Abstract of the Dissertation

Aspects of Positively Ricci Curved Spaces:

New Examples and the Fundamental Group

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For a simply connected nilpotent Lie group L, we construct a complete metric with positive Ricci curvature on the product manifold $L \times \mathbb{R}^p$, where p is taken sufficiently large. The construction uses a warped product method and involves subtle choices of functions. We endow L with a family of almost flat metrics, and the little "negativeness" of L can be compensated by warping the euclidean $\mathbb{R}^p$ factor. From the construction one also sees that the isometry group of the resulting manifold contains the original group L.

A basic consequence of this construction is that every finitely generated torsion-free discrete nilpotent group can be realized as the fundamental group of a complete manifold with positive Ricci curvature.

We also establish an a priori bound on the growth of the fundamental group for a class of compact near elliptic manifolds (in the sense of Gromov) whose volume is uniformly bounded from below.

To my parents.

Contents

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Introduction

One of the main themes in the development of global riemannian geometry is to understand the interplay between geometric quantities such as curvature and the topology of a riemannian manifold. We now have a rather satisfactory theory of the structure of riemannian manifolds with nonnegative sectional curvature. Among the most basic results are the Soul Theorem, the Splitting Theorem, and the uniform bounds on the Betti numbers (see next chapter for details). There have been major developments recently concerning the question whether or not various results for sectional curvature have analogues for Ricci curvature. Although it has been known for some time that the Splitting Theorem remains true, it turned out that without additional hypotheses, the other results above do not carry over to the case of nonnegative Ricci curvature [8, 20, 21] (compare also [1, 22]).

In this thesis we construct complete riemannian manifolds with $Ric > 0$ such that the isometry groups contain nilpotent Lie groups. A consequence of this construction is that every finitely generated, torsion-free, discrete nilpotent group can be realized as the fundamental group of a complete riemannian manifold with $Ric > 0$. Together with work of J. Milnor and M. Gromov, this leads to a fairly good understanding of the structure of the fundamental group of a nonnegativly Ricci curved manifold. Note that in the nonnegative sectional curvature case, every subgroup must be abelian up to finite index, as follows from the Soul Theorem and the Splitting Theorem.

There are several ways to construct manifolds with $Ric > 0$. Our construction is inspired by examples given by P. Nabonnand [16] and L. Bérard-Bergery [4]. The basic problem here is to put a positive Ricci curvature metric on $L \times \mathbb{R}^p$, where L is a simply connected nilpotent Lie group. L can be endowed with a family of almost flat metrics. To achieve positivity, we use a warped product metric on $L \times \mathbb{R}^p$ and compensate for the slight "negativeness" of L by the $\mathbb{R}^p$ factor. Variants of this construction will also be discussed, yielding some examples of near elliptic manifolds.

Motivated by Gromov's conjecture on the fundamental group of a near elliptic manifold, we give an upper bound on the growth of the fundamental group for a class of such manifolds with a uniform lower volume bound. The restriction on the volume may be a strong condition. So far, we do not see how to get around it.

Background

We refer to [5] for the basic facts in riemannian geometry that will be used here. We will now briefly discuss the three fundamental results concerning the topology of manifolds with nonnegative sectional curvature, which were mentioned in the Introduction.

1) **The Soul Theorem** (J. Cheeger & D. Gromoll [1972], [6]) *Let M be a complete riemannian manifold with sectional curvature $K \geq 0$. Then M contains a compact totally geodesic submanifold S without boundary which is also convex, $0 \leq \dim S < \dim M$, and M is diffeomorphic to the normal bundle $\nu(S)$ of S in M.*

This puts severe restrictions on the topology of a manifold if it is to admit a complete metric with nonnegative sectional curvature. For example, such a space must be of finite topological type and be actually homotopic to a closed manifold. Therefore, its cohomology must satisfy Poincaré-duality. From this fact, D. Gromoll and W. Meyer [8] first constructed complete open manifolds which admit metrics with nonnegative Ricci curvature but do not carry any metric with nonnegative sectional curvature.

2) **The Splitting Theorem** (J. Cheeger & D. Gromoll [1972], [7] Toponogov [1964]) *If $K_M \geq 0$ then M splits as an isometric product $\bar{M} \times \mathbb{R}^k$, where $\bar{M}$ contains no lines and $\mathbb{R}^k$ has its standard flat metric.*

A particular consequence of this theorem combined with the Soul Theorem is that the fundamental group of a manifold with $K \geq 0$ must be abelian up to finite index (having an abelian subgroup with finite index). However, as we will see, the situation is quite different in the Ricci curvature case, thus producing many other interesting examples of complete manifolds which admit metrics with nonnegative Ricci curvature but do not carry any metric with nonnegative sectional curvature.

3) **Theorem** (M. Gromov [1981], [11]) *There exists a constant $\mathcal{C} = \mathcal{C}(n)$ such that every complete n-dimensional riemannian manifold M of nonnegative sectional curvature satisfies*

$$\sum_{i=0}^{n} b_i \leq \mathcal{C},$$

where b_i is the i-th Betti number.

The question of whether such a uniform bound on the Betti numbers could also exist in the nonnegative Ricci curvature case has been studied by J. P. Sha and D. G. Yang recently. In fact, they constructed positive Ricci curvature metrics on the connected sums of arbitrarily many copies of $S^n \times S^m$ $(n, m \geq 2)$, showing that there is no uniform bound on the Betti numbers. They also showed that complete manifolds with positive Ricci curvature could be of infinite topological type.

Chapter 1

Examples of manifolds with positive Ricci curvature

Suppose M is a compact connected smooth manifold. For a positive number ϵ, a riemannian metric on M is called ϵ-flat, as introduced by M. Gromov [10], if its sectional curvature K and its diameter d obey the relation

$$|K|d^2 \leq \epsilon.$$

M is called almost flat if it admits such metrics for all ϵ.

The celebrated "almost flat manifold theorem" states:

Theorem 1 (M. Gromov [10]) *There exists an $\epsilon(n) \geq 0$ so that an $\epsilon(n)$-flat n-dimensional manifold is covered by a nilmanifold. More precisely,*
1) The fundamental group $\pi_1(M)$ contains a torsion-free nilpotent normal subgroup Γ of rank n.

2) The quotient $G = \pi_1(M)/\Gamma$ has order $\leq C(n)$ and is isomorphic to a subgroup of $O(n)$.

3) The finite covering of M with the fundamental group Γ and deckgroup G is diffeomorphic to a nilmanifold L/Γ.

4) The simply connected nilpotent Lie group L is uniquely determined by $\pi_1(M)$.

Later E. Ruh [19] proved a refinement of the almost flat manifold theorem. He showed that M itself, not just a finite covering, is diffeomorphic to L/Λ, where Λ is a uniform discrete group in the isometry group of L with respect to some left invariant metric. Such a manifold is called infra-nil.

In general nilmanifolds L/Γ can not be endowed with any left invariant metric with nonnegative Ricci curvature. However, we will show that one can put such a metric on $L/\Gamma \times \mathbb{R}^p$, or more generally on $M \times \mathbb{R}^p$ for M almost flat, provided p is sufficiently large. Thus almost flat manifolds are stabily diffeomorphic to complete manifolds with positive Ricci curvature.

In this chapter we shall prove the following main result:

Theorem 2 *Let L be an n-dimensional simply connected nilpotent Lie group. Then for all sufficiently large p, the product manifold $M^{p+n} = L \times \mathbb{R}^p$ admits complete riemannian metrics with strictly positive Ricci curvature such that the isometry group of M contains L.*

We first construct a family of almost flat left invariant metrics on L, then we show that some warped product metrics on $L \times \mathbb{R}^p$ have strictly positive Ricci curvature.

We will also give some examples of near elliptic manifolds.

1.1 Almost flat left invariant metrics on L

Let L be an n-dimensional simply connected nilpotent Lie group, and l its Lie algebra. We construct a family of almost flat left invariant metrics g_r on L, $0 \le r < \infty$.

It is well-known [23, 3.6.6] that any simply connected nilpotent analytic group G is isomorphic to a closed unipotent subgroup of $GL(V)$ for some finite-dimensional vector space V. Without loss of generality, we can assume $L = U(m)$, the closed unipotent subgroup of upper triangular matrices in $GL(m)$. Consider $X_i = X_{sl},\ s < l$, where X_{sl} is the $m \times m$-matrix such that every entry is 0, except the s-th row and l-th column spot which is 1. Then $\{X_1,\ \ldots,\ X_n\}$ forms a triangular basis for the Lie algebra l, i.e. $[X, X_i] \in l_{i-1}$, whenever $X \in l$, and l_{i-1} is spanned by $X_1,\ \ldots,\ X_{i-1}$. For $X = \sum_{i=1}^{n} a_i X_i$, set

$$\|X\|^2 = \sum_{i=1}^{n} h_i^2(r) a_i^2, \tag{1.1}$$

where

$$h_i(r) = (1 + r^2)^{-\alpha_i}, \tag{1.2}$$

$2\alpha_i - 4\alpha_{i+1} = \beta,\ \alpha_n = \alpha,\ \alpha_i = 2^{n-i}(\alpha + \frac{\beta}{2}) - \frac{\beta}{2}$ for $1 \le i \le n-1$, and

α, β are positive constants. β will be specified later (see below for the way of choosing the $h_i(r)$). The above norm gives rise to a corresponding almost flat left invariant metric g_r on L.

Proposition 1 *For the metric g_r, the curvature satisfies the following relations*

$$|K_L(Y_i, Y_j)| \leq c(1 + r^2)^{-\beta}, \tag{1.3}$$

$$< R(Y_i, Y_j)Y_j, Y_k >= 0, \qquad i \neq k, \tag{1.4}$$

where $Y_i = h_i^{-1}(r)X_i$, c is a constant depending on n and the structure constants.

To proof (1.3), we need the following lemma

Lemma 1 *If $\|[X, Y]\| \leq c\|X\|\|Y\|$, for any $X,\ Y \in l,\ c \geq 0$, then the sectional curvature satisfies $|K_L| \leq 6c^2$.*

This is elementry and is a consequence of the following basic curvature formulas for the left invariant metric of a Lie group [5].

$$< R(X, Y)Z, W >=< \bigtriangledown_X Z, \bigtriangledown_Y W > - < \bigtriangledown_Y Z, \bigtriangledown_X W > - < \bigtriangledown_{[X,Y]} Z, W >, \tag{1.5}$$

$$\bigtriangledown_X Y = \frac{1}{2}\{[X, Y] - (ad_X)^*(Y) - (ad_Y)^*(X)\}. \tag{1.6}$$

Q.E.D.

Now the commutator of a nilpotent group satisfies

$$[X_i, X_j] = \sum_{k<\min(i,j)} r_{ijk} X_k. \tag{1.7}$$

¿From Lemma 1 and (1.7) it is clear that if we scale the norm of X_i much faster than the norm of X_j when $i < j$, then the curvature K_L will be very small. This is exactly the way we choose the scale functions $h_i(r)$.

Let $c_1 = \max |r_{ijk}|$, then

$$||[X_i, X_j]||^2 \leq c_1^2 \sum_{k<\min(i,j)} h_k^2 \leq c_1^2 \frac{n-1}{(1+r^2)^\beta} h_i^2 h_j^2,$$

$$\begin{aligned} ||[\sum a_i X_i, \sum b_j X_j]|| &\leq ||\sum_{i,j} a_i h_i b_j h_j \frac{[X_i, X_j]}{h_i h_j}|| \\ &\leq c_1(n-1)^{\frac{1}{2}} \sum_{i,j} |a_i h_i||b_j h_j|/(1+r^2)^{\frac{\beta}{2}} \\ &\leq c_1 n(n-1)^{\frac{1}{2}} ||a_i h_i|| ||b_j h_j||/(1+r^2)^{\frac{\beta}{2}}. \end{aligned}$$

The norm of the Lie algebra is $\leq c_1 n(n-1)^{\frac{1}{2}}/(1+r^2)^{\frac{\beta}{2}}$, therefore we have proved (1.3).

To verify (1.4), first we calculate the Levi-Civita connection of g_r. By (1.6) and (1.7), we find

$$\bigtriangledown_{Y_j} Y_j = 0, \tag{1.8}$$

$$\bigtriangledown_{Y_i} Y_j = \frac{1}{2} h_i^{-1}(r) h_j^{-1}(r)\{[X_i, X_j] - (ad_{X_j})^*(X_i)\}, \qquad i < j, \tag{1.9}$$

$$\bigtriangledown_{Y_i} Y_j = \frac{1}{2} h_i^{-1}(r) h_j^{-1}(r)\{[X_i, X_j] - (ad_{X_i})^*(X_j)\}, \qquad i > j. \tag{1.10}$$

¿From (1.5) and (1.8)

$$< R(Y_i, Y_j)Y_j, Y_k > = < \bigtriangledown_{Y_i} Y_j, \bigtriangledown_{Y_j} Y_k > - < \bigtriangledown_{[Y_i, Y_j]} Y_j, Y_k > .$$

Without loss of generality, we assume $i < j < k$. Using (1.9) and (1.7), we have

$$
\begin{aligned}
&< \nabla_{Y_i} Y_j, \nabla_{Y_j} Y_k > \\
&\quad = \frac{1}{4} h_i^{-1}(r) h_j^{-2}(r) h_k^{-1}(r) \{< [X_i, X_j], [X_j, X_k] > - < [X_k, [X_i, X_j]], X_j > \\
&\quad - < [X_j, [X_j, X_k]], X_i > \} \\
&\quad = \frac{1}{4} h_i^{-1}(r) h_j^{-2}(r) h_k^{-1}(r) < [X_i, X_j], [X_j, X_k] >,
\end{aligned}
$$

$$
\begin{aligned}
< \nabla_{[Y_i, Y_j]} Y_j, Y_k > &= \frac{1}{2} h_i^{-1}(r) h_j^{-2}(r) h_k^{-1}(r) \{< [[X_i, X_j], X_j], X_k > \\
&\quad - < [X_i, X_j], [X_j, X_k] > \} \\
&= -\frac{1}{2} h_i^{-1}(r) h_j^{-2}(r) h_k^{-1}(r) < [X_i, X_j], [X_j, X_k] > .
\end{aligned}
$$

Therefore,

$$
< R(Y_i, Y_j) Y_j, Y_k >= \frac{3}{4} h_i^{-1}(r) h_j^{-2}(r) h_k^{-1}(r) < [X_i, X_j], [X_j, X_k] >, \quad i < j < k. \tag{1.11}
$$

Similarly we can find

$$
< R(Y_i, Y_j) Y_j, Y_k >= \frac{1}{2} h_i^{-1}(r) h_j^{-2}(r) h_k^{-1}(r) < [X_i, X_j], [X_j, X_k] >, \tag{1.12}
$$

for $j < i < k$, or $i < k < j$.

Hence it suffices to show

$$
< [X_i, X_j], [X_j, X_k] >= 0, \qquad i \neq k. \tag{1.13}
$$

Now

$$
\begin{aligned}
< [X_{ij}, X_{kl}], [X_{kl}, X_{pq}] > &= < \delta_{jk} X_{il} - \delta_{li} X_{kj}, \delta_{lp} X_{kq} - \delta_{qk} X_{pl} > \\
&= 0,
\end{aligned}
$$

unless $i = j$, or $i = p$ and $j = q$.

This yields (1.13) and proves (1.4).

Q.E.D.

Remark 1 For a given uniform discrete subgroup $\Gamma \subset L$, the diameter $d(L/\Gamma) \longrightarrow 0$ when $r \longrightarrow \infty$.

Remark 2 In fact we will only use the partial result that

$$Ric_L(Y_i) \geq -\frac{c}{1+r^2} \tag{1.14}$$

for the construction of metrics with $Ric > 0$ on $L \times \mathbb{R}^p$ in the next section.

Remark 3 To simplify expressions, we had scaled each X_i differently. Actually one can just scale each level (diagonal) differently. The metric g_r constructed in this way will be invariant under Λ, an extension of a lattice $\Gamma \subset L$ by a finite group H, if $\Lambda \subset Iso(L, g)$ for some left invariant metric g. This is because H preserves the levels. Note that g_r is not invariant under the whole isometry group of L with some left invariant metric.

1.2 Construction of metrics with $Ric > 0$

We define a warped product metric g on $M = L \times \mathbb{R}^p$ by

$$g = g_r + dr^2 + f^2(r)ds^2, \tag{1.15}$$

where g_r is the metric defined as before on L with $\beta = 1$, ds^2 is the canonical euclidean metric on the sphere $S^{p-1} \subset \mathbb{R}^p$, and

$$f(r) = r(1+r^2)^{-1/4}. \tag{1.16}$$

g is a complete metric on M, since $f(0) = 0,\ f'(0) = 1,\ f''(0) = 0,\ f(r) > 0$ for $r > 0,\ h_i(r) > 0$ for $r \geq 0,\ h_i'(0) = 0$ for $1 \leq i \leq n$.

It is clear that the isometry group of g contains L.

Now we will calculate the Ricci curvature of this metric and show $Ric_M > 0$.

Let $H = \partial/\partial r$, $U_j = f(r)^{-1}V_j$, where $1 \leq j \leq p-1$ and $V_1, \ \ldots, \ V_{p-1}$ is an orthonormal basis of S^{p-1} with cononical metric. Let

$$\omega_0, \ \omega_1', \ \ldots, \ \omega_n', \ \omega_{n+1}', \ \ldots, \ \omega_{n+p-1}'$$

be dual to the basis $H, \ X_1, \ \ldots, \ X_n, \ V_1, \ \ldots, \ V_{p-1}$ of M. Denote by

$$\omega_i = h_i(r)\omega_i' \qquad (1 \leq i \leq n),$$

$$\omega_{n+j} = f_i(r)\omega_{n+j}' \qquad (1 \leq j \leq p-1).$$

Then, (1.15) becomes the following,

$$g = \sum_{i=0}^{n+p-1} \omega_i^2.$$

By the Cartan structure equations, we have

$$\begin{aligned} d\omega_i &= h_i'(r)\omega_0 \wedge \omega_i' + h_i(r)d\omega_i' \\ &= \sum_{k=1}^{n} \omega_{ik} \wedge \omega_k + \omega_{i0} \wedge \omega_0 \qquad (1 \le i \le n), \qquad (1.17) \\ d\omega_{n+j} &= f'(r)\omega_0 \wedge \omega_{n+j}' + f(r)d\omega_{n+j}' \\ &= \sum_{k=1}^{p-1} \omega_{n+jk+n} \wedge \omega_{k+n} + \omega_{n+j0} \wedge \omega_0 \quad (1 \le j \le p-1), \quad (1.18) \\ d\omega_0 &= 0 = \sum_{k=1}^{n+p-1} \omega_{0k} \wedge \omega_k. \qquad (1.19) \end{aligned}$$

From (1.17), (1.18) and (1.19) we find

$$\begin{aligned} \omega_{i0} &= -\frac{h_i'(r)}{h_i(r)}\omega_i \qquad (1 \le i \le n), \qquad (1.20) \\ \omega_{n+j0} &= -\frac{f'(r)}{f(r)}\omega_{n+j} \qquad (1 \le j \le p-1), \qquad (1.21) \\ \omega_{n+jn+k} &= \omega_{n+jn+k}' \qquad (1 \le j, k \le p-1). \qquad (1.22) \end{aligned}$$

Therefore,

$$\begin{aligned} < R(Y_i, Y_j)Y_j, Y_k > &= 0 \qquad (i \ne k), \qquad (1.23) \\ < R(Y_i, H)H, Y_j > &= 0 \qquad (i \ne j), \qquad (1.24) \\ < R(Y_i, U_j)U_j, Y_k > &= 0 \qquad (i \ne k), \qquad (1.25) \end{aligned}$$

$$\begin{aligned} K(Y_i, Y_j) &= K_L(Y_i, Y_j) - \frac{h_i'(r)h_j'(r)}{h_i(r)h_j(r)} \qquad (i \ne j, 1 \le i, j \le n), \quad (1.26) \\ K(Y_i, H) &= -\frac{h_i''(r)}{h_i(r)} \qquad (1 \le i \le n), \qquad (1.27) \\ K(U_j, H) &= -\frac{f''(r)}{f(r)} \qquad (1 \le j \le p-1), \qquad (1.28) \end{aligned}$$

$$K(U_j, U_j) = \frac{1}{f(r)^2} - (\frac{f'(r)}{f(r)})^2 \qquad (1 \leq j \leq p-1), \tag{1.29}$$

$$K(Y_i, U_j) = -\frac{h_i'(r)f'(r)}{h_i(r)f(r)} \qquad (1 \leq i \leq n,\ 1 \leq j \leq p-1). \tag{1.30}$$

The Ricci curvature is the following:

$$Ric(H, U_j) = 0 \qquad (1 \leq j \leq p-1),$$

$$Ric(Y_i, H) = Ric(Y_i, U_j) = 0 \quad (1 \leq i \leq n,\ 1 \leq j \leq p-1),$$

$$Ric(Y_i, Y_j) = 0 \qquad (i \neq j, 1 \leq i, j \leq n),$$

$$Ric(Y_i, Y_i) = -\frac{h_i''(r)}{h_i(r)} - (p-1)\frac{h_i'(r)f'(r)}{h_i(r)f(r)} + Ric_L(Y_i) - \sum_{i \neq j} \frac{h_i'(r)h_j'(r)}{h_i(r)h_j(r)} \qquad (1 \leq i \leq n), \tag{1.31}$$

$$Ric(H, H) = -\sum_{i=1}^{n} \frac{h_i''(r)}{h_i(r)} - (p-1)\frac{f''(r)}{f(r)}, \tag{1.32}$$

$$Ric(U_j, U_j) = -\frac{f''(r)}{f(r)} + \frac{p-2}{f(r)^2} - (p-2)(\frac{f'(r)}{f(r)})^2 - \sum_{i=1}^{n} \frac{h_i'(r)f'(r)}{h_i(r)f(r)} \qquad (1 \leq j \leq p-1). \tag{1.33}$$

Since $1-(f'(r))^2 \geq 0$, $f''(r) \leq 0$, $f'(r) > 0$, $h'(r) \leq 0$, we have $Ric(U_j, U_j) > 0$ in (1.34). For the positivity of the Ricci curvature in the equations (1.32) and (1.33), we insert the functions $f(r)$ of (1.16), h_i of (1.2), and use the estimate in (1.14). We obtain

$$\begin{aligned} Ric(Y_i, Y_i) \geq\ & \{-2\alpha_i[(2\alpha_i+1)r^2 - 1] + (p-1)\alpha_i(2+r^2) \\ & -c(1+r^2) - \sum_{i \neq j} 4\alpha_i\alpha_j r^2\}/(1+r^2)^2, \end{aligned} \tag{1.34}$$

$$\begin{aligned} Ric(H, H) =\ & \{-\sum_{i=1}^{n} 2\alpha_i[(2\alpha_i+1)r^2 - 1] \\ & +(p-1)\frac{r^2+6}{4}\}/(1+r^2)^2. \end{aligned} \tag{1.35}$$

Positivity of the Ricci curvature in the equations (1.35) and (1.36) is equivalent to the following two inequalities,

$$(p-1)\alpha_i > 4\sum_{i\neq j}\alpha_i\alpha_j + 2\alpha_i + c, \tag{1.36}$$

$$p-1 > 4\sum_{i=1}^{n}(4\alpha_i^2 + 2\alpha_i). \tag{1.37}$$

Recall that $\alpha_i = (2\alpha+1)2^{n-i-1} - \frac{1}{2}$, for $\alpha > 0$. Clearly (1.37) and (1.38) hold for p sufficiently large. This completes the proof of Theorem 2.

Q.E.D.

Note there is no metric on $L \times \mathbb{R}^p$ invariant under L with $K \geq 0$ for any p, simply because the fundamental group of a complete manifold with $K \geq 0$ is abelian up to finite index (see the previous chapter).

Remark 4 The smallest p that yields positive Ricci curvature on $M^{p+n} = L \times \mathbb{R}^p$ by means of our construction is quite large in general. For example, in the case of the three-dimensional Heisenberg group $L = H^3$, we have to choose $p > 673$. (With a slightly refined choice of functions, $p > 26$ will already work, see the example below.) We do not known whether or not p can be chosen much smaller. However, by [2], no finitely generated subgroup of$\pi_1(M)$ is of polynomial growth of order $\geq n-2$ if M^n is a complete riemannian manifold of bounded geometry with $Ric_M > 0$. Therefore necessarily $p \geq 4$ when $L = H^3$.

Example. If

$$L = H^3 = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

is the three-dimensional Heisenberg group. Define a warped product metric g on $H^3 \times \mathbb{R}^p$ by

$$g = h^2(r)(dx^2 + dy^2) + g^2(r)(dz - xdy)^2 + dr^2 + f^2(r)ds^2,$$

where ds^2 is the cononical metric on $S^{p-1} \subset \mathbb{R}^p$, and $g,\ h > 0,\ g'(0) = h'(0) = 0,\ f(0) = 0,\ f'(0) = 1,\ f''(0) = 0$ and $f(r) > 0$ for $r > 0$. g is a complete metric on $H^3 \times \mathbb{R}^p$.

Denote by

$$\begin{aligned} X_1 &= h^{-1}(r)\partial/\partial x, \\ X_2 &= h^{-1}(r)(\partial/\partial y + x\partial/\partial z), \\ X_3 &= g^{-1}(r)\partial/\partial z, \\ X_4 &= \partial/\partial r, \\ X_5 &= f^{-1}(r)V, \end{aligned}$$

where V is an orthonormal basis of S^{p-1} with cononical metric. Now $X_1, \ldots, X_5$ is an orthonormal basis of $H^3 \times \mathbb{R}^p$ with respect to g.

Calculations as before yield

$$\begin{aligned} Ric(X_i, X_j) &= 0 \qquad (i \neq j), \\ Ric(X_i, X_i) &= -\frac{h''}{h} - (p-1)\frac{f'h'}{fh} - \frac{g^2}{2h^4} - (\frac{h'}{h})^2 - \frac{h'g'}{hg} \quad (i = 1, 2), \end{aligned}$$

$$
\begin{aligned}
Ric(X_3,X_3) &= -\frac{g''}{g} - (p-1)\frac{f'g'}{fg} + \frac{g^2}{2h^4} - 2\frac{h'g'}{hg}, \\
Ric(X_4,X_4) &= -2\frac{h''}{h} - \frac{g''}{g} - (p-1)\frac{f''}{f}, \\
Ric(X_5,X_5) &= -\frac{f''}{f} + \frac{p-2}{f^2} - (p-2)(\frac{f'}{f})^2 - 2\frac{f'h'}{fh} - \frac{f'g'}{fg}.
\end{aligned}
$$

Now let

$$
\begin{aligned}
f(r) &= \frac{r}{(1+r^2)^{1/4}}, \\
g(r) &= \frac{1}{(1+r^2)^{\alpha}}, \\
h(r) &= (f'(r))^{\epsilon},
\end{aligned}
$$

where $\alpha,\ \epsilon$ are positive constants satisfying $2\alpha-\epsilon=1$. And functions $f,\ g,\ h$ satisfy the initial conditions.

When choosing $\epsilon=1/2,\ \alpha=3/4$, it can be easily checked that $Ric>0$ if $p>26$.

Let M denote a complete n-diemensional riemannian manifold. By virtue of Ruh's refinement of Gromov's almost flat manifold theorem (see the beginning of this chapter), the following result is therefore a corollary of Theorem 2 and Remark 3.

Theorem 3 *If M is $\hat{\epsilon}(n)$-flat, then $M\times\mathbb{R}^p$ carries a complete metric with $Ric>0$ for p sufficiently large.*

Theorem 2 also shows that the double of X, where X is a compact manifold with boundary which carries a metric such that both the Ricci curvature of

X and the mean curvature of its boundary are positive, need not carry a metric with nonnegative Ricci curvature, as it would have in case of positive scalar curvature [13]. We just take $X = L/\Gamma \times D^n$, where Γ, L as defined before, and D^n is the unit disk. It has positive Ricci curvature by Theorem 2, and it can also be easily checked that the mean curvature of its boundary is positive. But the double of X is $L/\Gamma \times S^n$ which does not carry any metric with $Ric \geq 0$, since it is compact and the fundamental group of a compact manifold with $Ric \geq 0$ is abelian up to finite index by the Splitting Theorem of J. Cheeger and D. Gromoll [7].

1.3 Examples of near elliptic manifolds

Using a construction of J. Nash [17] concerning the existence of metric with $Ric > 0$ on principal bundles and results of Section 1.1, we can construct a large class of near elliptic manifods. We call a compact manifold *near elliptic* (*weak near elliptic*) if it admits a metric with $K\ d^2 > -\epsilon$ ($Ric\ d^2 > -\epsilon$) for any $\epsilon > 0$.

Proposition 2 *Let $\pi : P \to M^n$ be a principal L^m-bundle over a compact manifold M. If M admits a metric $<,>_M$ with $K_M \geq 0$ ($Ric_M \geq 0$), then P/Γ is a near elliptic (weak near elliptic) manifold for any uniform discrete subgroup $\Gamma \subset L$.*

Proof. For a fixed connection ω on P, define a family of metrics $<,>_r$ on P, $r > 0$, by

$$< X, Y >_r = < \pi(X), \pi(Y) >_M + (1 + r^2)^{-2} g_r, \tag{1.38}$$

for $X,\ Y \in T_pP$. Here g_r is the metric defined as before on L with $\beta = 3$. The map $\pi : P \to M^n$ becomes a riemannian submersion for $<,>_r$. These metrics are invariant under L, and the fibers in P are totally geodesic with respect to $<,>_r$.

We will show that K_r (Ric_r) is almost nonnegative when $r \to +\infty$, and the diameter of P/Γ is bounded independent of r. Actually P/Γ collapses to the base manifold M (in the sense of Cheeger-Gromov).Let $H_1,\ \ldots,\ H_n$ be an orthonormal basis of the horizontal subspace. We will denote various quantities associated to $<,>_r$ with a subscript or superscript. For $r=1$, the r will usually be deleted. Recall that $Y_1, \ldots, Y_m$ is an orthonormal basis of l with respect to g_r.

For $X \in T_pP$, $\|X\|_r = 1$, without loss of generality, we can assume $X = aY_1(1+r^2) + bH_1$ for some $a,\ b$ satisfying $a^2+b^2=1$. Then

$$X,\ bY_1(1+r^2) - aH_1,\ Y_2(1+r^2),\ \ldots,\ Y_m(1+r^2),\ H_2,\ \ldots,\ H_n$$

is an orthonormal basis of T_pP. Hence

$$Ric_r(X) = K_r(X, bY_1(1+r^2) - aH_1) + \sum_{i=2}^{m} K_r(X, Y_i(1+r^2)) + \sum_{j=2}^{n} K_r(X, H_j). \tag{1.39}$$

We find (see [17] for detail)

$$K_r(X, bY_1(1+r^2) - aH_1) = (1+r^2)^{-2}\|A_{H_1}Y_1\|^2 \geq 0, \tag{1.40}$$

$$K_r(X, Y_i(1+r^2)) \geq a^2(1+r^2)^2 K_L(Y_1, Y_i), \tag{1.41}$$

$$\begin{aligned} K_r(X, H_j) \geq\ & b^2[K_M(\pi(H_1), \pi(H_j))] - 3b^2(1+r^2)^{-2}\|A_{H_1}H_j\|^2_{g_r} \\ & + 2ab(1+r^2)^{-1} < R(H_1, H_j)H_j, Y_1 >_{g_r}. \end{aligned} \tag{1.42}$$

Here the (1,2)-tensor A is defined by

$$A_X Y = (\bigtriangledown_{X_h} Y_v)_h + (\bigtriangledown_{X_h} Y_h)_v$$

for any C^∞ vector field X, Y on P, where $X = X_h + X_v$ is the docomposition into horizontal and vertical components. The terms $< R(H_1, H_j)H_j, Y_1 >_{g_r}$ and $\|A_{H_1}H_j\|^2_{g_r}$ have bounds independent of Y_i and H_j. Thus

$$Ric_r(X) = b^2 Ric_M(\pi(H_1)) + a^2(1+r^2)^2 Ric_L(Y_1) + O(\frac{1}{1+r^2}).$$

With Proposition 1, we have

$$K_r \geq -\frac{c'}{1+r^2}$$

if $K_M \geq 0$, or

$$\begin{aligned} Ric_r(X) &\geq a^2(1+r^2)^2 \cdot (-\frac{c}{(1+r^2)^3}) + O(\frac{1}{1+r^2}) \\ &\geq -\frac{c''}{1+r^2} \end{aligned}$$

if $Ric_M \geq 0$, for some constants c', c''. The diameter of P/Γ is clearly bounded independently of r. Therefore P/Γ is a near elliptic (weak near elliptic) manifold.

Q.E.D.

A corollary of Theorem 2 and Proposition 2 is

Theorem 4 *Let $P \to M$ be a principal L-bundle over a compact manifold M. If M admits a metric with $Ric_M \geq 0$, then $P \times \mathbb{R}^p$ admits a complete metric which is invariant under L and with $Ric > 0$ for p sufficiently large.*

On the basis of these results, we believe that $M \times \mathbb{R}^p$ would still adimts a complete metric with $Ric > 0$ for p sufficiently large, if M is a weak near elliptic manifold. This would give an affirmative answer to a generalized conjecture of M. Gromov [10] (see the end of Section 2.2). But it looks difficult to construct the metric without knowing more about the structure of such manifolds. It seems possible that all weak near elliptic manifolds are somewhat like principal L-bundles over base manifolds admitting metrics with $Ric \geq 0$. Recently T. Yamaguchi developed some structure theory about weak near elliptic manifolds with an additonal condition that the sectional curvature is bounded from below [24].

Chapter 2

Ricci curvature and the fundamental group

2.1 The case $Ric > 0$

The first result about the fundamental group of a complete manifold M with $Ric > 0$ is due to S.B. Myers [15]. It says that if M is compact and $Ric_M > 0$ then $\pi_1(M)$ is finite. This can not be extended to complete non-compact manifolds as in the case of positive sectional curvature. P. Nabonnand [16] constructed a metric on $S^1 \times \mathbb{R}^3$ with positive Ricci curvature, showing that the fundamental group could be infinite. In fact we have

Corollary 1 *Every finitely generated torsion-free nilpotent group can be realized as the fundamental group of a complete riemannian manifold with strictly positive Ricci curvature.*

This follows immediately from Theorem 2 and the following classical result [18, p 40].

Theorem 5 (A. I. Malcev) *A group Γ is isomorphic to a lattice in a simply connected nilpotent Lie group if and only if*

1) Γ is finitely generated,

2) Γ is nilpotent, and

3) Γ has no torsion.

On the other hand, J. Milnor [14] studied the growth of finitely generated subgroups of the fundamental group of a manifold admitting a complete metric of nonnegative Ricci curvature. Let us recall the definition of the growth of a finitely generated group Γ. Choose a finite set of generators, say $g_1, \ldots, g_l$. Then every element of Γ can be expressed as a word in $g_1, \ldots, g_l$. Now define the growth function as

$$\#_\Gamma(N) = \text{ the number of distinct words in } \Gamma \text{ of length } \leq N.$$

Γ is said to be of polynomial growth if there exists an integer k and a constant C such that

$$\#_\Gamma(N) \leq CN^k.$$

By employing volume comparison, J. Milnor proved the following basic result.

Theorem 6 (J. Milnor [14]) *If M is complete and $Ric_M \geq 0$, then every finitely generated subgroup of $\pi_1(M)$ is of polynomial growth.*

This combined with the following remarkable result shows that every such subgroup of $\pi_1(M)$ is nilpotent up to finite index.

Theorem 7 (M. Gromov [9]) *A finitely generated group is of polynomial growth iff there is a nilpotent subgroup of finite index.*

Therefore, the structure of the torsion-free part of the fundamental group of a manifold with $Ric \geq 0$ is now more or less clear. However, the question whether or not it is finitely generated still remains open.

2.2 The case of almost nonnegative Ricci curvature

In this section we shall prove

Theorem 8 *For any constant $v > 0$, there exists $\epsilon = \epsilon(n, v) > 0$ such that if a complete manifold M^n admits a metric satisfying the conditions $Ric_M \geq -\epsilon$, $diam(M) = 1$, and $Vol(M) \geq v$, then the fundamental group of M is of polynomial growth with degree $\leq n$.*

Essential to our proof is a recent finiteness result of M. Anderson [3] for the fundamental groups of the class of compact n-dimensional riemannian manifolds M such that

$$Ric_M \geq (n-1)H, \; vol(M) \geq v, \; d(M) \leq D. \tag{2.1}$$

Actually we need the following more precise description of the fundamental group.

Theorem 9 (M. Anderson [3]) *Given M satisfying the bounds (2.1), then $\pi_1(M)$ has a presentation which obeys the following:*
1) The number of generators $g_1, \ldots, g_N$ is uniformly bounded with

$$N \leq N(v/D^n, HD^2),$$

2) $d(g_i(\tilde{x}_0), \tilde{x}_0) \leq 2D + \epsilon$, for any $\epsilon > 0$,
3) every relation is of the form $g_i g_j = g_k$.

The proof of Theorem 9 is closely related and useful to our proof of Theorem 8, so we will give the arguments. However, before going into the proof, let us mention a general lemma of M. Gromov which was pointed out in [12, 5.28].

Lemma 2 (M. Gromov) *Let M be a riemannian manifold with diameter D. There is a system of generators $\{g_i\}$ of $\pi_1(M) = \pi_1(M, x_0)$, and representatives γ_i of g_i such that $l(\gamma_i) \leq 2D + \epsilon$ and all relations among the generators are of the form $g_i g_j = g_k$.*

Proof of Lemma 2. First note that $\pi_1(M, x_0)$ is generated by geodesic loops of length $\leq 2D + \epsilon$, since any curve γ closed at x_0 can be subdivided into arcs of lengths $\leq \epsilon$. Thus γ is represented as a product of closed curves of lengths $\leq 2D+\epsilon$, which then are deformed via length decreasing homotopies to geodesic loops.

To show that all the relations can be reduced to the form $g_i g_j = g_k$, notice that if $\gamma(s)$ $(0 \leq s \leq 1)$ is nullhomotopic in M, and $l(\gamma) \leq \epsilon$. Join $\gamma(\frac{i}{3})$ by a minimizing geodesic g_i to x_0, $0 \leq i \leq 2$. Then each closed curve $g_i^{-1} \cup \gamma|_{[\frac{i}{3}, \frac{i+1}{3}]} \cup g_{i+1}$ is of length $\leq 2D+\epsilon$ and homotopic to a geodesic loop $\bar{\gamma}_i$ where i take values in $\mathbb{Z}_3$. We have $\bar{\gamma}_0 \bar{\gamma}_1 = \bar{\gamma}_2^{-1}$ representing the contractible loop $\gamma(s)$.

Now let $\gamma(t, s), 0 \leq t \leq 1$, be a piecewise differential homotopy from $\gamma = \gamma(0, s)$ to $\{x_0\} = \gamma(1, s)$. By uniform continuity we can choose N and subdivide $[0,1]\times[0,1]$ into small squares such that the curves

$$\begin{aligned}\gamma^1_{ij}:\ t &\rightarrow \gamma(\frac{i}{N}+t,\frac{j}{N}),\ 0\le t\le\frac{1}{N},\\ \gamma^2_{ij}:\ s &\rightarrow \gamma(\frac{i}{N},\frac{j}{N}+s),\ 0\le s\le\frac{1}{N},\\ \gamma^3_{ij}:\ t &\rightarrow \gamma(\frac{i}{N}+t,\frac{j}{N}+t),\ 0\le t\le\frac{1}{N},\end{aligned}$$

have lengths $\le \frac{1}{3}\epsilon$. Again join the points $\gamma(\frac{i}{N},\frac{j}{N})$ by minimizing geodesics g_{ij} to x_0. Then each curve

$$g_{ij}^{-1}\cup\gamma^1_{ij}\cup\gamma^2_{i+1j}\cup(\gamma^3_{ij})^{-1}\cup g_{ij},$$

$$g_{ij}^{-1}\cup\gamma^3_{ij}\cup(\gamma^1_{i+1j+1})^{-1}\cup(\gamma^2_{ij+1})^{-1}\cup g_{ij}$$

is of the form discussed before and represents a relation as in form 3) of Theorem 9. The product of these relations represents the contractible loop $\gamma(s)$.

Q.E.D.

Therefore we are only left to prove 1) of Theorem 9. Let

$$\Gamma=\{\text{homotopically distinct loops of length}\le 2D+\epsilon\},$$

$N=\#\Gamma$. Choose a base point $\tilde{x}_0$ in the universal covering $\tilde{M}\xrightarrow{p}M$, and let $x_0=p(\tilde{x}_0)$, and F a fundamental domain for the action of $\pi_1(M)$ on $\tilde{M}$ which contains $\tilde{x}_0$. For example, one may choose F to be the Dirichlet fundamental domain, i.e.

$$F=\cap_{\gamma\in\pi_1(M)}\{\tilde{x}\in\tilde{M};dist(\tilde{x},\tilde{x}_0)\le dist(\tilde{x},\gamma\tilde{x}_0)\}. \tag{2.2}$$

Let $B(\tilde{x}_0, r)$ (respectively $B(x_0, r)$) be the ball of radius r in $\tilde{M}$ (respectively M). Then it is easily verified that $B(\tilde{x}_0, r) \cap F$ is mapped isometrically onto $B(x_0, r)$ under the covering map , modulo a set of measure zero corresponding to ∂F. In particular, $vol(B(\tilde{x}_0, r) \cap F) = vol(B(x_0, r))$. Taking $r = d$ to be the diameter of M one has $vol(F) = vol(M)$, since it is clear from (2.2) that $F \subset B(\tilde{x}_0, d)$.

Now observe that

$$\cup_{\gamma \in \Gamma} \gamma(F) \subset B_{3D+\epsilon}(\tilde{x}_0),$$

or

$$N vol(M) \leq vol(B_{3D+\epsilon}(\tilde{x}_0)).$$

By volume comparison and (2.1), $N \leq N(v/D^n, HD^2)$. This proves Theorem 9.

Proof of Theorem 8. Choose a base point $\tilde{x}_0$ in the universal covering $\tilde{M} \xrightarrow{p} M$, and let $x_0 = p(\tilde{x}_0)$, and $g_1, \ldots, g_r$ a set of generators of the fundamental group $\pi_1(M)$ viewed as deck transformations in the isometry group of $\tilde{M}$. Denote by $\Gamma(s) = \{$ distinct words in $\pi_1(M)$ of length $\leq s$ $\}$, $\gamma(s) = \#\Gamma(s)$, and $l = \max_{1 \leq i \leq r}\{d(\tilde{x}_0, g_i(\tilde{x}_0))\}$.

Choose a fundamental domain F of $\pi_1(M)$ which contains $\tilde{x}_0$, then

$$\cup_{g \in \Gamma(s)} g(F) \subset B_{sl+d}(\tilde{x}_0),$$

where $d = d(M) = 1$. Therefore,

$$\gamma(s) \cdot vol(M) \leq vol(B_{sl+1}(\tilde{x}_0)). \tag{2.3}$$

Now suppose on the contrary, for any $\epsilon > 0$, there is a manifold M^n with a metric satisfying $Ric_M \geq -\epsilon,\ d(M) = 1,\ vol(M) \geq v$, and $\pi_1(M)$ is not of polynomial growth with degree $\leq n$. In particular, when taking the generators of Theorem 9, we can find real numbers s_i for all i such that

$$\gamma(s_i) > i s_i^n, \tag{2.4}$$

where s_i is independent of ϵ, since by Theorem 9 there are only finite many types of presentations for

$$\{\pi_1(M),\ M \text{ satisfying the bounds}(2.1)\},$$

if we choose the generators of Theorem 9. This is a crucial point here.

On the other hand, by (2.3) and volume comparison theorem we have

$$\gamma(s) \leq \frac{1}{v} \int_0^{3s+1} (\frac{\sinh \sqrt{\epsilon} t}{\sqrt{\epsilon}})^{n-1} dt.$$

For any fixed sufficiently large s_0, there is $\epsilon_0 = \epsilon(s_0)$ such that for all $s \leq s_0,\ \epsilon \leq \epsilon_0$,

$$\gamma(s) \leq \frac{6^n}{nv} s^n. \tag{2.5}$$

Now take $i_0 > 6^n/nv$. Then for $\epsilon < \epsilon(s_{i_0})$, using (2.4) and (2.5), we get a contradiction.

Q.E.D.

A conjecture of M. Gromov states that the fundamental group of a compact manifold with almost nonnegative sectional curvature (i.e. there exists a metric such that $K\,d^2 \geq -\epsilon$ for any $\epsilon > 0$) is of polynomial growth [10]. One would have proved Gromov's conjecture if the hypothesis of a lower volume bound in Theorem 8 could be removed. However this is not trivial at all. In fact the degree of the growth of the fundamental group of a compact manifold with almost nonnegative sectional curvature is not necassary bounded by the dimension (eg. nilmanifolds). Still, the following conjecture looks reasonable.

Conjecture 1 *The fundamental group of a weak near elliptic manifold is of polynomial growth.*

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