leq \frac{n}{2}$, this still does not work as Gromov's Betti number estimate [16] does not extend [15].

Example 2.1.2 (Gallot 1988) *For any $\epsilon > 0, D > 0, n \geq 3$, there are M_k^n such that*

$$\mathrm{diam}(M_k) \leq D, \quad \|K_-\|_{\frac{n}{2}} \leq \epsilon, \quad \overline{\|K_-\|}_{\frac{n}{2}} \leq \epsilon,$$

but $b_2(M_k) \to \infty$ as $k \to \infty$.

This is not the end of story. Most results extend when $\|Rm\|_p$, $\|K_-^H\|_p$, or $\|\mathrm{Ric}_-^H\|_p$ is small for $p > \frac{n}{2}$. Namely we need the error from pointwise curvature bound to be small in L^p for $p > \frac{n}{2}$. There is a gap phenomenon. Some of the basic tools for these extensions are volume comparison for integral curvature, use Ricci flow to deform the manifolds with integral curvature bounds to pointwise curvature bound (so called smoothing), [15, 44, 34, **?**, 10].

2.2 Mean Curvature Comparison Estimate

Recall the mean curvature comparison theorem (Theorem 1.2.2)states that if $\mathrm{Ric} \geq (n-1)H$, then $m \leq m_H$. In general, without any curvature bound, we can estimate $m_+^H = (m - m_H)_+$ (set it to zero whenever it is not defined), amount of mean curvature comparison failed in L^{2p}, in terms of Ric_-^H, amount of Ricci curvature lying below $(n-1)H$ in L^p when $p > \frac{n}{2}$ [34].

Theorem 2.2.1 (Mean Curvature Estimate, Petersen-Wei 1997) *For any $p > \frac{n}{2}$, $H \in \mathbb{R}$, and when $H > 0$ assume $r \leq \frac{\pi}{2\sqrt{H}}$, we have*

$$\|m_+^H\|_{2p}(r) \leq \left(\frac{(n-1)(2p-1)}{2p-n} \|\mathrm{Ric}_-^H\|_p(r) \right)^{\frac{1}{2}}. \tag{2.2}$$

Clearly this generalizes the mean curvature comparison theorem (Theorem 1.2.2). In fact we prove the following which also gives a pointwise estimate.

Proposition 2.2.2 (Mean Curvature Estimate, Petersen-Wei 1997) *For any $p > \frac{n}{2}$, $H \in \mathbb{R}$, and when $H > 0$ assume $r \leq \frac{\pi}{2\sqrt{H}}$, we have*

$$\int_0^r (m_+^H)^{2p} \mathcal{A}(t,\theta)\, dt \leq \left(\frac{(n-1)(2p-1)}{2p-n} \right)^p \int_0^r (\mathrm{Ric}_-^H)^p \mathcal{A}(t,\theta)\, dt \tag{2.3}$$

$$(m_+^H)^{2p-1}(r,\theta)\mathcal{A}(r,\theta) \leq (2p-1)^p \left(\frac{n-1}{2p-n} \right)^{p-1} \int_0^r (\mathrm{Ric}_-^H)^p \mathcal{A}\, dt, \tag{2.4}$$

where $\mathcal{A}(t,\theta)$ is the volume element of the volume form $d\,vol = \mathcal{A}(t,\theta) dt \wedge d\theta_{n-1}$ in polar coordinate.

Proof: By (2.2) and (2.4) we have

$$(m - m_H)' \leq -\frac{(m-m_H)(m+m_H)}{n-1} + (n-1)H - \mathrm{Ric}(\nabla r, \nabla r). \tag{2.5}$$

On the interval $m \leq m_H$, we have $m_+^H = 0$, on the interval where $m > m_H$, $m - m_H = m_+^H$ and $(n-1)H - \mathrm{Ric}(\nabla r, \nabla r) \leq \mathrm{Ric}_-^H$. Therefore we have

$$(m_+^H)' + \frac{(m_+^H)^2}{n-1} + 2\frac{m_+^H \cdot m_H}{n-1} \leq \mathrm{Ric}_-^H. \tag{2.6}$$

Multiply (2.6) by $(m_+^H)^{2p-2}\mathcal{A}$ we get

$$(m_+^H)'(m_+^H)^{2p-2}\mathcal{A} + \frac{(m_+^H)^{2p}}{n-1}\mathcal{A} + \frac{2(m_+^H)^{2p-1}}{n-1} m_H \mathcal{A} \leq \mathrm{Ric}_-^H (m_+^H)^{2p-2}\mathcal{A}.$$

To complete the integral of the first term we compute, using (4.1) and $m - m_H \leq m_+^H$,

$$\begin{aligned}
(2p-1)(m_+^H)'(m_+^H)^{2p-2}\mathcal{A} &= \left((m_+^H)^{2p-1}\mathcal{A}\right)' - (m_+^H)^{2p-1}\mathcal{A}' \\
&= \left((m_+^H)^{2p-1}\mathcal{A}\right)' - (m_+^H)^{2p-1}(m - m_H)\mathcal{A} - (m_+^H)^{2p-1} m_H \mathcal{A} \\
&\geq \left((m_+^H)^{2p-1}\mathcal{A}\right)' - (m_+^H)^{2p}\mathcal{A} - (m_+^H)^{2p-1} m_H \mathcal{A}.
\end{aligned}$$

Therefore we have the ODE

$$\begin{aligned}\left((m_+^H)^{2p-1}\mathcal{A}\right)' + \left(\frac{2p-1}{n-1} - 1\right)(m_+^H)^{2p}\mathcal{A} + \left(\frac{4p-2}{n-1} - 1\right)(m_+^H)^{2p-1} m_H \mathcal{A} \\ \leq (2p-1)\mathrm{Ric}_-^H (m_+^H)^{2p-2}\mathcal{A}. \qquad (2.7)\end{aligned}$$

When $p > \frac{n}{2}$, $\frac{2p-1}{n-1} - 1 = \frac{2p-n}{n-1} > 0$. Hence if $m_H \geq 0$ (which is true under our assumption), we can throw away the third term in (2.7) and integrate from 0 to r to get

$$\left((m_+^H)^{2p-1}\mathcal{A}\right)(r) + \frac{2p-n}{n-1}\int_0^r (m_+^H)^{2p}\mathcal{A}\,dt \leq (2p-1)\int_0^r \mathrm{Ric}_-^H (m_+^H)^{2p-2}\mathcal{A}\,dt.$$

This gives

$$\left((m_+^H)^{2p-1}\mathcal{A}\right)(r) \leq (2p-1)\int_0^r \mathrm{Ric}_-^H (m_+^H)^{2p-2}\mathcal{A}\,dt, \qquad (2.8)$$

$$\frac{2p-n}{n-1}\int_0^r (m_+^H)^{2p}\mathcal{A}\,dt \leq (2p-1)\int_0^r \mathrm{Ric}_-^H (m_+^H)^{2p-2}\mathcal{A}\,dt. \qquad (2.9)$$

By Hölder's inequality

$$\int_0^r \mathrm{Ric}_-^H (m_+^H)^{2p-2}\mathcal{A}\,dt \leq \left(\int_0^r (m_+^H)^{2p}\mathcal{A}\,dt\right)^{1-\frac{1}{p}} \left(\int_0^r (\mathrm{Ric}_-^H)^p \mathcal{A}\,dt\right)^{\frac{1}{p}}.$$

Plug this into (2.9) we get (2.3). Plug this into (2.8) and combine (2.3) we get (2.4). ■

When $H > 0$ and $r > \frac{\pi}{2\sqrt{H}}$, m_H is negative so we can not throw away the third term in (2.7). Following the above estimate with an integrating factor Aubry gets [2]

Proposition 2.2.3 *For $p > \frac{n}{2}, H > 0, \frac{\pi}{2\sqrt{H}} < r < \frac{\pi}{\sqrt{H}}$, we have*

$$\sin^{4p-n-1}(\sqrt{H}r)(m_+^H)^{2p-1}(r,\theta)\mathcal{A}(r,\theta) \leq (2p-1)^p \left(\frac{n-1}{2p-n}\right)^{p-1}\int_0^r (\mathrm{Ric}_-^H)^p\mathcal{A}\,dt. \qquad (2.10)$$

Proof: Write (2.7) as

$$\begin{aligned}&\left((m_+^H)^{2p-1}\mathcal{A}\right)' + \frac{(4p-n-1)m_H}{n-1}(m_+^H)^{2p-1}\mathcal{A} \\ &\quad + \left(\frac{2p-n}{n-1}\right)(m_+^H)^{2p}\mathcal{A} \leq (2p-1)\mathrm{Ric}_-^H (m_+^H)^{2p-2}\mathcal{A}.\end{aligned}$$

The integrating factor of the first two terms is $e^{\int \frac{(4p-n-1)m_H}{n-1}} = \sin^{4p-n-1}(\sqrt{H}r)$. Multiply by the integrating factor and integrate from 0 to r we get

$$\begin{aligned}0 &\leq \sin^{4p-n-1}(\sqrt{H}r)(m_+^H)^{2p-1}(r,\theta)\mathcal{A}(r,\theta) + \frac{2p-n}{n-1}\int_0^r (m_+^H)^{2p}\sin^{4p-n-1}(\sqrt{H}r)\mathcal{A} \\ &\leq (2p-1)\int_0^r \mathrm{Ric}_-^H (m_+^H)^{2p-2}\sin^{4p-n-1}(\sqrt{H}r)\mathcal{A}\end{aligned}$$

Using Hólder's inequality as before we get

$$\begin{aligned}
&\int_0^r (m_+^H)^{2p} \sin^{4p-n-1}(\sqrt{H}r)\mathcal{A}(t,\theta)\,dt \\
&\quad \leq \left(\frac{(n-1)(2p-1)}{2p-n}\right)^p \int_0^r (\mathrm{Ric}_-^H)^p \sin^{4p-n-1}(\sqrt{H}r)\mathcal{A}(t,\theta)\,dt
\end{aligned}$$

and this gives (2.10) as before. ∎

All estimates hold for the mean curvature of hypersurfaces.

2.3 Volume Comparison Estimate

From (4.1) one naturally expects that the mean curvature comparison estimates in the last section would give volume comparison estimate for integral Ricci lower bound.

First we give a comparison estimate for the area of geodesics spheres using the pointwise mean curvature estimate (2.4). Recall $A(x,r) = \int_{S^{n-1}} \mathcal{A}(r,\theta)d\theta_{n-1}$, the volume of the geodesic sphere $S(x,r) = \{y \in M | d(x,y) = r\}$, and $A_H(r)$ the volume of the geodesic sphere in the model space.

Theorem 2.3.1 *Let $x \in M^n, H \in \mathbb{R}$ and $p > \frac{n}{2}$ be given, and when $H > 0$ assume that $R \leq \frac{\pi}{2\sqrt{H}}$. For $r \leq R$, we have*

$$\left(\frac{A(x,R)}{A_H(R)}\right)^{\frac{1}{2p-1}} - \left(\frac{A(x,r)}{A_H(r)}\right)^{\frac{1}{2p-1}} \leq C(n,p,H,R)\left(\|\mathrm{Ric}_-^H\|_p(R)\right)^{\frac{p}{2p-1}}, \quad (3.11)$$

where $C(n,p,H,R) = \left(\frac{n-1}{(2p-1)(2p-n)}\right)^{\frac{p-1}{2p-1}} \int_0^R (A_H)^{-\frac{1}{2p-1}} dt$. Furthermore when $r = 0$ we obtain

$$A(x,R) \leq \left(1 + C(n,p,H,R) \cdot \left(\|\mathrm{Ric}_-^H\|_p(R)\right)^{\frac{p}{2p-1}}\right)^{2p-1} A_H(R). \qquad (3.12)$$

The proof below much simplifies the proof in [11].
Proof: Recall

$$\frac{d}{dt}\left(\frac{\mathcal{A}(t,\theta)}{\mathcal{A}_H(t)}\right) = (m - m_H)\cdot\frac{\mathcal{A}(t,\theta)}{\mathcal{A}_H(t)} \leq m_+^H \cdot \frac{\mathcal{A}(t,\theta)}{\mathcal{A}_H(t)}.$$

Hence

$$\begin{aligned}
\frac{d}{dt}\left(\frac{A(x,t)}{A_H(t)}\right) &= \frac{1}{\mathrm{Vol}S^{n-1}} \int_{S^{n-1}} \frac{d}{dt}\left(\frac{\mathcal{A}(t,\theta)}{\mathcal{A}_H(t)}\right) d\theta_{n-1} \\
&\leq \frac{1}{A_H(t)} \int_{S^{n-1}} m_+^H \mathcal{A}(t,\theta) d\theta_{n-1}.
\end{aligned}$$

Using Hölder's inequality and (2.4) yields

$$\begin{aligned}
&\int_{S^{n-1}} m_+^H \mathcal{A}(t,\theta) d\theta_{n-1} \\
&\quad\leq \left(\int_{S^{n-1}} (m_+^H)^{2p-1} \mathcal{A} d\theta_{n-1} \right)^{\frac{1}{2p-1}} (A(x,t))^{1-\frac{1}{2p-1}} \\
&\quad\leq C(n,p) \left(\|\mathrm{Ric}_-^H\|_p(t) \right)^{\frac{p}{2p-1}} (A(x,t))^{1-\frac{1}{2p-1}},
\end{aligned}$$

where $C(n,p) = \left((2p-1)^p \left(\frac{n-1}{2p-n}\right)^{p-1} \right)^{\frac{1}{2p-1}}$. Hence we have

$$\frac{d}{dt}\left(\frac{A(x,t)}{A_H(t)}\right) \leq C(n,p) \left(\|\mathrm{Ric}_-^H\|_p(t) \right)^{\frac{p}{2p-1}} (A_H)^{-\frac{1}{2p-1}} \left(\frac{A(x,t)}{A_H(t)}\right)^{1-\frac{1}{2p-1}}. \tag{3.13}$$

Separation of variables and integrate from r to R we get

$$\begin{aligned}
&\left(\frac{A(x,R)}{A_H(R)}\right)^{\frac{1}{2p-1}} - \left(\frac{A(x,r)}{A_H(r)}\right)^{\frac{1}{2p-1}} \\
&\quad\leq \left(\frac{n-1}{(2p-1)(2p-n)}\right)^{\frac{p-1}{2p-1}} \left(\|\mathrm{Ric}_-^H\|_p(R) \right)^{\frac{p}{2p-1}} \int_r^R (A_H)^{-\frac{1}{2p-1}} dt.
\end{aligned}$$

The integral $\int_r^R (A_H)^{-\frac{1}{2p-1}} dt \leq \int_0^R (A_H)^{-\frac{1}{2p-1}} dt$ converges when $p > \frac{n}{2}$. This gives (3.11). ∎

Using (3.13) we have

Theorem 2.3.2 (Volume Comparison Estimate, Petersen-Wei 1997) *Let $x \in M^n, H \in \mathbb{R}$ and $p > \frac{n}{2}$ be given, when $H > 0$ assume that $R \leq \frac{\pi}{2\sqrt{H}}$. For $r \leq R$ we have*

$$\left(\frac{\mathrm{Vol}\, B\,(x,R)}{\mathrm{Vol}_H\,(B(R))}\right)^{\frac{1}{2p-1}} - \left(\frac{\mathrm{Vol}\, B\,(x,r)}{\mathrm{Vol}_H\,(B(r))}\right)^{\frac{1}{2p-1}} \leq C\,(n,p,H,R) \cdot \left(\|\mathrm{Ric}_-^H\|_p(R) \right)^{\frac{p}{2p-1}}, \tag{3.14}$$

where $C(n,p,H,R) = \left(\frac{n-1}{(2p-1)(2p-n)}\right)^{\frac{p-1}{2p-1}} \int_0^R A_H(t) \left(\frac{t}{\mathrm{Vol}_H B(t)}\right)^{\frac{2p}{2p-1}} dt.$

Note that when $\|\mathrm{Ric}_-^H\|_p(R) = 0$, this gives the Bishop-Gromov relative volume comparison.

Proof of Theorem 2.3.2: Since $\frac{\mathrm{Vol}\, B(x,r)}{\mathrm{Vol}_H (B(r))} = \frac{\int_0^r A(x,t)dt}{\int_0^r A_H(t)dt}$, we have

$$\frac{d}{dr}\left(\frac{\mathrm{Vol}\, B\,(x,r)}{\mathrm{Vol}_H B(r)}\right) = \frac{A(x,r)\int_0^r A_H(t)dt - A_H(r)\int_0^r A(x,t)dt}{(\mathrm{Vol}_H B(r))^2}. \tag{3.15}$$

Integrate (3.13) from t to r gives

$$\frac{A(x,r)}{A_H(r)} - \frac{A(x,t)}{A_H(t)} \leq C(n,p) \int_t^r \left(\|\mathrm{Ric}_-^H\|_p(s) \right)^{\frac{p}{2p-1}} \frac{A(x,s)^{1-\frac{1}{2p-1}}}{A_H(s)} ds$$

$$
\begin{aligned}
&\leq C(n,p)\frac{\left(\|\mathrm{Ric}_-^H\|_p(r)\right)^{\frac{p}{2p-1}}}{A_H(t)}\int_t^r A(x,s)^{1-\frac{1}{2p-1}}ds \\
&\leq C(n,p)\frac{\left(\|\mathrm{Ric}_-^H\|_p(r)\right)^{\frac{p}{2p-1}}}{A_H(t)} r^{\frac{1}{2p-1}}\left(\mathrm{Vol}\,B\left(x,r\right)\right)^{1-\frac{1}{2p-1}}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&A(x,r)A_H(t)-A_H(r)A(x,t) \\
&\quad \leq C(n,p)\left(\|\mathrm{Ric}_-^H\|_p(r)\right)^{\frac{p}{2p-1}} A_H(r) r^{\frac{1}{2p-1}}\left(\mathrm{Vol}\,B\left(x,r\right)\right)^{1-\frac{1}{2p-1}}.
\end{aligned}
$$

Plug this into (3.15) gives

$$
\frac{d}{dr}\left(\frac{\mathrm{Vol}\,B\left(x,r\right)}{\mathrm{Vol}_H B(r)}\right)\leq C(n,p)\left(\|\mathrm{Ric}_-^H\|_p(r)\right)^{\frac{p}{2p-1}} A_H(r)\left(\frac{r}{\mathrm{Vol}_H B(r)}\right)^{\frac{2p}{2p-1}}\left(\frac{\mathrm{Vol}\,B\left(x,r\right)}{\mathrm{Vol}_H B(r)}\right)^{1-\frac{1}{2p-1}}.
$$

Separation of variables and integrate from r to R we get

$$
\begin{aligned}
&\left(\frac{\mathrm{Vol}B(x,R)}{\mathrm{Vol}_H(R)}\right)^{\frac{1}{2p-1}}-\left(\frac{\mathrm{Vol}B(x,r)}{\mathrm{Vol}_H(r)}\right)^{\frac{1}{2p-1}} \\
&\quad \leq\left(\frac{n-1}{(2p-1)(2p-n)}\right)^{\frac{p-1}{2p-1}}\left(\|\mathrm{Ric}_-^H\|_p(R)\right)^{\frac{p}{2p-1}}\int_r^R A_H(t)\left(\frac{t}{\mathrm{Vol}_H B(t)}\right)^{\frac{2p}{2p-1}}dt.
\end{aligned}
$$

The integral $\int_r^R A_H(t)\left(\frac{t}{\mathrm{Vol}_H B(t)}\right)^{\frac{2p}{2p-1}}dt \leq \int_0^R A_H(t)\left(\frac{t}{\mathrm{Vol}_H B(t)}\right)^{\frac{2p}{2p-1}}dt$ converges when $p>\frac{n}{2}$.

Corollary 2.3.3 (Volume Doubling Estimate)

2.4 Applications

Volume comparison is a powerful tool for studying manifolds with lower Ricci curvature bound and has many applications. As a result of (3.14), many results with pointwise Ricci lower bound (i.e. $\|\mathrm{Ric}_-^H\|_p(r)=0$) can be extended to the case when $\|\mathrm{Ric}_-^H\|_p(r)$ is very small [15, 34, 33, 10, 37, 35, 11, **?**].

Chapter 3

Comparison for Bakry-Emery Ricci Tensor

3.1 N-Bakry-Emery Ricci Tensor

The Bakry-Emery Ricci tensor is a Ricci tensor for smooth metric measure spaces, which are Riemannian manifold with a measure conformal to the Riemannian measure. Formally a smooth metric measure space is a triple $(M^n, g, e^{-f} dvol_g)$, where M is a complete n-dimensional Riemannian manifold with metric g, f is a smooth real valued function on M, and $dvol_g$ is the Riemannian volume density on M. This is also sometimes called a manifold with density. physics dilaton, analytical reasons. These spaces occur naturally as smooth collapsed limits of manifolds with lower Ricci curvature bound under the measured Gromov-Hausdorff convergence [13].

Definition 3.1.1 *We say (X_i, μ_i) converges in the measured Gromov-Hausdorff sense to (X_∞, μ_∞) if for all sequences of continuous functions $f_i : X_i \to \mathbb{R}$ converging to $f_\infty : X_\infty \to \mathbb{R}$, we have*

$$\int_{X_i} f_i d\mu_i \to \int_{X_\infty} f_\infty d\mu_\infty. \tag{1.1}$$

Example 3.1.2 *Let $(M^n \times F^N, g_\epsilon)$ be a product manifold with warped product metric $g_\epsilon = g_M + (\epsilon e^{-f})^2 g_F$, where f is a function on M. Then $(M \times F, g_\epsilon)$ converges to $(M^n, g_M, e^{-Nf} dvol_{g_M})$ under the measured Gromov-Hausdorff convergence.*

The N-Bakry-Emery Ricci tensor is

$$\mathrm{Ric}_f^N = \mathrm{Ric} + \mathrm{Hess} f - \frac{1}{N} df \otimes df \quad \text{for } N > 0. \tag{1.2}$$

As we will discuss below, N is related to the dimension of the model space. We allow N to be infinite, in this case we denote $\mathrm{Ric}_f = \mathrm{Ric}_f^\infty = \mathrm{Ric} + \mathrm{Hess} f$. Note that when f is a constant function $\mathrm{Ric}_f^N = \mathrm{Ric}$ for all N and we can take $N = 0$ in this case. Moreover, if $N_1 \geq N_2$ then $\mathrm{Ric}_f^{N_1} \geq \mathrm{Ric}_f^{N_2}$ so that $\mathrm{Ric}_f^N \geq \lambda g$ implies $\mathrm{Ric}_f \geq \lambda g$.

The Bakry Emery Ricci tensor (for N finite and and infinite) has a natural extension to non-smooth metric measure spaces [29, 38, 39] and diffusion operators [6]. Moreover, the equation $\mathrm{Ric}_f = \lambda g$ for some constant λ is exactly the gradient Ricci soliton equation, which plays an important role in the theory of Ricci flow; the equation $\mathrm{Ric}_f^N = \lambda g$, for N positive integer, corresponds to warped product Einstein metric on $M \times_{e^{-\frac{f}{N}}} F^N$. See [4] for a modification of the Ricci tensor which is conformally invariant.

We are interested in investigating what geometric and topological results for the Ricci tensor extend to the Bakry-Emery Ricci tensor. This was studied by Lichnerowicz [24, 25] almost forty years ago, though this work does not seem to be widely known. Recently this has been actively investigated and there are many interesting results in this direction which we will discuss below, see for example [22, 36, 28, 7, 31, 8, 21, **?**, **?**, **?**, 43, 42]. In this note we first recall the Bochner formulas for Bakry-Emery Ricci tensors (stated a little differently from how they have appeared in the literature). The derivation of these from the classical Bochner formula is elementary, so we present the proof. Then we quickly derive the first eigenvalue comparison from the Bochner formulas as in the classical case. In the rest of the paper we focus on mean curvature and volume comparison theorems and their applications. When N is finite, this work is mainly from [36, 8], and when N is infinite, it's mainly from our recent work [42].

3.2 Bochner formulas for the N-Bakry-Emery Ricci tensor

With respect to the measure $e^{-f}dvol$ the natural self-adjoint f-Laplacian is $\Delta_f = \Delta - \nabla f \cdot \nabla$. In this case we have

$$\begin{aligned} \Delta_f |\nabla u|^2 &= \Delta |\nabla u|^2 - 2\mathrm{Hess}\, u(\nabla u, \nabla f), \\ \langle \nabla u, \nabla(\Delta_f u)\rangle &= \langle \nabla u, \nabla(\Delta u)\rangle - \mathrm{Hess}\, u(\nabla u, \nabla f) - \mathrm{Hess} f(\nabla u, \nabla u). \end{aligned}$$

Plugging these into (1.1) we immediately get the following Bochner formula for the N-Bakry-Emery Ricci tensor.

$$\frac{1}{2}\Delta_f |\nabla u|^2 = |\mathrm{Hess}\, u|^2 + \langle \nabla u, \nabla(\Delta_f u)\rangle + \mathrm{Ric}_f^N(\nabla u, \nabla u) + \frac{1}{N}|\langle \nabla f, \nabla u\rangle|^2. \quad (2.3)$$

When $N = \infty$, we have

$$\frac{1}{2}\Delta_f |\nabla u|^2 = |\mathrm{Hess}\, u|^2 + \langle \nabla u, \nabla(\Delta_f u)\rangle + \mathrm{Ric}_f(\nabla u, \nabla u). \quad (2.4)$$

This formula is virtually the same as (1.1) except for the important fact that $tr(\mathrm{Hess}\, u) = \Delta u$ not $\Delta_f(u)$. In the case where N is finite, however, we can get around this difficulty by using the inequality

$$\frac{(\Delta u)^2}{n} + \frac{1}{N}|\langle \nabla f, \nabla u\rangle|^2 \geq \frac{(\Delta_f(u))^2}{N+n} \tag{2.5}$$

which implies

$$\frac{1}{2}\Delta_f|\nabla u|^2 \geq \frac{(\Delta_f(u))^2}{N+n} + \langle \nabla u, \nabla(\Delta_f u)\rangle + \mathrm{Ric}_f^N(\nabla u, \nabla u). \tag{2.6}$$

In other words, a Bochner formula holds for Ric_f^N that looks like the Bochner formula for the Ricci tensor of an $n+N$ dimensional manifold. Note that (2.5) is an equality if and only if $\Delta u = \frac{n}{N}\langle \nabla f, \nabla u\rangle$, so equality in (2.6) is seldom achieved when f is nontrivial. When f is constant, we can take $N = 0$ so (2.6) recovers (1.5).

3.3 Eigenvalue and Mean Curvature Comparison

From the Bochner formulas we can now prove eigenvalue and mean curvature comparisons which generalize the classical ones. First we consider the eigenvalue comparison.

Let M^n be a complete Riemannian manifold with $\mathrm{Ric}_f^N \geq (n-1)H > 0$. Applying (2.6) to the first eigenfunction u of Δ_f, $\Delta_f u = -\lambda_1 u$, and integrating with respect to the measure $e^{-f}dvol$, we have

$$0 \geq \int_M \left(\frac{(\lambda_1 u)^2}{N+n} - \lambda_1|\nabla u|^2 + (n-1)H|\nabla u|^2\right) e^{-f}dvol.$$

Since $\int_M |\nabla u|^2 e^{-f}dvol = \lambda_1 \int_M u^2 e^{-f}dvol$, we deduce the eigenvalue estimate [**?**]

$$\lambda_1 \geq (n-1)H\left(1 + \frac{1}{N+n-1}\right). \tag{3.1}$$

When f is constant, taking $N = 0$ gives the classical Lichnerowicz's first eigenvalue estimate $\lambda_1 \geq nH$ [23]. When $N = \infty$, we have [6]

$$\lambda_1 \geq (n-1)H. \tag{3.2}$$

This also can be derived from (2.4) directly. One may expect that the estimate (3.2) is weaker than the classical one. In fact (3.2) is optimal as the following example shows.

Example 3.3.1 *Let $M = \mathbb{R}^1 \times S^2$ with standard product metric g_0, $f(x,y) = \frac{1}{2}x^2$. Then $\mathrm{Hess} f(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = 1$ and zero on all other directions. We have $\mathrm{Ric}_f = 1g_0$. Now for the linear function $u(x,y) = x$, $\Delta_f u = -x$. So $\lambda_1 = 1$.*

On the other hand (3.2) is never optimal for compact manifolds since equality in (3.2) implies $\operatorname{Hess} u = 0$. Note that $\mathrm{Ric}_f \geq (n-1)H > 0$ on a compact manifold implies $\mathrm{Ric}_f^N \geq (n-1)H' > 0$ for some N big, hence one can use estimate (3.1).

Now we turn to the mean curvature (or Laplacian) comparison. Recall that the mean curvature measures the relative rate of change of the volume element. Therefore, for the measure $e^{-f}dvol$, the associated mean curvature is $m_f = m - \partial_r f$, where m is the mean curvature of the geodesic sphere with inward pointing normal vector. Also $m_f = \Delta_f(r)$, where r is the distance function.

Let m_H^k be the mean curvature of the geodesic sphere in the model space M_H^k, the complete simply connected k-manifold of constant curvature H. When we drop the superscript k and write m_H we mean the mean curvature from the model space whose dimension matches the dimension of the manifold. Since $\operatorname{Hess} r$ is zero along the radial direction, applying the Bochner formula (2.3) to the distance function r, the Schwarz inequality $|\mathrm{Hess} r|^2 \geq \frac{(\Delta r)^2}{n-1}$ and (2.5) gives

$$m_f' \leq -\frac{(m_f)^2}{n+N-1} - \mathrm{Ric}_f^N(\partial_r, \partial_r). \tag{3.3}$$

Thus, using the standard Sturm-Liouville comparison argument, one has the mean curvature comparison [8].

Theorem 3.3.2 (Mean curvature comparison for N-Bakry-Emery) *If* $\mathrm{Ric}_f^N \geq (n+N-1)H$, *then*

$$m_f(r) \leq m_H^{n+N}(r). \tag{3.4}$$

Namely the mean curvature is less or equal to the one of the model with dimension $n+N$. This does not give any information when N is infinite.

In fact, such a strong, uniform estimate is not possible when N is infinite. To see this note that, when $H > 0$, the model space M_H^{n+N} is a round sphere so that $m_H^{n+N}(r)$ goes to $-\infty$ as r goes to $\frac{\pi}{\sqrt{H}}$. Thus (3.4) implies that if N is finite and $\mathrm{Ric}_f^N \geq \lambda > 0$ then M is compact (See Theorem 3.4.5 in the next section for the diameter bound). On the other hand, this is not true when $N = \infty$ as the following example shows.

Example 3.3.3 *Let* $M = \mathbb{R}^n$ *with Euclidean metric* g_0, $f(x) = \frac{\lambda}{2}|x|^2$. *Then* $\mathrm{Hess} f = \lambda g_0$ *and* $\mathrm{Ric}_f = \lambda g_0$.

Thus, when N is infinite, one can not expect such a strong mean curvature comparison to be true. However, we can show a weaker, nonuniform estimate and also give some uniform estimates if we make additional assumptions on f such as f being bounded or $\partial_r f$ bounded from below. In these cases we have the following mean curvature comparisons [42] which generalizes the classical one.

Theorem 3.3.4 (Mean Curvature Comparison for ∞-Bakry-Emery) *Let $p \in M^n$. Assume $\mathrm{Ric}_f(\partial_r, \partial_r) \geq (n-1)H$,*
a) given any minimal geodesic segment and $r_0 > 0$,

$$m_f(r) \leq m_f(r_0) - (n-1)H(r - r_0) \quad \textit{for } r \geq r_0. \tag{3.5}$$

b) if $\partial_r f \geq -a$ along a minimal geodesic segment from p (when $H > 0$ assume $r \leq \pi/2\sqrt{H}$) then

$$m_f(r) - m_H(r) \leq a \tag{3.6}$$

along that minimal geodesic segment from p. Equality holds if and only if the radial sectional curvatures are equal to H and $f(t) = f(p) - at$ for all $t < r$.
c) if $|f| \leq k$ along a minimal geodesic segment from p (when $H > 0$ assume $r \leq \pi/4\sqrt{H}$) then

$$m_f(r) \leq m_H^{n+4k}(r) \tag{3.7}$$

along that minimal geodesic segment from p. In particular when $H = 0$ we have

$$m_f(r) \leq \frac{n + 4k - 1}{r} \tag{3.8}$$

Proof: From the Riccati inequality (2.2), equality (2.4), and assumption on Ric_f, we have

$$(m - m_H)' \leq -\frac{m^2 - m_H^2}{n-1} + \mathrm{Hess}\, f(\partial_r, \partial_r). \tag{3.9}$$

As in the third proof of the mean curvature comparison theorem (Theorem **??**), we compute

$$\begin{aligned}
\left(\mathrm{sn}_H^2(m - m_H)\right)' &= 2\mathrm{sn}_H'\mathrm{sn}_H(m - m_H) + sn_H^2(m - m_H)' \\
&\leq \frac{2}{n-1}\mathrm{sn}_H^2 m_H(m - m_H) - \frac{1}{n-1}\mathrm{sn}_H^2(m^2 - m_H^2) + sn_H^2 \mathrm{Hess}\, f(\partial_r, \partial_r) \\
&= -\frac{\mathrm{sn}_H^2}{n-1}(m - m_H)^2 + sn_H^2 \mathrm{Hess}\, f(\partial_r, \partial_r) \\
&\leq sn_H^2 \mathrm{Hess}\, f(\partial_r, \partial_r).
\end{aligned}$$

Here in the 2nd line we have used (3.9) and (2.5).

Therefore we have

$$\left(\mathrm{sn}_H^2 m\right)' \leq \left(\mathrm{sn}_H^2 m_H\right)' + \mathrm{sn}_H^2 \partial_t \partial_t f. \tag{3.10}$$

Integrating from 0 to r yields

$$\mathrm{sn}_H^2(r)m(r) \leq \mathrm{sn}_H^2(r)m_H(r) + \int_0^r \mathrm{sn}_H^2(t)\partial_t\partial_t f(t)dt. \tag{3.11}$$

When f is constant (the classical case) this gives the usual mean curvature comparison.

Proof or Part a. Using integration by parts on the last term we have

$$\mathrm{sn}_H^2(r)m_f(r) \le \mathrm{sn}_H^2(r)m_H(r) - \int_0^r (\mathrm{sn}_H^2(t))'\partial_t f(t)dt. \tag{3.12}$$

Under our assumptions $(\mathrm{sn}_H^2(t))' = 2\mathrm{sn}_H'(t)\mathrm{sn}_H(t) \ge 0$ so if $\partial_t f(t) \ge -a$ we have

$$\mathrm{sn}_H^2(r)m_f(r) \le \mathrm{sn}_H^2(r)m_H(r)+a\int_0^r (\mathrm{sn}_H^2(t))'dt = \mathrm{sn}_H^2(r)m_H(r)+\mathrm{sn}_H^2(r)a \tag{3.13}$$

This proves the inequality.

To see the rigidity statement suppose that $\partial_t f \ge -a$ and $m_f(r) = m_H(r)+a$ for some r. Then from (3.12) we see

$$a\mathrm{sn}_H^2 \le \int_0^r (\mathrm{sn}_H^2(t))'\partial_t f(t)dt \le a\mathrm{sn}_H^2. \tag{3.14}$$

So that $\partial_t f \equiv -a$. But then $m(r) = m_f - a = m_H(r)$ so that the rigidity follows from the rigidity for the usual mean curvature comparison.

Proof of Part b. Integrate (3.12) by parts again

$$\mathrm{sn}_H^2(r)m_f(r) \le \mathrm{sn}_H^2(r)m_H(r) - f(r)(\mathrm{sn}_H^2(r))' + \int_0^r f(t)(sn_H^2)''(t)dt. \tag{3.15}$$

Now if $|f| \le k$ and $r \in (0, \frac{\pi}{4\sqrt{H}}]$ when $H > 0$, then $(sn_H^2)''(t) \ge 0$ and we have

$$\mathrm{sn}_H^2(r)m_f(r) \le \mathrm{sn}_H^2(r)m_H(r) + 2k(\mathrm{sn}_H^2(r))'. \tag{3.16}$$

From (2.5) we can see that

$$(\mathrm{sn}_H^2(r))' = 2\mathrm{sn}_H'\mathrm{sn}_H = \frac{2}{n-1}m_H\mathrm{sn}_H^2.$$

so we have

$$m_f(r) \le \left(1 + \frac{4k}{n-1}\right)m_H(r) = m_H^{n+4k}(r). \tag{3.17}$$

■

Now when $H > 0$ and $r \in [\frac{\pi}{4\sqrt{H}}, \frac{\pi}{2\sqrt{H}}]$,

$$\begin{aligned}\int_0^r f(t)(sn_H^2)''(t)dt &\le k\left(\int_0^{\frac{\pi}{4\sqrt{H}}} (sn_H^2)''(t)dt - \int_{\frac{\pi}{4\sqrt{H}}}^r (sn_H^2)''(t)dt\right) \\ &= k\left(\frac{2}{\sqrt{H}} - sn_H(2r)\right).\end{aligned}$$

Hence

$$m_f(r) \le \left(1 + \frac{4k}{n-1}\cdot\frac{1}{\sin(2\sqrt{H}r)}\right)m_H(r). \tag{3.18}$$

This estimate will be used later to prove the Myers' theorem in Section 5. ■

In the case $H = 0$, we have $sn_H(r) = r$ so (3.15) gives the estimate in [12] that

$$m_f(r) \leq \frac{n-1}{r} - \frac{2}{r}f(r) + \frac{2}{r^2}\int_0^r f(t)dt. \tag{3.19}$$

These mean curvature comparisons can be used to prove some Myers' type theorems for Ric_f, and is related to volume comparison theorems, both of which we discuss in the next section.

3.4 Volume Comparison and Myers' Theorems

For $p \in M^n$, we use exponential polar coordinates around p and write the volume element $d\,vol = \mathcal{A}(r,\theta)dr \wedge d\theta_{n-1}$, where $d\theta_{n-1}$ is the standard volume element on the unit sphere $S^{n-1}(1)$. Let $\mathcal{A}_f(r,\theta) = e^{-f}\mathcal{A}(r,\theta)$. By the first variation of the area

$$\frac{\mathcal{A}'}{\mathcal{A}}(r,\theta) = (\ln(\mathcal{A}(r,\theta)))' = m(r,\theta). \tag{4.1}$$

Therefore

$$\frac{\mathcal{A}'_f}{\mathcal{A}_f}(r,\theta) = (\ln(\mathcal{A}_f(r,\theta)))' = m_f(r,\theta). \tag{4.2}$$

And for $r \geq r_0 > 0$

$$\frac{\mathcal{A}_f(r,\theta)}{\mathcal{A}_f(r_0,\theta)} = e^{\int_{r_0}^r m_f(r,\theta)}. \tag{4.3}$$

Combining this equation with the mean curvature comparisons we obtain volume comparisons. Let $\mathrm{Vol}_f(B(p,r)) = \int_{B(p,r)} e^{-f}dvol_g$, the weighted (or f-)volume, $\mathrm{Vol}_H^k(r)$ be the volume of the radius r-ball in the model space M_H^k.

Theorem 3.4.1 (Volume comparison for N-Bakry-Emery) *[36] If* $\mathrm{Ric}_f^N \geq (n+N-1)H$, *then* $\frac{\mathrm{Vol}_f(B(p,R))}{\mathrm{Vol}_H^{n+N}(R)}$ *is nonincreasing in* R.

In [28] Lott shows that if M is compact (or just $|\nabla f|$ is bounded) with $\mathrm{Ric}_f^N \geq \lambda$ for some positive integer $2 \leq N < \infty$, then, in fact, there is a family of warped product metrics on $M \times S^N$ with Ricci curvature bounded below by λ, recovering the comparison theorems for Ric_f^N.

When $N = \infty$ we have the following volume comparison results which generalize the classical one. Part a) is originally due to Morgan [**?**] where it follows from a hypersurface volume estimate(also see [**?**]). For the proofs of parts b) and c) see [42].

Theorem 3.4.2 (Volume Comparison for ∞-Bakry-Emery) *Let $(M^n, g, e^{-f}dvol_g)$ be complete smooth metric measure space with $\mathrm{Ric}_f \geq (n-1)H$. Fix $p \in M^n$.*
a) If $H > 0$, then $\mathrm{Vol}_f(M)$ is finite.
b) If $\partial_r f \geq -a$ along all minimal geodesic segments from p then for $R \geq r > 0$ (assume $R \leq \pi/2\sqrt{H}$ if $H > 0$) ,

$$\frac{\mathrm{Vol}_f(B(p,R))}{\mathrm{Vol}_f(B(p,r))} \leq e^{aR}\frac{\mathrm{Vol}_H^n(R)}{\mathrm{Vol}_H^n(r)}. \tag{4.4}$$

Moreover, equality holds if and only if the radial sectional curvatures are equal to H and $\partial_r f \equiv -a$. In particular if $\partial_r f \geq 0$ and $\mathrm{Ric}_f \geq 0$ then M has f-volume growth of degree at most n.
c) If $|f(x)| \leq k$ then for $R \geq r > 0$ (assume $R \leq \pi/4\sqrt{H}$ if $H > 0$),

$$\frac{\mathrm{Vol}_f(B(p,R))}{\mathrm{Vol}_f(B(p,r))} \leq \frac{\mathrm{Vol}_H^{n+4k}(R)}{\mathrm{Vol}_H^{n+4k}(r)}. \tag{4.5}$$

In particular, if f is bounded and $\mathrm{Ric}_f \geq 0$ then M has polynomial f-volume growth.

For Part a) we compare with a model space, however, we modify the measure according to a. Namely, the model space will be the pointed metric measure space $M_{H,a}^n = (M_H^n, g_H, e^{-h}dvol, O)$ where (M_H^n, g_H) is the n-dimensional simply connected space with constant sectional curvature H, $O \in M_H^n$, and $h(x) = -a \cdot d(x, O)$. We make the model a pointed space because the space only has $\mathrm{Ric}_f(\partial_r, \partial_r) \geq (n-1)H$ in the radial directions from O and we only compare volumes of balls centered at O.

Let $\mathcal{A}_H^a$ be the h-volume element in $M_{H,a}^n$. Then $\mathcal{A}_H^a(r) = e^{ar}\mathcal{A}_H(r)$ where $\mathcal{A}_H$ is the Riemannian volume element in M_H^n. By the mean curvature comparison we have $(\ln(\mathcal{A}_f(r,\theta))' \leq a + m_H = (\ln(\mathcal{A}_H^a))'$ so for $r < R$,

$$\frac{\mathcal{A}_f(R,\theta)}{\mathcal{A}_f(r,\theta)} \leq \frac{\mathcal{A}_H^a(R,\theta)}{\mathcal{A}_H^a(r,\theta)}. \tag{4.6}$$

Namely $\frac{\mathcal{A}_f(r,\theta)}{\mathcal{A}_H^a(r,\theta)}$ is nonincreasing in r. Using Lemma 3.2 in [45], we get for $0 < r_1 < r$, $0 < R_1 < R$, $r_1 \leq R_1$, $r \leq R$,

$$\frac{\int_{R_1}^R \mathcal{A}_f(t,\theta)dt}{\int_{r_1}^r \mathcal{A}_f(t,\theta)dt} \leq \frac{\int_{R_1}^R \mathcal{A}_H^a(t,\theta)dt}{\int_{r_1}^r \mathcal{A}_H^a(t,\theta)dt}. \tag{4.7}$$

Integrating along the sphere direction gives

$$\frac{\mathrm{Vol}_f(A(p,R_1,R))}{\mathrm{Vol}_f(A(p,r_1,r))} \leq \frac{\mathrm{Vol}_H^a(R_1,R)}{\mathrm{Vol}_H^a(r_1,r)}. \tag{4.8}$$

Where $\mathrm{Vol}_H^a(r_1,r)$ is the h-volume of the annulus $B(O,r) \setminus B(O,r_1) \subset M_H^n$. Since $\mathrm{Vol}_H(r_1,r) \leq \mathrm{Vol}_H^a(r_1,r) \leq e^{ar}\mathrm{Vol}_H(r_1,r)$ this gives (4.4) when $r_1 = R_1 = 0$ and proves Part b).

In the model space the radial function h is not smooth at the origin. However, clearly one can smooth the function to a function with $\partial_r h \geq -a$ and $\partial_r^2 h \geq 0$ such that the h-volume taken with the smoothed h is arbitrary close to that of the model. Therefore, the inequality (4.8) is optimal. Moreover, one can see from the equality case of the mean curvature comparison that if the annular volume is equal to the volume in the model then all the radial sectional curvatures are H and f is exactly a linear function.

Proof of Part b): In this case let $\mathcal{A}_H^{n+4k}$ be the volume element in the simply connected model space with constant curvature H and dimension $n+4k$.

Then from the mean curvature comparison we have $\ln(\mathcal{A}_f(r,\theta))' \leq \ln(\mathcal{A}_H^{n+4k}(r))'$. So again applying Lemma 3.2 in [45] we obtain

$$\frac{\mathrm{Vol}_f(A(p,R_1,R))}{\mathrm{Vol}_f(A(p,r_1,r))} \leq \frac{Vol_H^{n+4k}(R_1,R)}{Vol_H^{n+4k}(r_1,r)}. \tag{4.9}$$

With $r_1 = R_1 = 0$ this implies the relative volume comparison for balls

$$\frac{\mathrm{Vol}_f(B(p,R))}{\mathrm{Vol}_f(B(p,r))} \leq \frac{Vol_H^{n+4k}(R)}{Vol_H^{n+4k}(r)}. \tag{4.10}$$

Equivalently

$$\frac{\mathrm{Vol}_f(B(p,R))}{V_H^{n+4k}(R)} \leq \frac{\mathrm{Vol}_f(B(p,r))}{V_H^{n+4k}(r)}. \tag{4.11}$$

Since $n+4k > n$ we note that the right hand side blows up as $r \to 0$ so one does not obtain a uniform upper bound on $\mathrm{Vol}_f(B(p,R))$. Indeed, it is not possible to do so since one can always add a constant to f and not effect the Bakry-Emery tensor.

By taking $r=1$ we do obtain a volume growth estimate for $R > 1$

$$\mathrm{Vol}_f(B(p,R)) \leq \mathrm{Vol}_f(B(p,1)) Vol_H^{n+4k}(R). \tag{4.12}$$

Note that, from Part a) $\mathrm{Vol}_f(B(p,1)) \leq e^{-f(p)} e^a \omega_n$ if $\partial_r f \geq -a$ on $B(p,1)$. ■

Part a) should be viewed as a weak Myers' theorem for Ric_f. Namely if $\mathrm{Ric}_f > \lambda > 0$ then the manifold may not be compact but the measure must be finite. In particular the lifted measure on the universal cover is finite. Since this measure is invariant under the deck transformations, this weaker Myers' theorem is enough to recover the main topological corollary of the classical Myers' theorem.

Corollary 3.4.3 *If M is complete and $Ric_f \geq \lambda > 0$ then M has finite fundamental group.*

Using a different approach the second author has proven that the fundamental group is, in fact, finite for spaces satisfying $\mathrm{Ric} + \mathcal{L}_X g \geq \lambda > 0$ for some vector field X [43]. This had earlier been shown under the additional assumption that the Ricci curvature is bounded by Zhang [**?**]. See also [**?**]. When M is compact the finiteness of fundamental group was first shown by X. Li [22, Corollary 3] using a probabilistic method.

On the other hand, the volume comparison Theorem 3.4.1 and Theorem 3.4.2 Part c) also give the following generalization of Calabi-Yau's theorem [**?**].

Theorem 3.4.4 *If M is a noncompact, complete manifold with $\mathrm{Ric}_f^N \geq 0$, assume f is bounded when N is infinite, then M has at least linear f-volume growth.*

Theorem 3.4.2 Part a) and Theorem 3.4.4 then together show that any manifold with $\mathrm{Ric}_f^N \geq \lambda > 0$ and f bounded if N is infinite must be compact. In fact, from the mean curvature estimates one can prove this directly and obtain an upper bound on the diameter. For finite N this is due to Qian [36], for Part b) see [42].

Theorem 3.4.5 (Myers' Theorem) *Let M be a complete Riemannian manifold with $\mathrm{Ric}_f^N \geq (n-1)H > 0$,*
a) when N is finite, then M is compact and $\mathrm{diam}_M \leq \sqrt{\frac{n+N-1}{n-1}}\frac{\pi}{\sqrt{H}}$.
b) when N is infinite and $|f| \leq k$ then M is compact and $\mathrm{diam}_M \leq \frac{\pi}{\sqrt{H}} + \frac{4k}{(n-1)\sqrt{H}}$.

For some other Myers' Theorems for manifolds with measure see [**?**] and [**?**]. The relative volume comparison Theorem 3.4.2 also implies the following extensions of theorems of Gromov [17] and Anderson [1].

Theorem 3.4.6 *For the class of manifolds M^n with $\mathrm{Ric}_f \geq (n-1)H$, $\mathrm{diam}_M \leq D$ and $|f| \leq k$ ($|\nabla f| \leq a$), the first Betti number $b_1 \leq C(n+4k, HD^2)$ ($C(n, HD^2, aD)$).*

Theorem 3.4.7 *For the class of manifolds M^n with $\mathrm{Ric}_f \geq (n-1)H$, $\mathrm{Vol}_f \geq V$, $\mathrm{diam}_M \leq D$ and $|f| \leq k$ ($|\nabla f| \leq a$) there are only finitely many isomorphism types of $\pi_1(M)$.*

Chapter 4

Comparison in Ricci Flow

Perelman's reduced volume monotonicity [31], a basic and powerful tool in his work on Thurston's geometrization conjecture, is a generalization of Bishop-Gromov's volume comparison to Ricci flow.

4.1 Heuristic argument

4.2 Laplacian Comparison for Ricci Flow

4.3 Reduced Volume Monotonicity

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