MATH 3B
CALCULUS WITH APPLICATIONS 2
SOLUTIONS TO PRACTICE MIDTERM #1

Exercise 1. Find the following integrals.

$$\text{(a)} \quad \int e^s \cos(e^s)\, ds$$

$$\text{(b)} \quad \int_0^1 \frac{4x}{3x^2+4}\, dx$$

$$\text{(c)} \quad \int_0^1 (\sqrt[4]{u}+1)^2\, du$$

$$\text{(d)} \quad \int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$$

(a) Let $u = e^s$. Then $du = e^s\, ds$ and the integral becomes

$$\int \cos(u)\, du.$$

Since

$$\int \cos(u)\, du = \sin(u) + c,$$

it follows that

$$\int e^s \cos(e^s)\, ds = \sin(e^s) + c. \quad \square$$

(b) Let $u = x^2$. Then $du = 2x\, dx$ and the integral becomes

$$\int_0^1 \frac{2}{3u+4}\, du.$$

By the Second Part of the Fundamental Theorem of Calculus we have that

$$\begin{aligned} \int_0^1 \frac{2}{3u+4}\, du &= \left(\frac{2}{3}\ln(3u+4)\right)\Big]_0^1 \\ &= \frac{2}{3}\left[\ln(3+4) - \ln(4)\right] \\ &= \frac{2}{3}\ln\frac{7}{4}. \quad \square \end{aligned}$$

(c) We have that

$$\begin{aligned} \int_0^1 (\sqrt[4]{u}+1)^2\, du &= \int_0^1 (\sqrt[4]{u}+1)(\sqrt[4]{u}+1)\, du \\ &= \int_0^1 \sqrt{u} + 2\sqrt[4]{u} + 1\, du. \end{aligned}$$

By the Second Part of the Fundamental Theorem of Calculus we have that

$$\begin{aligned}\int_0^1 \sqrt{u} + 2\sqrt[4]{u} + 1\, du &= \left(\frac{2}{3}u^{3/2} + \frac{8}{5}u^{5/4} + u\right)\Big]_0^1 \\ &= \left(\frac{2}{3} + \frac{8}{5} + 1\right) - 0 \\ &= \frac{49}{15}. \quad \square\end{aligned}$$

(d) Let $u = \sqrt{x}$. Then

$$du = \frac{1}{2\sqrt{x}}\, dx$$

and the integral becomes

$$\int 2e^u\, du.$$

Since

$$\int 2e^u\, du = 2e^u + c,$$

it follows that

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx = 2e^{\sqrt{x}} + c. \quad \square$$

Exercise 2. Find the area of the region enclosed by the line $y = x - 1$ and the parabola $y^2 = -2x + 5$.

First we find the intersection points by setting the ys equal:

$$(x-1)^2 = -2x + 5$$

So $x = 2, -2$ and the intersection points are $(2, 1), (-2, -3)$.

By drawing the graph, we see it's easier to integrate with respect to y,

$$\begin{aligned}A &= \int_{-3}^{1} (x_R - x_L) dy \\ &= \int_{-3}^{1} \left[\frac{5 - y^2}{2} - (y+1)\right] dy \\ &= \left(\frac{3}{2}y - \frac{1}{6}y^3 - \frac{1}{2}y^2\right) \Big|_{-3}^{1} \\ &= \left(\frac{3}{2} - \frac{1}{6} - \frac{1}{2}\right) - \left(-\frac{9}{2} + \frac{9}{2} - \frac{9}{2}\right) \\ &= \frac{5}{6} + \frac{9}{2} = \frac{16}{3}.\end{aligned}$$

Exercise 3. Find the volume of the solid generated by rotating the region bounded by the curves $y^2 = x$ and $x = 2y$ about the y-axis.

We employ the method of washers. The volume is given by

$$\int_0^2 \pi(2y)^2 - \pi(y^2)^2\, dy = \int_0^2 4\pi y^2 - \pi y^4\, dy$$

$$= \left(\frac{4\pi}{3}y^3 - \frac{\pi}{5}y^5\right)\Big]_0^2$$

$$= \left(\frac{4\pi}{3}(8) - \frac{\pi}{5}(32)\right) - 0$$

$$= \frac{64\pi}{15}. \quad \square$$

Exercise 4. Find the derivative of the function.

$$\text{(a)}\ \ f(x) = \int_0^{x^3} \frac{t}{\sqrt{1+t^3}}\, dt.$$

$$\text{(b)}\ \ f(x) = \int_{2x}^{3x+1} \sin\left(t^4\right) dt.$$

(a) We define the functions

$$g(x) = \int_0^{x} \frac{t}{\sqrt{1+t^3}}\, dt \quad \text{and} \quad h(x) = x^3.$$

We observe that

$$f(x) = (g \circ h)(x).$$

It follows by the Chain Rule that

$$f^{'}(x) = g^{'}(h(x))h^{'}(x).$$

By the First Part of the Fundamental Theorem of Calculus we have that

$$g^{'}(x) = \frac{x}{\sqrt{1+x^3}}$$

and hence that

$$g^{'}(h(x)) = g^{'}(x^3) = \frac{x^3}{\sqrt{1+x^9}}.$$

Since $h^{'}(x) = 3x^2$, it follows that

$$f^{'}(x) = \frac{3x^5}{\sqrt{1+x^9}}. \quad \square$$

(b) By the properties of integrals we write that

$$f(x) = \int_{2x}^{3x+1} \sin\left(t^4\right) dt = \int_{2x}^{0} \sin\left(t^4\right) dt + \int_{0}^{3x+1} \sin\left(t^4\right) dt$$

$$= -\int_{0}^{2x} \sin\left(t^4\right) dt + \int_{0}^{3x+1} \sin\left(t^4\right) dt.$$

Applying the Chain Rule together with the First Part of the Fundamental Theorem of Calculus as before, we obtain that

$$f^{'}(x) = -2\sin\left((2x)^4\right) + 3\sin\left((3x+1)^4\right). \quad \square$$

Exercise 5. A particle moves along a line with velocity function $v(t) = t^2 - t - 12$, where v is measured in meters per second. Find (a) the displacement and (b) the distance traveled by the particle during the time interval $[1, 6]$.

(a) The displacement function is the antiderivative of the velocity. We have that the displacement $d(t)$ is given by

$$d(t) = \frac{1}{3}t^3 - \frac{1}{2}t^2 - 12t + d_0,$$

where d_0 is the initial displacement. It follows that the particle's displacement during the time interval $[1, 6]$ is given by

$$\begin{aligned} d(6) - d(1) &= \left(\frac{1}{3}(216) - \frac{1}{2}(36) - 72 + d_0\right) - \left(\frac{1}{3} - \frac{1}{2} - 12 + d_0\right) \\ &= -\frac{35}{6}. \quad \square \end{aligned}$$

(b) The distance the particle traveled during the time interval $[1, 6]$ is given by

$$\begin{aligned} \int_1^6 |t^2 - t - 12|\, dt &= \int_1^4 -t^2 + t + 12\, dt + \int_4^6 t^2 - t - 12\, dt \\ &= \left(-\frac{1}{3}t^3 + \frac{1}{2}t^2 + 12t\right)\Bigg]_1^4 + \left(\frac{1}{3}t^3 - \frac{1}{2}t^2 - 12t\right)\Bigg]_4^6 \\ &= \left(-\frac{1}{3}(64) + \frac{1}{2}(16) + 48\right) - \left(-\frac{1}{3} + \frac{1}{2} + 12\right) \\ &\quad + \left(\frac{1}{3}(216) - \frac{1}{2}(36) - 72\right) - \left(\frac{1}{3}(64) - \frac{1}{2}(16) - 48\right) \\ &= \frac{235}{6}. \quad \square \end{aligned}$$