POISSON STRUCTURES ON AFFINE SPACES AND FLAG VARIETIES. I. MATRIX AFFINE POISSON SPACE

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Abstract. The standard Poisson structure on the rectangular matrix variety $M_{m,n}(\mathbb{C})$ is investigated, via the orbits of symplectic leaves under the action of the maximal torus $T \subset GL_{m+n}(\mathbb{C})$. These orbits, finite in number, are shown to be smooth irreducible locally closed subvarieties of $M_{m,n}(\mathbb{C})$, isomorphic to intersections of dual Schubert cells in the full flag variety of $GL_{m+n}(\mathbb{C})$. Three different presentations of the T-orbits of symplectic leaves in $M_{m,n}(\mathbb{C})$ are obtained – (a) as pullbacks of Bruhat cells in $GL_{m+n}(\mathbb{C})$ under a particular map; (b) in terms of rank conditions on rectangular submatrices; and (c) as matrix products of sets similar to double Bruhat cells in $GL_m(\mathbb{C})$ and $GL_n(\mathbb{C})$. In presentation (a), the orbits of leaves are parametrized by a subset of the Weyl group S_{m+n}, such that inclusions of Zariski closures correspond to the Bruhat order. Presentation (b) allows explicit calculations of orbits. From presentation (c) it follows that, up to Zariski closure, each orbit of leaves is a matrix product of one orbit with a fixed column-echelon form and one with a fixed row-echelon form. Finally, decompositions of generalized double Bruhat cells in $M_{m,n}(\mathbb{C})$ (with respect to pairs of partial permutation matrices) into unions of T-orbits of symplectic leaves are obtained.

Introduction

0.1. We investigate the geometry of the affine variety $M_{m,n} = M_{m,n}(\mathbb{C})$ of complex $m \times n$ matrices in relation to its standard Poisson structure (see §1.5) and to the action of the torus of "row and column automorphisms". Specifically, let T denote the torus of diagonal matrices in GL_{m+n}, identified with $T_m \times T_n$ where T_ℓ denotes the corresponding torus in GL_ℓ. There is a natural action of T on $M_{m,n}$ which arises as the restriction of the natural left action of $GL_m \times GL_n$ on $M_{m,n}$: namely, $(a, b).x = axb^{-1}$ for $(a, b) \in T$ and $x \in M_{m,n}$. This action of T on $M_{m,n}$ is by Poisson isomorphisms; in particular, the action of each element of T maps symplectic leaves of $M_{m,n}$ to symplectic leaves. Thus, it is natural

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to look at T-orbits of symplectic leaves of $M_{m,n}$, which are regular Poisson submanifolds of $M_{m,n}$, rather than at individual symplectic leaves. (Here and throughout, we view the T-orbit of a symplectic leaf $\mathcal{L}$ as the set-theoretic union $\bigcup_{t\in T} t.\mathcal{L}$, rather than as the family $(t.\mathcal{L})_{t\in T}$ of symplectic leaves.) As advantages to this approach, we mention that T-orbits of symplectic leaves are easier to identify than single symplectic leaves, and these orbits exhibit direct relations with known geometric and Lie-theoretic structures. For example, we prove that the T-orbits of symplectic leaves in $M_{m,n}$ are isomorphic (as varieties) to intersections of dual Schubert cells in the full flag variety of GL_{m+n}, and each generalized double Bruhat cell in $M_{m,n}$ (corresponding to a pair of partial permutation matrices) is a disjoint union of T-orbits of symplectic leaves, containing one such orbit as an open dense subset. One thus sees that the Poisson structure of $M_{m,n}$ is in some ways similar to, but also more intricate than, that of the group GL_n – for instance, as follows from the analysis of Hodges and Levasseur [**15**], the orbits of symplectic leaves in GL_n under left translation by the standard maximal torus are precisely the double Bruhat cells.

In a sequel to this paper, we will investigate the relation between the standard Poisson structures on different (partial) flag varieties of a semisimple algebraic group. We will also relate the restriction of the Poisson structure to various Poisson subvarieties with known quadratic Poisson structures on affine spaces. A detailed study of Poisson structures of the latter type associated to arbitrary Schubert cells in flag varieties of semisimple groups will be presented as well.

0.2. Recall that a Poisson group structure on an algebraic group G is thought of as the "semiclassical limit" of a quantization of G, a viewpoint promulgated in particular by Drinfeld and his school (cf. [**6**]). Relationships between the symplectic foliation of such a Poisson structure and the primitive spectrum of a quantized coordinate ring $\mathcal{O}_q(G)$ are viewed under the heading of a generalized version of the Kirillov–Kostant orbit method. In the work of Soibelman (e.g., [**27**]) on compact groups G, this led to bijections between the symplectic leaves of G and the primitive ideals of $\mathcal{O}_q(G)$. Hodges and Levasseur [**15**, **16**] then established analogous results for $G = SL_n$, which were extended to all semisimple groups by Joseph [**19**]. We take the corresponding viewpoint that the Poisson structure on $M_{m,n}$ is the semiclassical limit of the structure of $\mathcal{O}_q(M_{m,n})$, and argue that the results of the present paper should correspond to the framework of the primitive ideals in $\mathcal{O}_q(M_{m,n})$. Specifically, we conjecture that the sets of minors which define the T-orbits of symplectic leaves in $M_{m,n}$ (obtainable from Theorem 4.2) should match the sets of quantum minors which generate the prime ideals of $\mathcal{O}_q(M_{m,n})$ invariant under winding automorphisms (cf. [**13**]). Some relations between $M_{m,n}$ and $\mathcal{O}_q(M_{m,n})$ are already known. In particular, the set of T-orbits of symplectic leaves in $M_{m,n}$, partially ordered by inclusions of closures, is anti-isomorphic to the poset $T\text{-Spec}\,\mathcal{O}_q(M_{m,n})$ of winding-invariant prime ideals in $\mathcal{O}_q(M_{m,n})$ – our work shows that the former poset is anti-isomorphic to the set

$$S_{m+n}^{\leq w_\circ^{m,n}} = \left\{ y \in S_{m+n} \;\middle|\; y \leq \begin{pmatrix} 1 & 2 & \cdots & n & n{+}1 & n{+}2 & \cdots & n{+}m \\ m{+}1 & m{+}2 & \cdots & m{+}n & 1 & 2 & \cdots & m \end{pmatrix} \right\}$$

under the Bruhat order, while Launois [**22**, Theorem 5.6] has proved that $S_{m+n}^{\leq w_\circ^{m,n}}$ is isomorphic to $T\text{-Spec}\,\mathcal{O}_q(M_{m,n})$.

0.3. Before summarizing our main results, we indicate some notation, beginning with $N = m + n$. By $\mathrm{Gr}(n, N)$ we denote the Grassmannian of n-dimensional subspaces of an N-dimensional space. We write B_ℓ^+ and B_ℓ^- for the standard Borel subgroups of any GL_ℓ (consisting of upper, respectively lower, triangular matrices), and identify the Weyl group of GL_ℓ with both the symmetric group S_ℓ and the group of permutation matrices in GL_ℓ. The symbol $w_\circ^\ell$ denotes the longest element of S_ℓ. For $0 \le t \le \ell$, let S_t^1 and $S_{\ell-t}^2$ denote the natural copies of S_t and $S_{\ell-t}$ inside S_ℓ, acting on the numbers $1, \dots, t$ and $t+1, \dots, \ell$, respectively. Finally, for a Weyl group W and a subgroup W_1 generated by simple reflections, W^{W_1} denotes the set of minimal length representatives for left cosets in W/W_1. Recall that each such coset has a unique representative in W^{W_1} (cf. [**3**, Proposition 2.3.3(i)]).

The following theorem summarizes Theorems 3.9 and 3.13.

Theorem A. (a) *There are only finitely many T-orbits of symplectic leaves in $M_{m,n}$, and they are smooth irreducible locally closed subvarieties.*

(b) *The T-orbits of symplectic leaves in $M_{m,n}$ can be described as the sets*

$$\mathcal{P}_w = \left\{ x \in M_{m,n} \;\middle|\; \begin{bmatrix} w_\circ^n & 0 \\ x & w_\circ^m \end{bmatrix} \in B_N^+ w B_N^+ \right\},$$

where $w \in S_N$ and $w \ge \left[\begin{smallmatrix} w_\circ^n & 0 \\ 0 & w_\circ^m \end{smallmatrix}\right]$ in the Bruhat order.

(c) *The closure of $\mathcal{P}_w$ equals the disjoint union of the $\mathcal{P}_z$ for $z \le w$.*

(d) *As an algebraic variety, $\mathcal{P}_w$ is isomorphic to the intersection of dual Schubert cells*

$$B_N^- . \begin{bmatrix} w_\circ^n & 0 \\ 0 & w_\circ^m \end{bmatrix} B_N^+ \cap B_N^+ . w B_N^+$$

in the full flag variety GL_N / B_N^+. □

0.4. Fulton [**10**] has given computational descriptions of Bruhat cells $B_N^+ w B_N^+$ in terms of ranks of rectangular submatrices. We apply his results to the sets $\mathcal{P}_w$, to characterize exactly which matrices x lie in each $\mathcal{P}_w$, in terms of ranks of rectangular submatrices of x. See Theorem 4.2 for the precise statement.

0.5. The results of Theorem A are obtained by embedding $M_{m,n}$ in the Grassmannian $\mathrm{Gr}(n, N)$ which, equipped with an appropriate Poisson structure, becomes a Poisson homogeneous space for the standard Poisson algebraic group GL_N. (For details on Poisson homogeneous spaces for Poisson algebraic groups see [**7**] or Section 1.) This approach provides, in addition, a natural Poisson compactification of $M_{m,n}$ which, in particular, suggests an approach to the problem of studying the spectrum of $\mathcal{O}_q(M_{m,n})$ via noncommutative projective geometry.

A completely different viewpoint is obtained by focussing, as we do in Sections 5 and 6, on the sets $\mathcal{O}_t^{m,n}$ of matrices in $M_{m,n}$ with a fixed rank t. Each $\mathcal{O}_t^{m,n}$ is a Poisson homogeneous space for the natural action of $GL_m \times GL_n$ (equipped with an appropriate Poisson group structure). The latter group is the Levi factor of the maximal parabolic subgroup of GL_N defining $\mathrm{Gr}(n, N)$. The key results of this approach, taken from Theorems 5.11, 6.1, 6.4 and Corollary 6.5, are as follows.

Theorem B. *Fix a nonnegative integer $t \le \min\{m,n\}$.*

(a) *The T-orbits of symplectic leaves within $\mathcal{O}_t^{m,n}$ can be described as the sets*

$$\mathcal{P}^t_{(y,v,z,u)} = \bigcup_{\substack{\tau \in S_t^1 \\ z\tau \le y,\ v\tau^{-1} \le u}} \big(B_m^+ y B_m^+ \cap B_m^- z\tau\big). \begin{bmatrix} I_t & 0 \\ 0 & 0 \end{bmatrix} .\big(\tau^{-1} B_n^- u^{-1} B_n^- \cap v^{-1} B_n^+\big),$$

where $(y,v,z,u) \in S_m^{S^2_{m-t}} \times S_n^{S_t^1} \times S_m^{S_t^1} \times S_n^{S^2_{n-t}}$ and $z \le y$, $v \le u$.

(b) *For (y,v,z,u) as in* (a)*, the set*

$$\mathcal{C}_{y,z}.\mathcal{R}_{u,v} = \big(B_m^+ y B_m^+ \cap B_m^- z\big). \begin{bmatrix} I_t & 0 \\ 0 & 0 \end{bmatrix} .\big(B_n^- u^{-1} B_n^- \cap v^{-1} B_n^+\big)$$

is dense in $\mathcal{P}^t_{(y,v,z,u)}$.

(c) *The sets $\mathcal{C}_{y,z} = \big(B_m^+ y B_m^+ \cap B_m^- z\big). \begin{bmatrix} I_t \\ 0 \end{bmatrix}$ are $(T_m \times T_t)$-orbits of symplectic leaves of $M_{m,t}$, and each of the sets consisting of all matrices in $M_{m,t}$ with rank t and a given column-echelon form is a disjoint union of certain $\mathcal{C}_{y,z}$.*

(d) *The sets $\mathcal{R}_{u,v} = [\, I_t \quad 0\,].\big(B_n^- u^{-1} B_n^- \cap v^{-1} B_n^+\big)$ are $(T_t \times T_n)$-orbits of symplectic leaves of $M_{t,n}$, and each of the sets consisting of all matrices in $M_{t,n}$ with rank t and a given row-echelon form is a disjoint union of certain $\mathcal{R}_{u,v}$.* $\square$

The descriptions of torus orbits of symplectic leaves in $M_{m,n}$ given in part (b) of Theorem A and part (a) of Theorem B are matched in Theorem 5.11 and Proposition 5.9.

0.6. Finally, we study the decomposition of $M_{m,n}$ into generalized double Bruhat cells

$$\mathcal{B}^{w_1,w_2} = B_m^+ w_1 B_n^+ \cap B_m^- w_2 B_n^-,$$

for partial permutation matrices w_1, w_2. If w_1 and w_2 have the same rank t (which is necessary for $\mathcal{B}^{w_1,w_2}$ to be nonempty), there are unique decompositions

$$w_1 = y \begin{bmatrix} I_t & 0 \\ 0 & 0 \end{bmatrix} v^{-1} \qquad\qquad w_2 = z \begin{bmatrix} I_t & 0 \\ 0 & 0 \end{bmatrix} u^{-1}$$

where $y \in S_m^{S^2_{m-t}}$, $v \in S_n^{S_t^1 S^2_{n-t}}$, $z \in S_m^{S_t^1 S^2_{m-t}}$, and $u \in S_n^{S^2_{n-t}}$ (see Lemma 7.3). The following results are given in Theorem 7.4.

Theorem C. *Let $w_1, w_2 \in M_{m,n}$ be partial permutation matrices with rank t, decomposed as above.*

(a) *$\mathcal{B}^{w_1,w_2}$ is nonempty if and only if $z \le y$ and $v \le u$, in which case it is a T-stable Poisson subvariety of $M_{m,n}$, and a smooth irreducible locally closed subvariety.*

(b) *The partition of $\mathcal{B}^{w_1,w_2}$ into T-orbits of symplectic leaves is given by*

$$\mathcal{B}^{w_1,w_2} = \bigsqcup \left\{ \mathcal{P}^t_{(y,v\tau_2,z\tau_1,u)} \;\middle|\; \begin{matrix} \tau_1 \in S^2_{m-t} \subseteq S_m,\ z\tau_1 \le y \\ \tau_2 \in S^2_{n-t} \subseteq S_n,\ v\tau_2 \le u \end{matrix} \right\}.$$

(c) *$\mathcal{P}^t_{(y,v,z,u)}$ is Zariski open and dense in $\mathcal{B}^{w_1,w_2}$.* $\square$

0.7. Let us also note that the standard Poisson algebraic group GL_m is a T-stable Poisson subvariety of $M_{m,m}$. Thus the T-orbits of symplectic leaves of GL_m (which are the same as the T_m-orbits of leaves) comprise a subset of the T-orbits of symplectic leaves of $M_{m,m}$. The former are the double Bruhat cells $B_m^+ w_1 B_m^+ \cap B_m^- w_2 B_m^-$ of GL_m, for $w_1, w_2 \in S_m$. They were studied in detail by Fomin and Zelevinsky in [**8**], who in a joint work with Berenstein also proved [**1**] that their rings of regular functions provide important examples of upper *cluster algebras* [**9**]. Our results in particular show that the double Bruhat cells in GL_m are special cases of intersections of dual Schubert cells on the full flag variety of GL_{2m}. It would be very interesting to understand whether any intersection of dual Schubert cells on the full flag variety of an arbitrary reductive algebraic group gives rise to a cluster algebra in the sense of Fomin and Zelevinsky [**9**]. If this is true, it will imply that any T-orbit of symplectic leaves of $M_{m,n}$ is the spectrum of a cluster algebra.

0.8. We conclude the introduction with some remarks on our notation and conventions. All manifolds and algebraic varieties considered in this paper are over the field of complex numbers.

Given an algebraic group G with tangent Lie algebra $\mathfrak{g}$, we denote by $L(\gamma)$ and $R(\gamma)$ the left and right invariant multi-vector fields on G corresponding to $\gamma \in \wedge\mathfrak{g}$. If G acts on a smooth quasiprojective variety M, we will denote by

$$\chi : \wedge\mathfrak{g} \to \Gamma(M, \wedge TM) \tag{0.1}$$

the extension of the infinitesimal action of $\mathfrak{g}$ on M to $\wedge\mathfrak{g}$. In the special case of the left and right multiplication actions of G on itself ($g.a = ga$ and $g.a = ag^{-1}$), the above infinitesimal actions will be denoted by

$$\chi^R, \chi^L : \wedge\mathfrak{g} \to \Gamma(G, \wedge TG). \tag{0.2}$$

Note that for $\gamma \in \wedge\mathfrak{g}$,

$$\chi^L(\gamma) = R(\gamma) \qquad \chi^R(\gamma) = (-1)^{\epsilon(\gamma)} L(\gamma), \tag{0.3}$$

where $\epsilon(\gamma)$ is the parity of γ.

If Y is a locally closed subvariety of an algebraic variety X and $Z \subseteq Y$, we will denote the closure of Z in Y by $\mathrm{Cl}_Y(Z)$. By a *stratification* of an algebraic variety X we mean a partition of X into smooth, irreducible, locally closed subvarieties, $X = \bigsqcup_{\alpha \in A} X_\alpha$, such that for each $\alpha \in A$, we have $\overline{X}_\alpha = \bigsqcup_{\beta \in A(\alpha)} X_\beta$ for some index set $A(\alpha) \subseteq A$.

We will use the following convention to distinguish double cosets from orbits of cosets. *For any subgroups C and D of a group G*:

(i) *The (C, D) double coset of $g \in G$ will be denoted by CgD*;
(ii) *The C-orbit of gD in G/D will be denoted by $C.gD$.*

The adjoint action of $g \in G$ on $h \in G$ will be written as $\mathrm{Ad}_g(h) = ghg^{-1}$.

1. Poisson algebraic groups and Poisson homogeneous spaces

We begin with background and notation for Poisson algebraic groups and Poisson homogeneous spaces, and then characterize the symplectic leaves and their orbits in certain Poisson homogeneous spaces.

1.1. Poisson varieties. Recall that a *Poisson manifold* is a pair (X, π) consisting of a smooth manifold X together with a Poisson bivector field $\pi \in \wedge^2 TX$, that is, $[\pi, \pi] = 0$ where $[.,.]$ denotes the Schouten bracket. A (not necessarily closed) submanifold Y of X is called a *Poisson submanifold* if $\pi_y \in \wedge^2 T_y Y$ for all $y \in Y$. In this case $(Y, \pi|_Y)$ is a Poisson manifold as well. A (not necessarily closed) submanifold Y of X is called a *complete Poisson submanifold* if it is stable under all Hamiltonian flows. Any complete Poisson submanifold is a Poisson submanifold. The converse is not necessarily true but, if (X, π) is a Poisson manifold which is partitioned into a disjoint union of Poisson submanifolds $X = \bigsqcup_{\alpha \in A} Y_\alpha$, then all Y_α are complete Poisson submanifolds, see [**17**, Lemma 3.2].

The Poisson manifold (X, π) is *regular*, respectively *symplectic*, if $\operatorname{rank} \pi$ is constant, respectively $\operatorname{rank} \pi = \dim X$. A *symplectic leaf* of (X, π) is a maximal connected (not necessarily closed) symplectic submanifold. It is well known that any Poisson manifold (X, π) can be decomposed into a disjoint union of its symplectic leaves, see e.g. [**29**, **28**]. Note that a (not necessarily closed) submanifold Y of X is a complete Poisson submanifold if and only if it is a union of symplectic leaves of (X, π).

Let us also recall that a map $\phi : (X, \pi) \to (Z, \pi')$ between two Poisson manifolds is called a *Poisson map* if $\phi_*(\pi_x) = \pi'_{\phi(x)}$ for all $x \in X$. For instance if Y is a Poisson submanifold of (X, π), the natural inclusion $i : (Y, \pi|_Y) \hookrightarrow (X, \pi)$ is Poisson.

All Poisson manifolds considered in this paper will be (*complex*) *smooth quasiprojective Poisson varieties. The symplectic leaves of a smooth quasiprojective Poisson variety are not necessarily algebraic, i.e., smooth irreducible locally closed subvarieties. We will see below that this is the case for many Poisson varieties admitting appropriate transitive algebraic group actions.*

1.2. Poisson algebraic groups and Manin triples. A *Poisson algebraic group* is an algebraic group G equipped with a Poisson bivectorfield $\pi \in \wedge^2 TG$ such that the map

$$(G, \pi) \times (G, \pi) \to (G, \pi)$$

is Poisson. The tangent Lie algebra $\mathfrak{g} = \operatorname{Lie}(G)$ of a Poisson algebraic group (G, π) has a canonical Lie bialgebra structure; see [**4**, §1.3] and [**21**, §3.3] for details.

Recall that a *Manin triple of Lie algebras* is a triple $(\mathfrak{d}, \mathfrak{a}, \mathfrak{b})$ with the following properties:

(1) $\mathfrak{d}$ is a Lie algebra, $\mathfrak{a}$ and $\mathfrak{b}$ are Lie subalgebras of $\mathfrak{d}$, and $\mathfrak{d}$ is the vector space direct sum of $\mathfrak{a}$ and $\mathfrak{b}$.

(2) $\mathfrak{d}$ is equipped with a nondegenerate invariant bilinear form with respect to which both $\mathfrak{a}$ and $\mathfrak{b}$ are Lagrangian (i.e., maximal isotropic) subspaces.

To any Lie bialgebra $\mathfrak{g}$ one associates the Manin triple $(D(\mathfrak{g}), \mathfrak{g}, \mathfrak{g}^*)$. Here $D(\mathfrak{g})$ and $\mathfrak{g}^*$ are the underlying Lie algebras of the double and the dual Lie bialgebras of $\mathfrak{g}$. The bilinear form on $D(\mathfrak{g})$ is given by $\langle x + \alpha, y + \beta \rangle = \beta(x) + \alpha(y)$ for $x, y \in \mathfrak{g}$, $\alpha, \beta \in \mathfrak{g}^*$.

1.3. Definition. *A Manin triple of algebraic groups is a triple (D, A, B) of algebraic groups such that A and B are algebraic subgroups of D and $(\mathrm{Lie}(D), \mathrm{Lie}(A), \mathrm{Lie}(B))$ is a Manin triple of Lie algebras.*

Fix a Manin triple of algebraic groups (D, A, B). Then D has a canonical Poisson algebraic group structure with a Poisson bivector field given by

$$\pi^D = L(r) - R(r) = \chi^R(r) - \chi^L(r) \quad \text{where} \quad r = \sum_i x_i \wedge x^i \in \wedge^2 \mathrm{Lie}\, D$$

in the notation (0.2)–(0.3) for left and right invariant multi-vector fields $L(.)$, $R(.)$ on D and infinitesimal actions $\chi^L(.)$, $\chi^R(.)$ of $\mathrm{Lie}\, D$ on D. Here $\{x_i\}$ and $\{x^i\}$ are dual bases of $\mathrm{Lie}(A)$ and $\mathrm{Lie}(B)$, respectively, with respect to the nondegenerate bilinear form on $\mathrm{Lie}(D)$.

The groups A and B are Poisson subvarieties of D. The Poisson algebraic group (D, π^D) is a *double* of $(A, \pi^D|_A)$, and $(B, -\pi^D|_B)$ is a *dual* Poisson algebraic group of $(A, \pi^D|_A)$; cf. [**4**, §1.4] and [**21**, §3.3].

We will say that a Poisson algebraic group (G, π) is a part of a Manin triple of algebraic groups (D, G, F) if the Poisson structure π coincides with the Poisson structure $\pi^D|_G$ induced from D.

1.4. Standard Poisson structures on reductive algebraic groups. Let G be a complex reductive algebraic group. The standard Poisson structure on G, turning it into a Poisson algebraic group, is defined as follows. Fix two opposite Borel subalgebras $\mathfrak{b}^\pm$ of $\mathfrak{g} = \mathrm{Lie}\, G$ and set $\mathfrak{h} = \mathfrak{b}^+ \cap \mathfrak{b}^-$ for the corresponding Cartan subalgebra of $\mathfrak{g}$. Fix a nondegenerate bilinear invariant form $\langle ., . \rangle$ on $\mathfrak{g}$ for which the square of the length of a long root is equal to 2. Choose sets of root vectors $\{e_\alpha\}$ and $\{f_\alpha\}$, spanning respectively the nilradicals $\mathfrak{n}^+$ and $\mathfrak{n}^-$ of $\mathfrak{b}^+$ and $\mathfrak{b}^-$, normalized by $\langle e_\alpha, f_\alpha \rangle = 1$.

The standard r-matrix of $\mathfrak{g}$ is given by

$$r = \sum_\alpha e_\alpha \wedge f_\alpha \tag{1.1}$$

and the corresponding standard Poisson structure on G is defined by

$$\pi = L(r) - R(r) = \chi^R(r) - \chi^L(r), \tag{1.2}$$

in the notation (0.2)–(0.3).

The standard r-matrix on $G = GL_N$ is

$$r^N = \sum_{1 \le i < j \le n} E_{ij} \wedge E_{ji} \in \wedge^2 \mathfrak{gl}_N \tag{1.3}$$

where the E_{ij} are the standard elementary matrices.

By abuse of notation, GL_N will denote the algebraic group GL_N equipped with the standard Poisson structure π^N from (1.2), associated to the r-matrix r^N (1.3). By $GL_N^\bullet$ we will denote the Poisson algebraic group $(GL_N, -\pi^N)$.

Any standard (complex) reductive Poisson algebraic group (G, π) is a part of the Manin triple $(G \times G, \Delta(G), F)$ where $\Delta(G)$ is the diagonal of $G \times G$ and

$$F = \{(hu^+, h^{-1}u^-) \mid h \in T,\, u^\pm \in U^\pm\} \subseteq B^+ \times B^-, \tag{1.4}$$

where $B^\pm$ are the Borel subgroups of G corresponding to $\mathfrak{b}^\pm$, $U^\pm$ are their unipotent radicals, and $T = B^+ \cap B^-$ is the corresponding maximal torus of G. For the standard Poisson structure on G,

$$\mathfrak{g}^* = \operatorname{Lie} F = \{(h + n^+, -h + n^-) \mid h \in \mathfrak{h},\, n^\pm \in \mathfrak{n}^\pm\} \subseteq \mathfrak{b}^+ \oplus \mathfrak{b}^-. \tag{1.5}$$

The nondegenerate invariant bilinear form on $\mathrm{Lie}(G \times G) \cong \mathfrak{g} \oplus \mathfrak{g}$, used in the Manin triple of Lie algebras $(\mathfrak{g} \oplus \mathfrak{g}, \Delta(\mathfrak{g}), \mathfrak{g}^*)$, is

$$\langle (x_1, x_2), (y_1, y_2) \rangle = \langle x_1, y_1 \rangle - \langle x_2, y_2 \rangle, \tag{1.6}$$

where in the right hand side $\langle ., . \rangle$ denotes the bilinear form on $\mathfrak{g}$, fixed above.

1.5. Matrix affine Poisson spaces. The $m \times n$ *matrix affine Poisson space* is the affine space $\mathbb{A}^{mn}$, identified with the space $M_{m,n}$ of all $m \times n$ complex matrices. The standard Poisson structure on $M_{m,n}$ is given by

$$\pi^{m,n} = \sum_{i,k=1}^{m} \sum_{j,l=1}^{n} \big(\operatorname{sign}(k-i) + \operatorname{sign}(l-j)\big) x_{il} x_{kj} \frac{\partial}{\partial x_{ij}} \wedge \frac{\partial}{\partial x_{kl}} \tag{1.7}$$

in terms of the standard coordinate functions x_{ij} on $M_{m,n}$. *By abuse of notation, $M_{m,n}$ will denote the matrix affine Poisson space, thus dropping the symbol for the Poisson structure* (1.7) *on $M_{m,n}$.*

Note that GL_m acts on $M_{m,n}$ by left multiplication ($g.x = gx$ for $g \in GL_m$, $x \in M_{m,n}$), and GL_n acts on $M_{m,n}$ by (inverted) right multiplication ($g.x = xg^{-1}$ for $g \in GL_m$, $x \in M_{m,n}$). The extensions of the corresponding infinitesimal actions of $\mathfrak{gl}_m$ and $\mathfrak{gl}_n$ on M_n to $\wedge \mathfrak{gl}_m$ and $\wedge \mathfrak{gl}_n$ will be denoted by

$$\chi^L : \wedge \mathfrak{gl}_m \to \Gamma(M_{m,n}, TM_{m,n}) \quad \text{and} \quad \chi^R : \wedge \mathfrak{gl}_n \to \Gamma(M_{m,n}, TM_{m,n}).$$

Note that in the case $m = n$ these extend the infinitesimal actions χ^L and χ^R of $\mathfrak{gl}_m$ on $GL_m \subseteq M_m$, defined in (0.2).

By direct computation one shows that the Poisson structure (1.7) on $M_{m,n}$ is also given by the formula

$$\pi^{m,n} = \chi^R(r^n) - \chi^L(r^m) \tag{1.8}$$

in terms of the standard r-matrix r^N for GL_N, see (1.3).

Note that GL_n is a Poisson subvariety of $M_{n,n}$.

1.6. Poisson homogeneous spaces. Fix a Poisson algebraic group (G,π) and set $\mathfrak{g} = \mathrm{Lie}(G)$. A *Poisson* (G,π)*-space* is a smooth quasiprojective Poisson variety (M,π_M) equipped with a morphic G-action for which

$$(G,\pi)\times(M,\pi_M)\to(M,\pi_M)$$

is a Poisson morphism.

A *Poisson homogeneous space for* (G,π) is a Poisson (G,π)-space (M,π^M) for which M is a homogeneous G-space. (Recall that any homogeneous space of an algebraic group is a smooth quasiprojective variety [**2**, Theorem 6.8].) To each $m\in M$, one associates the *Drinfeld subalgebra* [**7**]

$$\mathfrak{l}_m = \{x+\alpha\in D(\mathfrak{g}) \mid x\in\mathfrak{g},\ \alpha\in\mathfrak{g}^*,\ \alpha|_{\mathfrak{g}_m}=0,\ \alpha\rfloor\pi^M_m = x+\mathfrak{g}_m\}$$

of the double $D(\mathfrak{g})$, where $\mathfrak{g}_m$ denotes the Lie algebra of the stabilizer $G_m = \mathrm{Stab}_G(m)$, the tangent space T_mM is identified with $\mathfrak{g}/\mathfrak{g}_m$, and the Poisson bivectorfield π^M_m is thought of as an element of $\wedge^2(\mathfrak{g}/\mathfrak{g}_m)$. Note that

$$\mathfrak{g}_m = \mathfrak{g}\cap\mathfrak{l}_m. \tag{1.9}$$

The Drinfeld subalgebras $\mathfrak{l}_m$ are moreover Lagrangian subalgebras of the double $D(\mathfrak{g})$, equipped with the canonical nondegenerate invariant bilinear form [**7**; **21**, Proposition 6.2.15]. The map, associating to $m\in M$ its Drinfeld subalgebra $\mathfrak{l}_m\subseteq D(\mathfrak{g})$, is G-equivariant:

$$\mathfrak{l}_{gm} = \mathrm{Ad}_g(\mathfrak{l}_m)$$

where Ad_g refers to the adjoint action of G on $D(\mathfrak{g})$.

1.7. Definition. *A Poisson homogeneous* (G,π)*-space* (M,π^M) *will be called* <u>*algebraic*</u> *if the Drinfeld subalgebra of some* $m\in M$ *is the tangent Lie algebra of an algebraic subgroup* $L_m\subseteq D$.

Because of the G-equivariance of the map $m\mapsto\mathfrak{l}_m$, if the condition in the definition is satisfied for one point $m\in M$, then it holds for any $m\in M$.

An important type of Poisson homogeneous space (M,π^M) is the class of those for which π^M vanishes at some point of M. In the rest of this subsection we describe those.

An algebraic subgroup Q of a Poisson algebraic group (G,π) will be called an *almost Poisson algebraic subgroup* if

$$\pi_q\in T_qQ\wedge T_qG$$

for all $q\in Q$. (Recall that if $\pi_q\in\wedge^2T_qQ$ for all $q\in Q$, then Q is called a Poisson algebraic subgroup of (G,π).) Fix an almost Poisson algebraic subgroup Q of (G,π), and consider the projection

$$p: G\to G/Q,\quad p(g)=gQ.$$

Then

$$\pi_{gq} - R_q(\pi_g)\in L_g(T_qQ)\wedge T_{gq}G$$

for all $g \in G$, $q \in Q$, and the rule

$$\pi^{G/Q}_{gQ} = p_*(\pi_g), \quad g \in G \tag{1.10}$$

gives a well-defined Poisson structure $\pi^{G/Q}$ on G/Q. The pair $(G/Q, \pi^{G/Q})$ is a Poisson homogeneous space of (G, π) and $\pi^{G/Q}$ vanishes at the base point eQ of G/Q.

1.8. Theorem. *Fix a Poisson algebraic group (G, π).*

(a) *Any Poisson homogeneous (G, π)-space (M, π^M) with the property that the Poisson bivectorfield π^M vanishes at some point $m \in M$ is isomorphic to $(G/Q, \pi^{G/Q})$ for $Q = \mathrm{Stab}_G(m)$ which is an almost Poisson algebraic subgroup of (G, π).*

(b) *For an almost Poisson algebraic subgroup Q of G, the Drinfeld Lagrangian subalgebra of the base point eQ of the Poisson homogeneous space $(G/Q, \pi^{G/Q})$ is*

$$\mathfrak{l} = \mathfrak{q} + \mathfrak{q}^\perp \tag{1.11}$$

where $\mathfrak{q} = \mathrm{Lie}\, Q$ and $\mathfrak{q}^\perp$ refers to the orthogonal subspace to $\mathfrak{q} \subseteq \mathfrak{g}$ in $\mathfrak{g}^*$.

(c) *A connected algebraic subgroup Q of (G, π) is an almost Poisson algebraic subgroup if and only if the orthogonal complement $\mathfrak{q}^\perp \subseteq \mathfrak{g}^*$ is a subalgebra of the dual Lie bialgebra $\mathfrak{g}^*$ of $\mathfrak{g}$ (as in part* (b)*, $\mathfrak{q} = \mathrm{Lie}\, Q$).*

(d) *A connected algebraic subgroup Q of (G, π) is a Poisson algebraic subgroup if and only if $\mathfrak{q}^\perp$ is an ideal in $\mathfrak{g}^*$.* □

Parts (a) and (d) of this theorem can be found, e.g., in [**21**, page 52 and Proposition 6.2.3]; parts (b) and (c) are well known.

Below we gather some results on symplectic leaves of algebraic Poisson homogeneous spaces. Fix a Poisson algebraic group (G, π) which is a part of a Manin triple of algebraic groups (D, G, F), as defined in §1.3. Fix also an algebraic Poisson homogeneous (G, π)-space with connected stabilizer subgroups G_m (see §1.6). Such a homogeneous space has the form G/N where N is a connected subgroup of G and the Drinfeld Lagrangian subalgebra of $\mathrm{Lie}(D)$ corresponding to the base point $eN \in G/N$ integrates to an algebraic subgroup $L \subseteq D$. Note that

$$N = (G \cap L)^\circ,$$

the identity component of $G \cap L$, because of (1.9) and the connectedness of N. Consider the composition of maps

$$\Pi : G/N \xrightarrow{\mu} G/(G \cap L) \xrightarrow{\cong} G.L \subseteq D/L, \tag{1.12}$$

where μ is the map $gN \mapsto g(G \cap L)$.

1.9. Theorem. *Assume that (G, π) is a Poisson algebraic group which is a part of a Manin triple of algebraic groups (D, G, F). Let $(G/N, \pi')$ be an algebraic Poisson homogeneous (G, π)-space with connected stabilizer subgroups for which the Drinfeld Lagrangian subalgebra of the base point eN is $\mathrm{Lie}\, L$ for an algebraic subgroup L of G.*

Then the symplectic leaves of G/N are the connected components (i.e., irreducible components) of the inverse images under Π of the F-orbits on D/L, and all of them are smooth irreducible locally closed subvarieties of G/N.

Note that some F-orbits on D/L might not intersect the image of Π, but when an F-orbit on D/L intersects the image of Π, the intersection is transversal since the Lie algebras of G and F span $\operatorname{Lie} D$. Below we will consider only those F-orbits on D/L that intersect the image of Π.

Proof. Any F-orbit on D/L is a smooth locally closed subvariety, see e.g. [**2**, Proposition 1.8]. Thus, its inverse image under Π (if it is nontrivial) is a locally closed subvariety of G/N. Each intersection of an F-orbit on D/L with $\operatorname{Im}\Pi = G.L$ is a transversal intersection of group orbits and therefore is smooth. As a consequence its inverse image under the étale map $\Pi : G/N \to G.L$ is smooth as well.

Finally, the connected components of the (nontrivial) inverse images of F-orbits are known to be symplectic leaves of $(G/N, \pi)$ due to results of Lu [**23**] and Karolinsky [**20**] in the differential category. Since [**23**, **20**] assume that $D = FG$, we sketch another approach. Consider the bivector field $\chi(r) \in \Gamma(D/L, \wedge^2 TD/L)$ where $r \in \wedge^2 \operatorname{Lie} D$ is the r-matrix for the Poisson structure on D, see Definition 1.3, and $\chi(.)$ refers to the natural infinitesimal action of $\operatorname{Lie} D$ on D/L. It was proved in [**24**] that $\chi(r)$ is a Poisson bivectorfield and that the connected components of the intersections of any F and G orbits on D/L are symplectic leaves of $\chi(r)$. It is straightforward to show that the map $\Pi : (G/N, \pi') \to (D/L, \chi(r))$ is Poisson. The statement now follows from the fact that $\Pi : G/N \to G.L$ is étale. □

In the remainder of this section, we gather some results on orbits of symplectic leaves in Poisson homogeneous spaces. In the setting of Theorem 1.9, assume that H is a subgroup of G that normalizes $F \subseteq D$. Then the Poisson structure π on G vanishes on H, see [**24**], and as a consequence H acts by Poisson isomorphisms on any Poisson homogeneous (G, π)-space (M, π^M). This in particular means that each element $h \in H$ maps symplectic leaves of (M, π^M) to symplectic leaves. The H-orbits of symplectic leaves are characterized in the following theorem which is adapted from [**24**]. Let us first note that since H normalizes $F \subseteq D$, the product HF is an algebraic subgroup of D.

1.10. Theorem. *In the setting of Theorem* 1.9*, the H-orbits of symplectic leaves of the Poisson homogeneous space G/N are the irreducible components of the inverse images under Π of the HF-orbits on D/L (see* (1.12)*), and all of them are smooth irreducible locally closed subvarieties of G/N.*

Proof. Fix $y \in G.L = \operatorname{Im}(\Pi) \subseteq D/L$. The intersection of $\operatorname{Im}\Pi = G.L$ with Fy is transversal because the Lie algebras of HG and F span $\operatorname{Lie}(D)$. Therefore $\operatorname{Im}\Pi \cap Fy$ is a smooth and locally closed subset of D/L. The second statement follows from the fact that both $G.L$ and Fy are locally closed subsets of D/L (as orbits of algebraic groups). Let $\mathcal{P}$ be an irreducible component of $\Pi^{-1}(HFy)$. It is a smooth, irreducible, locally closed subset of G/N because $\Pi : G/N \to G.L$ is an étale morphism, recall (1.12).

We need to show that $\mathcal{P} = H\mathcal{S}$ for some irreducible component $\mathcal{S}$ of $\Pi^{-1}(Fy)$. First, note that for two distinct irreducible components $\mathcal{S}_1$ and $\mathcal{S}_2$ of $\Pi^{-1}(Fy)$, either $H\mathcal{S}_1 = H\mathcal{S}_2$

or $H\mathcal{S}_1$ and $H\mathcal{S}_2$ are disjoint. Since the map Π is H-equivariant,

$$\Pi^{-1}(HFy) = H\Pi^{-1}(Fy). \tag{1.13}$$

As a consequence,

$$\mathcal{P} = H\mathcal{S}_1 \sqcup \ldots \sqcup H\mathcal{S}_m$$

for some irreducible components $\mathcal{S}_i$ of $\Pi^{-1}(Fy)$, lying inside $\mathcal{P}$. All that we need to show now is that $m = 1$. Since $\mathcal{P}$ is irreducible it is sufficient to show that

For each irreducible component $\mathcal{S}$ of $\Pi^{-1}(Fy)$, the set $H\mathcal{S}$ is an open subset of $\mathcal{P}$.

We show this in the rest of the proof. Let $x' \in G.L = \operatorname{Im}\Pi$. Since H normalizes F,

$$T_{x'}(HFx') = T_{x'}(Hx') + T_{x'}(Fx').$$

The intersections $HFx' \cap Gx'$ and $Fx' \cap Gx'$ in D/L are transversal because the Lie algebras of F and G span Lie(D). Taking into account this and the facts that H is a subgroup of G and $\operatorname{Im}\Pi = G.x' \supset H.x'$ gives

$$T_{x'}(HFx' \cap \operatorname{Im}\Pi) = T_{x'}(Hx') + T_{x'}(Fx' \cap \operatorname{Im}\Pi).$$

Since Π is an étale map, recall (1.12), we obtain

$$T_x\mathcal{P} = T_x(Hx) + T_x\mathcal{S} \qquad \text{for all } x \in \mathcal{S}.$$

If $f : H \times \mathcal{S} \to \mathcal{P}$ denotes the map $(h, x) \mapsto hx$, then the above equality implies that df is surjective at any point of $H \times \mathcal{S}$. As a consequence of this, the morphism f is smooth and thus flat, because $H \times \mathcal{S}$ and $\mathcal{P}$ are nonsingular, see [**14**, §III, Proposition 10.4]. The latter implies that f is open, see [**14**, §III, Problem 9.1]. Therefore the image of f (which is nothing but $H\mathcal{S}$) is an open subset of $\mathcal{P}$.

In fact, since we work over $\mathbb{C}$ the last statement is almost immediate: the fact that the differential of $f : H \times \mathcal{S} \to \mathcal{P}$ is surjective everywhere implies that the image of f is open in the classical topology. But $\operatorname{Im} f$ is also a constructible subset of $\mathcal{P}$, thus it is a Zariski open subset. $\square$

2. Intersections of Bruhat and Schubert cells

Our main results rely on certain combinatorial and geometric information about intersections of Bruhat and Schubert cells, which we develop in this section.

2.1. Bruhat and Schubert cells. Let G be a complex reductive algebraic group. As in §1.4, fix two opposite Borel subgroups $B^{\pm}$ of G and set $T = B^+ \cap B^-$ for the corresponding maximal torus of G. Denote the projection to the flag variety by

$$\eta : G \to G/B^+. \tag{2.1}$$

Recall that the $(B^{\pm}, B^{\pm})$-double cosets of G are called *Bruhat cells* of G and the $B^{\pm}$-orbits on G/B^+ are called *Schubert cells* of G/B^+.

Let $U^\pm$ be the unipotent radical of $B^\pm$. Denote by W the Weyl group of (G,T), by $\leq$ the Bruhat order on W, and by $l(.)$ the length function on W. For each $w \in W$, fix a representative $\dot{w}$ in the normalizer of T. When the result of a formula involving some $\dot{w}$ does not depend on the particular representative $\dot{w}$ of w, the notation for such a representative will be omitted. As a consequence of the Bruhat lemma, all Bruhat cells of G are uniquely represented in the form $B^\pm w B^\pm$ for some $w \in W$ and all Schubert cells of G/B^+ are uniquely represented in the form $B^\pm.wB^+$ for some $w \in W$.

For each $w \in W$, define the following subgroups of $U^\pm$:

$$U_w^- = U^- \cap \mathrm{Ad}_w(U^-) \quad \text{and} \quad U_w^0 = \mathrm{Ad}_w^{-1}(U^-) \cap U^+. \tag{2.2}$$

Recall that U^-, U_w^-, and U_w^0 are affine spaces (and closed subvarieties of G), and as such,

$$U_w^- \times \mathrm{Ad}_w(U_w^0) \cong U^-, \tag{2.3}$$

with the isomorphism given by group multiplication (e.g., see [**2**, §14.12, p. 193]).

In Theorem 2.3, for all $y, z \in W$ we describe the structure of the locally closed subvarieties

$$B^- z \cap B^+ y B^+, \quad U^- \dot{z} \cap B^+ y B^+, \quad \text{and} \quad U_z^- \dot{z} \cap B^+ y B^+ \tag{2.4}$$

of the intersection of Bruhat cells $B^- z B^+ \cap B^+ y B^+$ in terms of the intersection of the dual Schubert cells

$$\mathcal{B}_{z,y} = B^-.zB^+ \cap B^+.yB^+ \subseteq G/B^+. \tag{2.5}$$

The first two varieties in (2.4) are smooth due to the transversality of the intersections $(\mathrm{Lie}\, U^- + \mathrm{Lie}\, B^+ = \mathrm{Lie}\, G)$. It will be shown in Theorem 2.3 that the third variety in (2.4) is also smooth. In Theorem 2.5, we describe the Zariski closures in G of the sets in (2.4).

First recall the following result of Deodhar, [**5**, Corollary 1.2]:

2.2. Proposition. [Deodhar] *For $y, z \in W$, the intersection $\mathcal{B}_{z,y} = B^-.zB^+ \cap B^+.yB^+$ of dual Schubert cells is nonempty if and only if $y \geq z$ in the Bruhat order of W. In that case, the intersection is a smooth irreducible locally closed subvariety of G/B^+ of dimension $l(y) - l(z)$.* □

The smoothness in the second part of the Proposition is a direct consequence of the transversality of the intersection. The harder result in the second part is the irreducibility. It follows from a stratification of the intersection by smooth irreducible locally closed subvarieties isomorphic to $\mathbb{C}^n \times (\mathbb{C}^\times)^m$, $n, m \in \mathbb{Z}$, obtained by Deodhar [**5**, Theorem 1.1], in which only one set has dimension equal to $l(y) - l(z)$. A direct consequence of the first part of the Proposition is that the intersection of Bruhat cells $B^- z B^+ \cap B^+ y B^+$ is nonempty if and only $y \geq z$.

2.3. Theorem. *Let $y, z \in W$ with $y \geq z$.*

(a) *The projection $\eta : G \to G/B^+$ restricts to a biregular isomorphism of affine spaces*

$$\eta : U_z^- \dot{z} \xrightarrow{\cong} B^-.zB^+. \tag{2.6}$$

The set $U_z^- \dot{z} \cap B^+ y B^+$ is a smooth irreducible locally closed subset of G, and η further restricts to a biregular isomorphism of quasiprojective varieties

$$\eta : U_z^- \dot{z} \cap B^+ y B^+ \xrightarrow{\cong} B^-.zB^+ \cap B^+.yB^+. \tag{2.7}$$

(b) *The group multiplication in G restricts to biregular isomorphisms of quasiprojective varieties*

$$\left(U_z^- \dot{z} \cap B^+ y B^+\right) \times U_z^0 \xrightarrow{\cong} U^- \dot{z} \cap B^+ y B^+ \tag{2.8}$$

and

$$\left(U_z^- \dot{z} \cap B^+ y B^+\right) \times U_z^0 \times T \xrightarrow{\cong} B^- z \cap B^+ y B^+. \tag{2.9}$$

Proof. (a) The first statement (2.6) is well known. (E.g., see [**2**, Theorem 14.12(b)] for the analogous isomorphism $U^+ \cap \mathrm{Ad}_w(U^-) \to B^+.wB^+$.) Because U_z^- is a closed subvariety of G, to complete the proof of part (a), all that we need to show is

$$\eta\left(U_z^- \dot{z} \cap B^+ y B^+\right) = B^-.zB^+ \cap B^+.yB^+. \tag{2.10}$$

It is obvious that

$$\eta\left(U_z^- \dot{z} \cap B^+ y B^+\right) \subseteq B^-.zB^+ \cap B^+.yB^+.$$

But

$$\eta\left(B^- z B^+ \cap B^+ y B^+\right) = B^-.zB^+ \cap B^+.yB^+,$$

and $B^- z B^+ \subseteq U_z^- \dot{z} B^+$ because of (2.3), so that $B^- z B^+ \cap B^+ y B^+ \subseteq (U_z^- \dot{z} \cap B^+ y B^+) B^+$. The surjectivity in (2.10) now follows from the isomorphism (2.6).

(b) First note that the right action of $U_z^0 \subseteq B^+ \cap \mathrm{Ad}_z^{-1} U^-$ on G preserves the intersection on the right hand side of (2.8), that is,

$$U^- \dot{z} \cap B^+ y B^+ \supset \left(U_z^- \dot{z} \cap B^+ y B^+\right) U_z^0.$$

To show the opposite inclusion, let

$$g \in U^- \dot{z} \cap B^+ y B^+.$$

Multiplying (2.3) on the right by $\dot{z}$, we get that

$$g = g_1 u \quad \text{for some} \quad g_1 \in U_z^- \dot{z} \text{ and } u \in U_z^0.$$

Since $B^+ y B^+ U_z^0 = B^+ y B^+$, we obtain that $g_1 = g u^{-1} \in U_z^- \dot{z} \cap B^+ y B^+$ and thus

$$g = g_1 u \in \left(U_z^- \dot{z} \cap B^+ y B^+\right) U_z^0.$$

Therefore

$$U^- \dot{z} \cap B^+ y B^+ = \left(U_z^- \dot{z} \cap B^+ y B^+\right) U_z^0,$$

which together with (2.3) implies (2.8).

In a similar way one proves (2.9), using (2.8) and

$$B^- z \cap B^+ y B^+ = \left(U^- \dot{z} \cap B^+ y B^+\right) T. \qquad \square$$

The following Theorem combines and summarizes Proposition 2.2 and Theorem 2.3.

2.4. Theorem. *For any $y, z \in W$ with $y \le z$, the sets $U_z^- \dot{z} \cap B^+ y B^+$, $U^- \dot{z} \cap B^+ y B^+$ and $B^- z \cap B^+ y B^+$ are smooth irreducible locally closed subvarieties of the intersection of Bruhat cells $B^- z B^+ \cap B^+ y B^+ \subseteq G$. They are related to the intersection of dual Schubert cells $\mathcal{B}_{z,y} = B^-.zB^+ \cap B^+.yB^+ \subseteq G/B^+$ by the following biregular isomorphisms, obtained as compositions of the isomorphisms* (2.7)–(2.9)*:*

$$\begin{aligned} &U_z^- \dot{z} \cap B^+ y B^+ \cong \mathcal{B}_{z,y} \\ &U^- \dot{z} \cap B^+ y B^+ \cong \mathcal{B}_{z,y} \times U_z^0 \\ &B^- z \cap B^+ y B^+ \cong \mathcal{B}_{z,y} \times U_z^0 \times T. \qquad \square \end{aligned}$$

The first of the intersections above will play an important role in the following section. We label it as follows

$$\mathcal{U}_{z,y} = U_z^- \dot{z} \cap B^+ y B^+ \tag{2.11}$$

for $y, z \in W$.

2.5. Theorem. *For any $y, z \in W$ with $y \le z$, the Zariski closures of the three locally closed subsets of G considered in Theorem* 2.4 *are given by*

(a) $$\overline{U_z^- \dot{z} \cap B^+ y B^+} = U_z^- \dot{z} \cap \overline{B^+ y B^+} = \bigsqcup_{\substack{w \in W \\ z \le w \le y}} U_z^- \dot{z} \cap B^+ w B^+$$

(b) $$\overline{U^- \dot{z} \cap B^+ y B^+} = U^- \dot{z} \cap \overline{B^+ y B^+} = \bigsqcup_{\substack{w \in W \\ z \le w \le y}} U^- \dot{z} \cap B^+ w B^+$$

(c) $$\overline{B^- z \cap B^+ y B^+} = B^- z \cap \overline{B^+ y B^+} = \bigsqcup_{\substack{w \in W \\ z \le w \le y}} B^- z \cap B^+ w B^+.$$

In the proof of Theorem 2.5, we will need the following algebrogeometric fact.

2.6. Lemma. *Let $\bigsqcup_{\alpha \in A} X_\alpha$ be a stratification* (*cf.* §0.8) of a smooth algebraic variety X, and Y a smooth, irreducible, locally closed subvariety of X that intersects all the strata X_α transversely. Then

$$\mathrm{Cl}_Y(Y \cap X_\alpha) = Y \cap \overline{X}_\alpha$$

for all $\alpha \in A$.

Proof. Fix $\alpha \in A$. Then $\overline{X}_\alpha = \bigsqcup_{\beta \in A(\alpha)} X_\beta$ for some subset $A(\alpha) \subseteq A$, and $\dim X_\beta < \dim X_\alpha$ for all $\beta \in A(\alpha) \setminus \{\alpha\}$.

Because $\overline{X}_\alpha$ is a closed subvariety of X that contains X_α, the set $\mathrm{Cl}_Y(Y \cap X_\alpha)$ equals the union of those irreducible components of

$$Y \cap \overline{X}_\alpha = \bigsqcup_{\beta \in A(\alpha)} Y \cap X_\beta$$

that meet $Y \cap X_\alpha$. On one hand, the dimension of any irreducible component of $Y \cap \overline{X}_\alpha$ is greater than or equal to $\dim Y + \dim X_\alpha - \dim X$; see [**14**, Chapter I, Proposition 7.1 and Theorem 7.2]. On the other hand, for all $\beta \in A(\alpha) \setminus \{\alpha\}$,

$$\dim(Y \cap X_\beta) = \dim Y + \dim X_\beta - \dim X < \dim Y + \dim X_\alpha - \dim X$$

because of the transversality of the intersection of Y with X_β. Therefore each irreducible component of $Y \cap \overline{X}_\alpha$ meets $Y \cap X_\alpha$, which completes the proof of the lemma. $\square$

Proof of Theorem 2.5. The second equalities in (a)–(c) follow from Proposition 2.2, Theorem 2.3, and the well known fact for the closures of Bruhat cells,

$$\overline{B^+yB^+} = \bigsqcup_{\substack{w \in W \\ w \leq y}} B^+wB^+.$$

The first equalities in (b) and (c) are obtained by applying Lemma 2.6 to the Bruhat decomposition $G = \bigsqcup_{w \in W} B^+wB^+$ of the group G and taking $Y = U^-\dot{z}$ and $Y = B^-z$, respectively. In both cases, the intersection of Y with any Bruhat cell B^+wB^+ is transversal since $\operatorname{Lie} U^-$ and $\operatorname{Lie} B^+$ span $\operatorname{Lie} G$. Moreover, in both cases Y is a closed subvariety of G and $\operatorname{Cl}_Y(Z)$ coincides with $\overline{Z}$ for any subset Z of Y.

The first equality in (a) cannot be proved in exactly the same way because $\operatorname{Lie} U_z^-$ and $\operatorname{Lie} B^+$ do not span G. We apply Lemma 2.6 to the stratification of the flag variety G/B^+ by Schubert cells $B^+.wB^+$, and take $Y = B^-.zB^+$. This gives us

$$\operatorname{Cl}_{B^-.zB^+}(B^-.zB^+ \cap B^+.yB^+) = B^-.zB^+ \cap \overline{B^+.yB^+}.$$

Applying the biregular isomorphisms (2.6) and (2.7), one obtains

$$\operatorname{Cl}_{U_z^-\dot{z}}(U_z^-\dot{z} \cap B^+yB^+) = U_z^-\dot{z} \cap \overline{B^+yB^+}.$$

Since $U_z^-.z$ is a closed subvariety of G we can replace the left hand side with $\overline{U_z^-\dot{z} \cap B^+yB^+}$. This completes the proof of (a). $\square$

3. A first approach to $M_{m,n}$ through a Poisson structure on $\operatorname{Gr}(n, m+n)$

Throughout this section, fix positive integers m and n, with

$$N = m + n.$$

We derive a description of the orbits of symplectic leaves in $M_{m,n}$ under a natural action of the maximal torus of GL_N, by embedding $M_{m,n}$ in a Grassmannian Poisson homogeneous space, $\operatorname{Gr}(n, N)$.

3.1. Generalities on GL_N. The Borel subgroups of GL_N consisting of upper and lower triangular matrices will be respectively denoted by B^+ and B^-. Let $U^{\pm}$ be their unipotent radicals. The maximal torus of GL_N consisting of diagonal matrices will be denoted by T. In situations where it is helpful to indicate that we are working with subgroups of the $N \times N$ general linear group, we will label the above Borel and Cartan subgroups of GL_N as $B_N^{\pm}$ and T_N. However, we reserve subscripts on $U^{\pm}$ for a different purpose – see (3.3) below.

We will need to describe a number of sets of matrices given in block form, and it will be convenient to use a block form of set notation for the purpose. For example, if A, B, C, D are subsets of M_n, $M_{n,m}$, $M_{m,n}$, M_m respectively, we set

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_N \;\middle|\; a \in A,\, b \in B,\, c \in C,\, d \in D \right\}.$$

In case one of the sets A, B, C, D is a singleton, we may omit the corresponding braces from the notation. Thus, for instance, the notation $\begin{bmatrix} I_n & M_{n,m} \\ 0 & I_m \end{bmatrix}$ indicates the unipotent subgroup $\left\{ \begin{bmatrix} I_n & b \\ 0 & I_m \end{bmatrix} \;\middle|\; b \in M_{n,m} \right\}$ of GL_N, where I_n and I_m are the identity matrices of sizes n and m.

Define the following maximal parabolic subgroup of GL_N:

$$P_n = \begin{bmatrix} GL_n & M_{n,m} \\ 0 & GL_m \end{bmatrix}. \tag{3.1}$$

Let L_n be the Levi factor of P_n containing T, and U_n^+ the unipotent radical of P_n. Denote by U_n^- the unipotent radical of the parabolic subgroup of GL_N opposite to P_n. Explicitly, $L_n = L_n^1 L_m^2$ where

$$L_n^1 = \begin{bmatrix} GL_n & 0 \\ 0 & I_m \end{bmatrix} \cong GL_n \qquad L_m^2 = \begin{bmatrix} I_n & 0 \\ 0 & GL_m \end{bmatrix} \cong GL_m \tag{3.2}$$

and

$$U_n^+ = \begin{bmatrix} I_n & M_{n,m} \\ 0 & I_m \end{bmatrix} \qquad U_n^- = \begin{bmatrix} I_n & 0 \\ M_{m,n} & I_m \end{bmatrix}. \tag{3.3}$$

Let $\mathfrak{b}^{\pm}$, $\mathfrak{h}$, $\mathfrak{p}_n$, $\mathfrak{l}_n$, $\mathfrak{l}_n^1$, $\mathfrak{l}_m^2$, and $\mathfrak{n}_n^{\pm}$ denote the Lie algebras of $B^{\pm}$, T, P_n, L_n, L_n^1, L_m^2, and $U_n^{\pm}$. The Lie algebras $\mathfrak{n}_n^+$ and $\mathfrak{n}_n^-$ are naturally identified as vector spaces with $M_{n,m}$ and $M_{m,n}$. The exponential maps $\exp : \mathfrak{n}_n^{\pm} \to U_n^{\pm}$ are bijective and are explicitly given by

$$\mathfrak{n}_n^- \cong M_{m,n} \ni x \mapsto \begin{bmatrix} I_n & 0 \\ x & I_m \end{bmatrix} \qquad \mathfrak{n}_n^+ \cong M_{n,m} \ni y \mapsto \begin{bmatrix} I_n & y \\ 0 & I_m \end{bmatrix}. \tag{3.4}$$

The Weyl group of GL_N is isomorphic to the symmetric group S_N. The maximal length element of S_N will be denoted by $w_\circ^N$. Explicitly, we have $w_\circ^N = \begin{pmatrix} 1 & 2 & \cdots & N \\ N & N-1 & \cdots & 1 \end{pmatrix}$.

For $k = 1, \dots, N$, we will denote by S^1_k and S^2_k the subgroups of S_N that are isomorphic to S_k and permute respectively the first and the last k indices. In other words:

$$
\begin{aligned}
S^1_k &= \{w \in S_N \mid w(i) = i \text{ for all } i > k\} \\
S^2_k &= \{w \in S_N \mid w(i) = i \text{ for all } i \le N - k\}.
\end{aligned}
\tag{3.5}
$$

In this notation, the Weyl groups of L^1_n and L^2_m are identified respectively with the subgroups S^1_n and S^2_m of the Weyl group S_N of GL_N. The Weyl group of the Levi factor L_n is identified with the subgroup $S^1_n S^2_m$ of S_N.

Denote by $(w^n_\circ, w^m_\circ) \in S_N$ the product of the maximal length elements of S^1_n and S^2_m. In other words, this is the maximal length element of the Weyl group of the Levi factor L_n. Set

$$
w^{m,n}_\circ = w^N_\circ (w^n_\circ, w^m_\circ).
\tag{3.6}
$$

It is the maximal length representative in S_N of the coset $w^N_\circ (S^1_n S^2_m)$.

For a given $w \in S_N$, define the following subsets of S_N:

$$
\begin{aligned}
S_N^{\le w} &= \{y \in S_N \mid y \le w\} \\
S_N^{\ge w} &= \{y \in S_N \mid y \ge w\}
\end{aligned}
\tag{3.7}
$$

$$
S_N^{[-n,m]} = \{y \in S_N \mid -n \le s(i) - i \le m \text{ for all } i = 1, 2, \dots, N\}.
\tag{3.8}
$$

In Lemma 3.12, we will show that the subsets $S_N^{[-n,m]}$ and $S_N^{\le w^{m,n}_\circ}$ of S_N coincide. This set will enter as a parametrizing set for the set of T-orbits of symplectic leaves of the matrix affine Poisson space $M_{m,n}$.

Finally, consider the embedding

$$
S_N \hookrightarrow N(T), \qquad S_N \ni w \mapsto (a_{ij}) = (\delta_{iw(j)}) \in N(T),
\tag{3.9}
$$

which is a section for the projection $N(T) \to N(T)/T \cong S_N$. *By abuse of notation we will identify S_N with its image in $N(T)$, and thus use the same letter w to denote the permutation matrix in $N(T)$ corresponding to a permutation $w \in S_N$.* Under this identification, the maximal length element $w^N_\circ \in S_N$ corresponds to the (unit) anti-diagonal matrix. Moreover, we have

$$
w^{m,n}_\circ = \begin{bmatrix} 0 & w^m_\circ \\ w^n_\circ & 0 \end{bmatrix} \begin{bmatrix} w^n_\circ & 0 \\ 0 & w^m_\circ \end{bmatrix} = \begin{bmatrix} 0 & I_m \\ I_n & 0 \end{bmatrix}.
$$

3.2. GL_N/P_n and $\mathrm{Gr}(n, N)$. Recall the natural isomorphism $\mathrm{Gr}(n, N) \cong GL_N/P_n$.

Proposition. (a) *The orthogonal complement of* $\mathfrak{p}_n$ *in the dual Lie bialgebra* $\mathfrak{gl}_N^*$ *for the standard Lie bialgebra structure on* $\mathfrak{gl}_N$ *(recall* (1.5)*) is* $\mathfrak{p}_n^\perp = \mathfrak{n}_n^+ \oplus \{0\}$.

(b) *The parabolic subgroup* P_n *of* GL_N *is a Poisson algebraic subgroup for the standard Poisson structure on* GL_N.

(c) *The pair* $\big(\mathrm{Gr}(n,N) \cong GL_N/P_n, -\chi(r^N)\big)$ *is a Poisson homogeneous space for the standard Poisson algebraic group* GL_N. *Here* r^N *is the standard* r*-matrix* (1.3) *for* $\mathfrak{gl}_N$ *and* χ *denotes the infinitesimal action for the left multiplication of* GL_N *on* $\mathrm{Gr}(n,N)$.

(d) *The Drinfeld Lagrangian subalgebra of the base point* eP_n *of the Poisson homogeneous space* $\big(GL_N/P_n, -\chi(r^N)\big)$ *is*

$$\overline{\mathfrak{l}}_n = \left\{ \left(\begin{bmatrix} a & b_1 \\ 0 & c \end{bmatrix}, \begin{bmatrix} a & b_2 \\ 0 & c \end{bmatrix} \right) \,\middle|\, a \in \mathfrak{gl}_n,\, c \in \mathfrak{gl}_m,\, b_i \in M_{n,m} \right\} \subseteq \mathfrak{gl}_N \oplus \mathfrak{gl}_N \cong D(\mathfrak{gl}_N). \tag{3.10}$$

It is the tangent Lie algebra of the algebraic subgroup

$$\overline{L}_n = \left\{ \left(\begin{bmatrix} a & b_1 \\ 0 & c \end{bmatrix}, \begin{bmatrix} a & b_2 \\ 0 & c \end{bmatrix} \right) \,\middle|\, a \in GL_n,\, c \in GL_m,\, b_i \in M_{n,m} \right\} \tag{3.11}$$

of $GL_N \times GL_N$*; in particular,* $\big(\mathrm{Gr}(n,N), -\chi(r^N)\big)$ *is an algebraic Poisson homogeneous space for the standard Poisson algebraic group* GL_N*. Moreover,*

$$\Delta(GL_N) \cap \overline{L}_n = \Delta(P_n). \tag{3.12}$$

(e) *Each intersection of a* B^+*- and a* B^-*-orbit on* $\mathrm{Gr}(n,N)$ *is a locally closed Poisson subvariety of* $\big(\mathrm{Gr}(n,N), -\chi(r^N)\big)$.

Proof. (a) It is straightforward to check that $\mathfrak{n}_n^+ \oplus \{0\} \subseteq \mathfrak{gl}_N^* \subseteq \mathfrak{gl}_N \oplus \mathfrak{gl}_N$ is orthogonal to $\Delta(\mathfrak{p}_n)$ with respect to the bilinear form (1.6), recall §1.4. The statement now follows from the fact that the sum of the dimensions of $\mathfrak{p}_n$ and $\mathfrak{n}_n^+$ is equal to $\dim \mathfrak{gl}_N$.

Part (b) follows from Theorem 1.8(d) and the first part.

(c) Consider the projection $p : GL_N \to GL_N/P_n$ and the Poisson structure (1.10) for GL_N/P_n. Since the standard matrices E_{ij} belong to $\mathfrak{p}_n$ for $i < j$, we have

$$p_*(\chi^L(r^N)) = 0.$$

Thus in the present situation the Poisson structure (1.10) is exactly $-\chi(r^N)$. Now part (c) follows from the discussion before Theorem 1.8.

(d) Since the Poisson structure $-\chi(r^N)$ vanishes at the base point eP_n of GL_N/P_n, according to Theorem 1.8(b) the Drinfeld Lagrangian subalgebra of the double $D(\mathfrak{gl}_n) \cong \mathfrak{gl}_n \oplus \mathfrak{gl}_n$ is $\Delta(\mathfrak{p}_n) + \mathfrak{p}_n^\perp$. A simple computation leads to (3.10). The rest of part (d) is straightforward and will be omitted.

(e) Observe that the subgroup $T \subseteq GL_N \subseteq D(GL_N)$ normalizes the subgroup $F \subseteq D(GL_N)$ (recall (1.4)), and that $TF = B^+ \times B^-$. Theorem 1.10 implies that the T-orbits of symplectic leaves of $\big(GL_{n+m}/P_n,\ -\chi(r^N)\big)$ are the irreducible components of the inverse images of the $(B^+ \times B^-)$-orbits on $D(GL_N)/\overline{L}_n$ under the map

$$GL_N/P_n \xrightarrow{\Delta} D(GL_N)/\overline{L}_n$$

(cf. (1.12)), which is an embedding because of (3.12). It is obvious (because $\overline{L}_n \subseteq P_n \times P_n$) that each such inverse image falls within a single intersection of a B^+- and B^--orbit on $\mathrm{Gr}(n, N)$. Thus, the latter are finite unions of T-orbits of symplectic leaves and hence Poisson subvarieties of $\big(\mathrm{Gr}(n, N),\ -\chi(r^N)\big)$. $\square$

Throughout the remainder of the section, we shall always assume that $\mathrm{Gr}(n, N) \cong GL_N/P_n$ has been equipped with the Poisson structure $-\chi(r^N)$.

3.3. The open B^--orbit on $\mathrm{Gr}(n, N)$. The B^--orbit through the base point of GL_N/P_n is a Zariski open subvariety. According to Proposition 3.2(e), it is a Poisson subvariety of GL_N/P_n. Moreover, the open orbit $B^-.P_n \subseteq GL_N/P_n$ is an affine space which is isomorphic to U_n^- by

$$U_n^- \ni u \mapsto uP_n;$$

in particular, $B^-.P_n = U_n^-.P_n$. Composing this map with the exponential map

$$\exp : M_{m,n} \cong \mathfrak{n}_n^- \xrightarrow{\cong} U_n^-$$

induces an isomorphism of affine spaces

$$M_{m,n} \xrightarrow{\cong} U_n^-.P_n \subseteq GL_N/P_n, \qquad x \mapsto \begin{bmatrix} I_n & 0 \\ x & I_m \end{bmatrix} P_n. \tag{3.13}$$

We consider a twisted version of this isomorphism:

$$\Psi : M_{m,n} \xrightarrow{\cong} U_n^-.P_n, \qquad x \mapsto \exp(xw_\circ^n)P_n = \begin{bmatrix} I_n & 0 \\ xw_\circ^n & I_m \end{bmatrix} P_n. \tag{3.14}$$

Recall that $w_\circ^n$ denotes the maximal length element of S_n and its representative in the normalizer of the diagonal subgroup of GL_n, as fixed in §3.1.

The restriction of the Poisson structure $-\chi(r^N)$ to $U_n^-.P_n$ was computed by Gekhtman, Shapiro, and Vainshtein in [**12**]. The following result can be deduced from their computations, but we offer a more geometric proof.

3.4. Proposition. *The map $\Psi : M_{m,n} \to U_n^-.P_n$ is a Poisson isomorphism between the matrix affine Poisson space $M_{m,n}$ and the Poisson subvariety $U_n^-.P_n$ of GL_N/P_n.*

We break the standard r-matrix for $\mathfrak{gl}_N$ into three terms as follows

$$r^N = \sum_{1\le i<j\le n} E_{ij} \wedge E_{ji} + \sum_{n<i<j\le N} E_{ij} \wedge E_{ji} + \sum_{i\le n<j} E_{ij} \wedge E_{ji} \in \wedge^2 \mathfrak{gl}_N \tag{3.15}$$

and denote them by r_1^N, r_2^N, and r_3^N, respectively. First we establish an auxiliary result.

3.5. Lemma. *In the above notation,*

$$\chi(r_3^N)|_{U_n^-.P_n} = 0.$$

Proof. We shall use the label $(3.13)^{-1}$ to refer to the inverse isomorphism $U_n^-.P_n \to M_{m,n}$ of the isomorphism (3.13). Since U_n^- is abelian and $E_{j+n,i} \in \mathfrak{n}_n^-$ for $i \le n$, $j \le m$, under the isomorphism $(3.13)^{-1}$ we have

$$\chi(E_{j+n,i})|_{U_n^-.P_n} \mapsto \frac{\partial}{\partial x_{ji}} \quad \text{for} \quad i \le n,\, j \le m. \tag{3.16}$$

By a direct computation, one checks that for $x \in M_{m,n}$, $y \in M_{n,m}$, and a small $\epsilon \in \mathbb{C}$,

$$\begin{bmatrix} I_n & \epsilon y \\ 0 & I_m \end{bmatrix} . \begin{bmatrix} I_n & 0 \\ x & I_m \end{bmatrix} P_n = \begin{bmatrix} I_n & 0 \\ x(I_n + \epsilon yx)^{-1} & I_m \end{bmatrix} P_n = \begin{bmatrix} I_n & 0 \\ x - \epsilon xyx + O(\epsilon^2) & I_m \end{bmatrix} P_n.$$

This implies that under the isomorphism $(3.13)^{-1}$,

$$\chi(E_{i,j+n})|_{U_n^-.P_n} \mapsto -\sum_{k=1}^{m}\sum_{l=1}^{n} x_{ki}x_{jl}\frac{\partial}{\partial x_{kl}} \quad \text{for} \quad i \le n,\, j \le m. \tag{3.17}$$

Combining (3.16) and (3.17), we see that under the isomorphism $(3.13)^{-1}$,

$$\chi(r_3^N)|_{U_n^-.P_n} \mapsto -\sum_{k,j=1}^{m}\sum_{i,l=1}^{n} x_{ki}x_{jl}\frac{\partial}{\partial x_{kl}} \wedge \frac{\partial}{\partial x_{ji}} = 0. \qquad \square$$

3.6. Actions of L_n on $U_n^-.P_n$ and $M_{m,n}$, and a proof of Proposition 3.4. Since the Levi factor L_n normalizes U_n^-, it preserves the open B^--orbit $U_n^-.P_n$ on $\mathrm{Gr}(n,N)$ (recall §3.1 for notation). Via the isomorphism (3.13), this induces an action of $GL_m \times GL_n \cong L_n^2 \times L_n^1 = L_n$ on the affine space $M_{m,n}$. It is given by $(a,b).x = axb^{-1}$ for $a \in GL_m$, $b \in GL_n$, $x \in M_{m,n}$, which is checked by a direct computation:

$$\begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix} . \begin{bmatrix} I_n & 0 \\ x & I_m \end{bmatrix} P_n = \begin{bmatrix} I_n & 0 \\ axb^{-1} & I_m \end{bmatrix} P_n.$$

This action of $GL_m \times GL_n$ on $M_{m,n}$ breaks into the actions of GL_m and GL_n on $M_{m,n}$ from §1.5, used to define the standard Poisson structure $\pi_{m,n}$ on $M_{m,n}$. In §5.2, we will consider this from a Poisson point of view.

In the rest of §3.6 *we prove Proposition* 3.4. The terms r_1^N and r_2^N of the standard r-matrix on $\mathfrak{gl}_N$, see (3.15), are respectively equal to the pushforwards of r^n and r^m under $\mathfrak{gl}_n \cong \mathfrak{l}_n^1 \hookrightarrow \mathfrak{gl}_N$ and $\mathfrak{gl}_m \cong \mathfrak{l}_m^2 \hookrightarrow \mathfrak{gl}_N$. From the above discussion it follows that under the isomorphism $(3.13)^{-1}$,

$$-\chi(r_1^N + r_2^N)|_{U_n^-.P_n} \mapsto -\chi^L(r^m) - \chi^R(r^n).$$

(Recall from §1.5 that $\chi^L(.)$ and $\chi^R(.)$ denote the infinitesimal actions of $\mathfrak{gl}_m$ and $\mathfrak{gl}_n$ on $M_{m,n}$.) Since the maximal length element $w_\circ^n \in S_n$ satisfies $\mathrm{Ad}_{w_\circ^n}(E_{ij}) = E_{n+1-i,n+1-j}$, we have $\mathrm{Ad}_{w_\circ^n}(r^n) = -r^n$, and thus

$$\Psi_*(\pi^{m,n}) = -\chi(r_1^N + r_2^N)|_{U_n^-.P_n}$$

(see (1.8) and (3.14) for the definitions of $\pi^{m,n}$ and Ψ). Now Proposition 3.4 follows from Lemma 3.5. $\square$

3.7. A Poisson homogeneous space of B^-. One can use Proposition 3.4 to identify $M_{m,n}$ with a (full) Poisson homogeneous space of B^-. First, recall the well known fact that B^- is a Poisson algebraic subgroup of GL_N. Since P_n is also a Poisson algebraic subgroup of GL_N (cf. Proposition 3.2 (b)), we get that

$$B^- \cap L_n = B^- \cap P_n$$

is a Poisson algebraic subgroup of GL_N (and thus of $(B^-, \pi^N|_{B^-})$ as well). According to Theorem 1.8, one obtains a natural structure of a Poisson homogeneous space for $(B^-, \pi^N|_{B^-})$ on $B^-/(B^- \cap L_n)$ by equipping it with the Poisson bivectorfield $\nu_*(\pi^N|_{B^-})$ where ν is the projection $\nu : B^- \to B^-/(B^- \cap L_n)$.

Corollary. *The map*

$$M_{m,n} \cong \mathfrak{n}_n^- \ni x \mapsto \exp(x w_\circ^n)(B^- \cap L_n)$$

is a Poisson isomorphism between the matrix affine Poisson space $M_{m,n}$ and the Poisson homogeneous space $(B^-/(B^- \cap L_n), \nu_*(\pi^N|_{B^-}))$ of $(B^-, \pi^N|_{B^-})$.

One can use this corollary instead of Proposition 3.4 in obtaining the results in §3.8, but Proposition 3.4 is conceptually more important since it provides a natural compactification of the matrix affine Poisson space.

Proof. The map Ψ provides a Poisson isomorphism of $M_{m,n}$ with the complete Poisson subvariety $U_n^-.P_n$ of $(GL_N/P_n, -\chi(r^N))$, cf. Proposition 3.4. The latter is a B^--orbit with stabilizer $B^- \cap L_n$ of the base point P_n and thus can be identified with the homogeneous space $B^-/(B^- \cap L_n)$. Under this identification, the Poisson structure $-\chi(r^N)|_{U_n^-.P_n}$ is matched with the Poisson structure $\nu_*(\pi^N|_{B^-})$ because both are push-forwards of the standard Poisson structure π^N on GL_N. The corollary now follows from the fact that $x \mapsto \exp(x w_\circ^n)(B^- \cap L_n)$ is nothing but the map Ψ when we identify $U_n^-.P_n$ and $B^-/(B^- \cap L_n)$. $\square$

3.8. Recall the notation W^V for the set of minimal length representatives for left cosets of a subgroup V of a Weyl group W.

Lemma. *The set $S_N \times S_N^{S_n^1 S_m^2}$ is a complete, irredundant set of representatives for the $(B^+ \times B^-, \overline{L}_n)$ double cosets in $GL_N \times GL_N$.*

Proof. We apply Theorem A.1. For that purpose, let $G = GL_N \times GL_N$, choose $B^+ \times B^-$ and $B^- \times B^+$ to be the positive and negative Borel subgroups of G, respectively, and consider the parabolic subgroup $P = P_n \times P_n$ of G, which contains $B^+ \times B^+$. There is a Levi decomposition $P = L_0 N$ where $L_0 = L_n \times L_n \supset T \times T$ and $N = U_n^+ \times U_n^+$, and we put $L_0 = L_n^\ell L_n^r$ where $L_n^\ell = L_n \times \{I\}$ and $L_n^r = \{I\} \times L_n$. There is an isomorphism $\Theta : L_n^\ell \to L_n^r$ given by $\Theta(a, I) = (I, a)$, and we observe that the simple factors $F \times \{I\}$ of L_n^ℓ (where $F = L_n^1, L_m^2$) satisfy

$$\Theta\big((F \times \{I\}) \cap (B^- \times B^+)\big) = (\{I\} \times F) \cap (B^+ \times B^-).$$

Let $\pi_j : P \to P/N \cong L_0 \to L_n^j$ (for $j = \ell, r$) denote the natural projections, and observe that the subgroup

$$R = \{p \in P \mid \Theta\pi_\ell(p) = \pi_r(p)\}$$

coincides with $\overline{L}_n$. Since the Weyl group of L_n^r, considered as a subgroup of the Weyl group of G, is just $\{1\} \times (S_n^1 S_m^2) \subseteq S_N \times S_N$, Theorem A.1 implies that the set

$$(S_N \times S_N)^{\{1\}\times(S_n^1 S_m^2)} = S_N \times S_N^{S_n^1 S_m^2}$$

is a complete, irredundant set of representatives for the $(B^+ \times B^-, \overline{L}_n)$ double cosets in G. $\square$

3.9. T-orbits of symplectic leaves in $M_{m,n}$. Since the image of $GL_m \times GL_n \cong L_n \subseteq GL_N$ contains the torus T, the action of $GL_m \times GL_n$ on $M_{m,n}$ given in §3.6 incorporates an action of T on $M_{m,n}$. Specifically, if T_m and T_n denote the maximal tori consisting of diagonal matrices in GL_m and GL_n respectively, then $(a, b).x = axb^{-1}$ for $a \in T_m$, $b \in T_n$, $x \in M_{m,n}$.

Theorem. *There are only finitely many T-orbits of symplectic leaves on the matrix affine Poisson space $M_{m,n}$. They are smooth irreducible locally closed subvarieties of $M_{m,n}$, and they are parametrized by $S_N^{\geq(w_\circ^n, w_\circ^m)}$, recall* (3.7)*. The T-orbit of symplectic leaves corresponding to $w \in S_N^{\geq(w_\circ^n, w_\circ^m)}$ is explicitly given by*

$$\mathcal{P}_w = \left\{ x \in M_{m,n} \;\middle|\; \begin{bmatrix} w_\circ^n & 0 \\ x & w_\circ^m \end{bmatrix} \in B^+ w B^+ \right\}. \tag{3.18}$$

As an algebraic variety, $\mathcal{P}_w$ is biregularly isomorphic to $\mathcal{B}_{(w_\circ^n, w_\circ^m), w}$.

Proof. We will make use of the isomorphism Ψ (see (3.14)) of Proposition 3.4 between the matrix affine Poisson space $M_{m,n}$ and the T-stable Poisson subvariety $U_n^-.P_n$ of GL_N/P_n. Recall that $U_n^-.P_n = B^-.P_n$ is open in GL_N/P_n. The isomorphism Ψ is not T-equivariant, but we have

$$\begin{aligned} \Psi((a,b).x) &= \begin{bmatrix} I_n & 0 \\ axb^{-1}w_\circ^n & I_m \end{bmatrix} P_n \\ &= \begin{bmatrix} I_n & 0 \\ axw_\circ^n(w_\circ^n b w_\circ^n)^{-1} & I_m \end{bmatrix} P_n = \begin{bmatrix} w_\circ^n b w_\circ^n & 0 \\ 0 & a \end{bmatrix} .\Psi(x) \end{aligned}$$

for $a \in T_m$, $b \in T_n$, $x \in M_{m,n}$, whence Ψ and Ψ^{-1} preserve T-orbits. Consequently, Ψ maps T-orbits of symplectic leaves in $M_{m,n}$ to T-orbits of symplectic leaves in $U_n^-.P_n$.

Firstly, Theorem 1.10 (applied with $H = T$, as in the proof of Proposition 3.2(e)) implies that the T-orbits of symplectic leaves of $U_n^-.P_n$ are smooth locally closed subvarieties, and so the same is true for $M_{m,n}$. The map (1.12) in the present situation is

$$\Delta : GL_N/P_n \hookrightarrow (GL_N \times GL_N)/\overline{L}_n, \qquad \Delta(gP_n) = (g, g)\overline{L}_n$$

(see the proof of Proposition 3.2(e)). From Theorem 1.10, we also know that the T-orbits of symplectic leaves of $U_n^-.P_n$ are those irreducible components of inverse images of $(B^+ \times B^-)$-orbits on $(GL_N \times GL_N)/\overline{L}_n$ under Δ that lie inside $U_n^-.P_n$.

The set of $(B^+ \times B^-)$-orbits on $(GL_N \times GL_N)/\overline{L}_n$ is in one to one correspondence with the set of $(B^+ \times B^-, \overline{L}_n)$ double cosets in $GL_N \times GL_N$. According to Lemma 3.8, the latter set is parametrized by $S_N \times S_N^{S_n^1 S_m^2}$. Therefore, the $(B^+ \times B^-)$-orbits on $(GL_N \times GL_N)/\overline{L}_n$ are the sets

$$(B^+ \times B^-).(w_1, w_2)\overline{L}_n, \qquad w_1 \in S_N,\ w_2 \in S_N^{S_n^1 S_m^2}, \tag{3.19}$$

and all such sets are distinct. Observe that

$$\Delta^{-1}\big((B^+ \times B^-).(w_1, w_2)\overline{L}_n\big) \subseteq B^-.w_2 P_n.$$

If $w_2 \in S_N^{S_n^1 S_m^2}$ and $w_2 \neq 1$, then $B^- w_2 P_n \cap B^- P_n = \emptyset$ because of the Bruhat lemma. Thus, only the Δ-inverse images of the sets (3.19) with $w_2 = 1$ might intersect $U_n^-.P_n$ nontrivially.

The intersection with $U_n^-.P_n$ of the Δ-inverse image of the set (3.19) with $w_2 = 1$ consists of $uP_n \in GL_N/P_n$ for those $u \in U_n^-$ for which

$$u = b^+ w_1 l u_1^+ = b^- l u_2^+ \tag{3.20}$$

for some $b^\pm \in B^\pm$, $l \in L_n$, $u_i^+ \in U_n^+$. From these equalities, one obtains $l \in L_n \cap B^-$ and $u_2^+ = e$. Conversely, if $u = b^+ w_1 l u_1^+$ for some $b^+ \in B^+$, $l \in L_n \cap B^-$, $u_1^+ \in U_n^+$, we can also write $u = b^- l$ where $b^- = ul^{-1} \in B^-$. Thus,

$$\Delta^{-1}\big((B^+ \times B^-).(w_1, 1)\overline{L}_n\big) \cap U_n^-.P_n = \big(U_n^- \cap B^+ w_1 (L_n \cap B^-) U_n^+\big).P_n.$$

Next, observe that $(L_n \cap B^-)U_n^+ = (w_\circ^n, w_\circ^m)B^+(w_\circ^n, w_\circ^m)$ and that $U_n^- = U^-_{(w_\circ^n, w_\circ^m)}$ (recall (2.2)). Thus, setting $w = w_1(w_\circ^n, w_\circ^m)$ and recalling the notation (2.11), we have

$$\begin{aligned} \Delta^{-1}\big((B^+ \times B^-).(w_1, 1)\overline{L}_n\big) \cap U_n^-.P_n &= \big(U_n^- \cap B^+ w_1 (L \cap B^-) U_n^+\big).P_n \\ &= \Big(U^-_{(w_\circ^n, w_\circ^m)} \cap B^+ w B^+ (w_\circ^n, w_\circ^m)\Big).P_n \\ &= \mathcal{U}_{(w_\circ^n, w_\circ^m), w}.P_n \end{aligned} \tag{3.21}$$

(since $(w_\circ^n, w_\circ^m) \in P_n$). According to Theorem 2.4, $\mathcal{U}_{(w_\circ^n, w_\circ^m), w}$ is irreducible. Therefore, the set (3.21) is a single T-orbit of symplectic leaves of $U_n^-.P_n$. The fact that the T-orbits of symplectic leaves of the matrix affine Poisson space are the sets (3.18) follows by applying the Poisson isomorphism $\Psi : M_{m,n} \to U_n^-.P_n$ to (3.21). Namely, since $U_n^- \cap P_n = \{I\}$, we compute that

$$\begin{aligned} \Psi^{-1}\big(\mathcal{U}_{(w_\circ^n, w_\circ^m), w}.P_n\big) &= \left\{ x \in M_{m,n} \;\middle|\; \begin{bmatrix} I_n & 0 \\ x w_\circ^n & I_m \end{bmatrix} P_n \in \big(U_n^- \cap B^+ w B^+ (w_\circ^n, w_\circ^m)\big).P_n \right\} \\ &= \left\{ x \in M_{m,n} \;\middle|\; \begin{bmatrix} I_n & 0 \\ x w_\circ^n & I_m \end{bmatrix} \in U_n^- \cap B^+ w B^+ (w_\circ^n, w_\circ^m) \right\} \\ &= \left\{ x \in M_{m,n} \;\middle|\; \begin{bmatrix} I_n & 0 \\ x w_\circ^n & I_m \end{bmatrix} \begin{bmatrix} w_\circ^n & 0 \\ 0 & w_\circ^m \end{bmatrix} \in B^+ w B^+ \right\} = \mathcal{P}_w. \end{aligned} \tag{3.22}$$

Moreover, $\mathcal{P}_w \cong \mathcal{U}_{(w_\circ^n, w_\circ^m),w} \cong \mathcal{B}_{(w_\circ^n, w_\circ^m),w}$ by Theorem 2.4. Irreducibility thus follows from Proposition 2.2 of Deodhar. Finally, $\mathcal{P}_w$ is nonempty if and only if $\mathcal{B}_{(w_\circ^n, w_\circ^m),w}$ is nonempty, which occurs precisely when $w \geq (w_\circ^n, w_\circ^m)$, by Proposition 2.2. □

3.10. Since $\begin{bmatrix} w_\circ^n & 0 \\ M_{m,n} & w_\circ^m \end{bmatrix} \subseteq B^-(w_\circ^n, w_\circ^m)B^-$, the set $\mathcal{P}_w$ described in (3.18) can be written as the inverse image of $B^-(w_\circ^n, w_\circ^m)B^- \cap B^+wB^+$ under the map $\Omega : M_{m,n} \to GL_N$ given by $x \mapsto \begin{bmatrix} w_\circ^n & 0 \\ x & w_\circ^m \end{bmatrix}$. It is known that the T-orbits of symplectic leaves in GL_N coincide with the double Bruhat cells $B^-yB^- \cap B^+wB^+$ (e.g., this follows from the results of [**15**, Appendix A]). The following statement is thus an immediate consequence: *The T-orbits of symplectic leaves in $M_{m,n}$ are precisely the nonempty Ω-inverse images of the T-orbits of symplectic leaves in GL_N.* The lifting Ω of Ψ is neither T-equivariant nor Poisson, and because of this one cannot approach Theorem 3.9 directly using Ω.

3.11. Alternative descriptions of $S_N^{\geq (w_\circ^n, w_\circ^m)}$. It is convenient to describe the Bruhat order on S_N in terms of relations between sets of integers, as follows. First, if I and J are t-element subsets of $\{1, \ldots, N\}$, list their elements in ascending order, say

$$I = \{i_1 < i_2 < \cdots < i_t\} \qquad\qquad J = \{j_1 < j_2 < \cdots < j_t\},$$

and then define $I \leq J$ if and only if $i_l \leq j_l$ for $l = 1, \ldots, t$. For $y, z \in S_N$, we have

$$y \leq z \iff \{y(1), \ldots, y(p)\} \leq \{z(1), \ldots, z(p)\} \text{ for } p = 1, \ldots, N \tag{3.23}$$

(e.g., [**11**, Exercise 8, p. 175]). For I and J as above, it is clear that $I \leq J$ if and only if $w_\circ^N(I) \geq w_\circ^N(J)$. Hence,

$$y \leq z \iff w_\circ^N y \geq w_\circ^N z$$

for any $y, z \in S_N$.

In particular, a permutation $w \in S_N$ satisfies $w \geq (w_\circ^n, w_\circ^m)$ if and only if $w_\circ^N w \leq w_\circ^N (w_\circ^n, w_\circ^m) = w_\circ^{m,n}$ (recall (3.6)). Thus,

$$S_N^{\geq (w_\circ^n, w_\circ^m)} = w_\circ^N S_N^{\leq w_\circ^{m,n}}. \tag{3.24}$$

The following description of $S_N^{\leq w_\circ^{m,n}}$ is known in the case $m = n$; we thank Jon McCammond for bringing the result to our attention. This result also appears in [**22**, Proposition 1.3]; we provide a proof for the reader's convenience. Recall (3.8) for the notation $S_N^{[-n,m]}$.

3.12. Lemma. $S_N^{\leq w_\circ^{m,n}} = S_N^{[-n,m]}$ *and*

$$S_N^{\geq (w_\circ^n, w_\circ^m)} = \{y \in S_N \mid n \leq y(i) + i - 1 \leq m + 2n \textit{ for all } i = 1, \ldots, N\}.$$

Proof. Since the second statement follows immediately from the first via (3.24), we need only prove the first statement.

First, consider $s \in S_N^{\leq w_\circ^{m,n}}$ and $j \in \{1, \dots, N\}$. If $j \leq n$, then

$$s(\{1, \dots, j\}) \leq w_\circ^{m,n}(\{1, \dots, j\}) = \{m+1, \dots, m+j\}, \tag{3.25}$$

whence $s(j) \leq m+j$. On the other hand, if $j > n$, then

$$s(\{1, \dots, j-1\}) \leq w_\circ^{m,n}(\{1, \dots, j-1\}) = \{1, \dots, j-1-n, m+1, \dots, N\},$$

from which we see that $\{1, \dots, j-1-n\} \subseteq s(\{1, \dots, j-1\})$, and consequently $s(j) \geq j-n$. We automatically have $s(j) \geq j-n$ when $j \leq n$, and $s(j) \leq m+j$ when $j > n$. Thus, $s \in S_N^{[-n,m]}$.

Conversely, let $s \in S_N^{[-n,m]}$ and $j \in \{1, \dots, N\}$. If $j \leq n$, then $s(i) \leq i+m \leq j+m$ for $i \leq j$, whence $s(\{1, \dots, j\}) \subseteq \{1, \dots, j+m\}$, and consequently (3.25) holds. On the other hand, if $j > n$, then $s(i) \geq i-n > j-n$ for $i > j$, whence $\{1, \dots, j-n\} \subseteq s(\{1, \dots, j\})$, and consequently

$$s(\{1, \dots, j\}) \leq \{1, \dots, j-n, m+1, \dots, N\} = w_\circ^{m,n}(\{1, \dots, j\}).$$

Therefore $s \leq w_\circ^{m,n}$. $\square$

In the last result of this section, we describe the Zariski closures of the T-orbits of symplectic leaves of the matrix affine Poisson space.

3.13. Theorem. *The Zariski closures of the T-orbits of symplectic leaves of the matrix affine Poisson space $M_{m,n}$, see Theorem* 3.9, *are given by*

$$\overline{\mathcal{P}}_w = \bigsqcup_{\substack{z \in S_N \\ (w_\circ^n, w_\circ^m) \leq z \leq w}} \mathcal{P}_z = \left\{ x \in M_{m,n} \;\middle|\; \begin{bmatrix} w_\circ^n & 0 \\ x & w_\circ^m \end{bmatrix} \in \overline{B^+ w B^+} \right\}. \tag{3.26}$$

Consequently, the inclusions between the Zariski closures of the T-orbits of symplectic leaves (3.18) *on $M_{m,n}$ correspond to the Bruhat order on $S_N^{\geq (w_\circ^n, w_\circ^m)}$.*

Proof. As noted in the proof of Theorem 3.9, $U_n^- = U_{(w_\circ^n, w_\circ^m)}^-$. Since $(w_\circ^n, w_\circ^m) \in P_n$, the isomorphism between U_n^- and $U_n^-.P_n$ (recall §3.3) yields a corresponding isomorphism between $U_n^-(w_\circ^n, w_\circ^m)$ and $U_n^-.P_n$.

Now let $w \in S_N^{\geq (w_\circ^n, w_\circ^m)}$. According to (3.22), we have

$$\mathcal{P}_w = \Psi^{-1}(\mathcal{U}_{(w_\circ^n, w_\circ^m), w}.P_n).$$

Invoking the isomorphisms $\Psi : M_{m,n} \to U_n^-.P_n$ and $U_n^-(w_\circ^n, w_\circ^m) \to U_n^-.P_n$, we obtain

$$\begin{aligned} \overline{\mathcal{P}}_w &= \Psi^{-1}\big(\mathrm{Cl}_{U_n^-.P_n}(\mathcal{U}_{(w_\circ^n, w_\circ^m), w}.P_n)\big) \\ &= \Psi^{-1}\big(\big[\mathrm{Cl}_{U_n^-(w_\circ^n, w_\circ^m)}(\mathcal{U}_{(w_\circ^n, w_\circ^m), w})\big].P_n\big) \\ &= \Psi^{-1}\big(\overline{\mathcal{U}}_{(w_\circ^n, w_\circ^m), w}.P_n\big). \end{aligned} \tag{3.27}$$

By Theorem 2.5(a),

$$\overline{\mathcal{U}}_{(w_\circ^n, w_\circ^m),w} = \bigsqcup_{\substack{z\in S_N \\ (w_\circ^n, w_\circ^m)\le z\le w}} \mathcal{U}_{(w_\circ^n, w_\circ^m),z}. \tag{3.28}$$

The first equality of (3.26) follows from (3.27) and (3.28). Since $\overline{B^+wB^+}$ is the disjoint union of the cells B^+zB^+ for $z \le w$, we have

$$\left\{ x \in M_{m,n} \;\middle|\; \begin{bmatrix} w_\circ^n & 0 \\ x & w_\circ^m \end{bmatrix} \in \overline{B^+wB^+} \right\} = \bigsqcup_{\substack{z\in S_N \\ z\le w}} \mathcal{P}_z\,,$$

which yields the second equality of (3.26) because $\mathcal{P}_z$ is empty when $z \not\ge (w_\circ^n, w_\circ^m)$ (recall the end of the proof of Theorem 3.9). □

4. Computational description of T-orbits of symplectic leaves

As in the previous section, we fix positive integers m, n, and $N = m + n$. We derive a description of the T-orbits $\mathcal{P}_w$ of symplectic leaves in $M_{m,n}$ in terms of ranks of rectangular submatrices.

4.1. Descriptions of B^+wB^+ and B^-wB^-. In order to give computational descriptions of the sets $\mathcal{P}_w$ in (3.18) and $\overline{\mathcal{P}}_w$ in (3.26), we rely on the computational descriptions of B^-wB^+ and its closure given by Fulton in [**10**]; these descriptions are easily modified to deal with B^+wB^+. Since we will also make use of the corresponding descriptions in $M_{m,n}$ and $M_{n,m}$, we give a general version of these results.

Let $1 \le a \le b \le k$ and $1 \le c \le d \le l$. For $x \in M_{k,l}$, we write $x_{[a,\dots,b;c,\dots,d]}$ to denote the submatrix of x with rows $a, \dots, b$ and columns $c, \dots, d$. Recall from §3.1 that we use $B_k^\pm$ and $B_l^\pm$ to denote the standard Borel subgroups of GL_k and GL_l. The closures in the proposition below denote Zariski closures in the matrix variety $M_{k,l}$.

Proposition. [Fulton] *Let k and l be positive integers and $x, w \in M_{k,l}$.*

(a) $x \in B_k^+wB_l^+$ *if and only if* $\operatorname{rank}(x_{[p,\dots,k;1,\dots,q]}) = \operatorname{rank}(w_{[p,\dots,k;1,\dots,q]})$ *for all* $p = 1, \dots, k$ *and* $q = 1, \dots, l$.

(b) $x \in \overline{B_k^+wB_l^+}$ *if and only if* $\operatorname{rank}(x_{[p,\dots,k;1,\dots,q]}) \le \operatorname{rank}(w_{[p,\dots,k;1,\dots,q]})$ *for all* $p = 1, \dots, k$ *and* $q = 1, \dots, l$.

(c) $x \in B_k^-wB_l^-$ *if and only if* $\operatorname{rank}(x_{[1,\dots,p;q,\dots,l]}) = \operatorname{rank}(w_{[1,\dots,p;q,\dots,l]})$ *for all* $p = 1, \dots, k$ *and* $q = 1, \dots, l$.

(d) $x \in \overline{B_k^-wB_l^-}$ *if and only if* $\operatorname{rank}(x_{[1,\dots,p;q,\dots,l]}) \le \operatorname{rank}(w_{[1,\dots,p;q,\dots,l]})$ *for all* $p = 1, \dots, k$ *and* $q = 1, \dots, l$.

Proof. (a) Observe that $x \in B_k^+wB_l^+$ if and only if $w_\circ^k x \in B_k^- w_\circ^k w B_l^+$. The result of [**10**, p. 390, second display] shows that $w_\circ^k x \in B_k^- w_\circ^k w B_l^+$ if and only if

$$\operatorname{rank}\big((w_\circ^k x)_{[1,\dots,p;1,\dots,q]}\big) = \operatorname{rank}\big((w_\circ^k w)_{[1,\dots,p;1,\dots,q]}\big)$$

for $p = 1, \dots, k$ and $q = 1, \dots, l$. Part (a) follows.

(b) This follows from [**10**, Proposition 3.3(a)] in the same manner as (a).

(c) and (d) follow similarly. □

4.2. Description of $\mathcal{P}_w$. Recall the notation $\mathcal{P}_w$ from (3.18) for T-orbits of symplectic leaves in $M_{m,n}$.

It will be convenient to write some matrices $w \in M_N$ in the following block form:

$$w = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}, \qquad \begin{pmatrix} w_{11} \in M_n & w_{12} \in M_{n,m} \\ w_{21} \in M_{m,n} & w_{22} \in M_m \end{pmatrix}. \tag{4.1}$$

Theorem. *Let $x \in M_{m,n}$ and $w \in S_N^{\geq (w_\circ^n, w_\circ^m)}$, and write $w = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$ as in* (4.1). *Then $x \in \mathcal{P}_w$ if and only if the following four conditions hold:*

(a) $\operatorname{rank}(x_{[p,\dots,m;1,\dots,q]}) = \operatorname{rank}\big((w_{21})_{[p,\dots,m;1,\dots,q]}\big)$ *for $p = 1, \dots, m$, $q = 1, \dots, n$.*

(b) $\operatorname{rank}(x_{[1,\dots,p;q,\dots,n]}) = \operatorname{rank}\big((w_\circ^m w_{12}^{\mathrm{tr}} w_\circ^n)_{[1,\dots,p;q,\dots,n]}\big)$ *for $p = 1, \dots, m$, $q = 1, \dots, n$.*

(c) $\operatorname{rank}(x_{[1,\dots,m;p,\dots,q]}) = q + 1 - p - \operatorname{rank}\big((w_\circ^n w_{11})_{[p,\dots,n;p,\dots,q]}\big)$ *for $2 \leq p \leq q \leq n$.*

(d) $\operatorname{rank}(x_{[p,\dots,q;1,\dots,n]}) = q + 1 - p - \operatorname{rank}\big((w_{22} w_\circ^m)_{[p,\dots,q;1,\dots,q]}\big)$ *for $1 \leq p \leq q \leq m-1$.*

Furthermore, $x \in \overline{\mathcal{P}_w}$ if and only if conditions (a)–(d) *hold with each rank equality replaced by $\leq$.*

Proof. We shall repeatedly use the following easy observation: whenever a partial permutation matrix u is partitioned into blocks, the rank of u equals the sum of the ranks of the blocks.

Set $\overline{x} = \begin{bmatrix} w_\circ^n & 0 \\ x & w_\circ^m \end{bmatrix}$. In view of Theorem 3.9 and Proposition 4.1(a), we have $x \in \mathcal{P}_w$ if and only if

$$\operatorname{rank}(\overline{x}_{[r,\dots,N;1,\dots,s]}) = \operatorname{rank}(w_{[r,\dots,N;1,\dots,s]}) \tag{4.2}$$

for all $r, s = 1, \dots, N$. Observe that (4.2) holds automatically if $r = 1$ (in which case both sides equal s), or if $s = N$ (in which case both sides equal $N + 1 - r$). We shall consider (4.2) in a number of separate cases.

Case 1: $s \leq n < r$. Set $p = r - n$ and $q = s$, and note that

$$\overline{x}_{[r,\dots,N;1,\dots,s]} = x_{[p,\dots,m;1,\dots,q]} \qquad w_{[r,\dots,N;1,\dots,s]} = (w_{21})_{[p,\dots,m;1,\dots,q]}.$$

Hence, (4.2) holds for $s \leq n < r$ if and only if (a) holds.

Case 2: $r, s \leq n$ and $r + s \leq n + 1$. Since $r \leq n + 1 - s$, we have

$$\overline{x}_{[r,\dots,N;1,\dots,s]} = \begin{bmatrix} 0 \\ w_\circ^s \\ x_{[1,\dots,m;1,\dots,s]} \end{bmatrix}$$

(where the 0 block is present only if $r < n + 1 - s$). It follows that $\overline{x}_{[r,\dots,N;1,\dots,s]}$ has rank s in this case. Since $w \in S_N^{\geq (w_\circ^n, w_\circ^m)}$, Lemma 3.12 says that $w(j) \geq n + 1 - j$ for all j. For $j \leq s$, we obtain $w(j) \geq n + 1 - s \geq r$, and so $w_{[r,\dots,N;1,\dots,s]}$ has a 1 in each of its s columns. Hence, $w_{[r,\dots,N;1,\dots,s]}$ has rank s, and thus (4.2) always holds in the present case, independent of x.

Case 3: $r, s \le n$ and $r+s > n+1$. Set $p = n+2-r$ and $q = s$, so that $2 \le p \le q$. We have

$$\overline{x}_{[r,\dots,N;1,\dots,s]} = \begin{bmatrix} w_\circ^{p-1} & 0 \\ x_{[1,\dots,m;1,\dots,p-1]} & x_{[1,\dots,m;p,\dots,q]} \end{bmatrix},$$

and so $\operatorname{rank}(\overline{x}_{[r,\dots,N;1,\dots,s]}) = p-1+\operatorname{rank}(x_{[1,\dots,m;p,\dots,q]})$. For $j \le p-1$, we have $w(j) \ge n+1-j \ge n+2-p = r$, which implies that $w_{[r,\dots,N;1,\dots,p-1]}$ has rank $p-1$. Hence,

$$\begin{aligned} \operatorname{rank}(w_{[r,\dots,N;1,\dots,s]}) &= p-1+\operatorname{rank}(w_{[r,\dots,N;p,\dots,q]}) \\ &= p-1+q+1-p-\operatorname{rank}(w_{[1,\dots,r-1;p,\dots,q]}) \\ &= q-\operatorname{rank}\big((w_{11})_{[1,\dots,n+1-p;p,\dots,q]}\big) \\ &= q-\operatorname{rank}\big((w_\circ^n w_{11})_{[p,\dots,n;p,\dots,q]}\big). \end{aligned}$$

Therefore, (4.2) holds for $r, s \le n$ and $r+s > n+1$ if and only if (c) holds.

Case 4: $r, s > n$ and $r+s > m+2n$. Set $t = N+1-r$. Then $s \ge n+t$, and so

$$\overline{x}_{[r,\dots,N;1,\dots,s]} = \begin{bmatrix} x_{[r-n,\dots,m;1,\dots,n]} & w_\circ^t & 0 \end{bmatrix}$$

(where the 0 block is present only if $s > n+t$). Hence, $\overline{x}_{[r,\dots,N;1,\dots,s]}$ has rank t in this case. Lemma 3.12 says that $w(j) \le m+2n+1-j$ for all j, and so for $j \ge s+1$, we get $w(j) \le m+2n-s \le r-1$. Consequently, the nonzero entries in rows $r, \dots, N$ of w must occur in columns $1, \dots, s$, from which we obtain $\operatorname{rank}(w_{[r,\dots,N;1,\dots,s]}) = N+1-r = t$. Therefore (4.2) always holds in the present case.

Case 5: $r, s > n$ and $r+s \le m+2n$. Set $p = r-n$ and $q = N-s$, so that $1 \le p \le q \le m-1$. Now

$$\overline{x}_{[r,\dots,N;1,\dots,s]} = \begin{bmatrix} x_{[p,\dots,q;1,\dots,n]} & 0 \\ x_{[q+1,\dots,m;1,\dots,n]} & w_\circ^{s-n} \end{bmatrix},$$

and so $\operatorname{rank}(\overline{x}_{[r,\dots,N;1,\dots,s]}) = s-n+\operatorname{rank}(x_{[p,\dots,q;1,\dots,n]})$. As in Case 4, for $j \ge s+1$, we have $w(j) \le m+2n-s = n+q$, whence $w_{[n+q+1,\dots,N;1,\dots,s]}$ has rank $(N+1)-(n+q+1) = m-q = s-n$. Hence,

$$\begin{aligned} \operatorname{rank}(w_{[r,\dots,N;1,\dots,s]}) &= s-n+\operatorname{rank}(w_{[r,\dots,n+q;1,\dots,s]}) \\ &= s-n+n+q+1-r-\operatorname{rank}(w_{[r,\dots,n+q;s+1,\dots,N]}) \\ &= s-n+q+1-p-\operatorname{rank}\big((w_{22})_{[p,\dots,q;m+1-q,\dots,m]}\big) \\ &= s-n+q+1-p-\operatorname{rank}\big((w_{22}w_\circ^m)_{[p,\dots,q;1,\dots,q]}\big) \end{aligned}$$

Therefore, (4.2) holds for $r, s > n$ and $r+s \le m+2n$ if and only if (d) holds.

Case 6: $2 \le r \le n+1$ and $n \le s < N$. Set $p = N-s$ and $q = n+2-r$, so that $1 \le p \le m$ and $1 \le q \le n$. We have

$$\overline{x}_{[r,\dots,N;1,\dots,s]} = \begin{bmatrix} w_\circ^{q-1} & 0 & 0 \\ x_{[1,\dots,p;1,\dots,q-1]} & x_{[1,\dots,p;q,\dots,n]} & 0 \\ x_{[p+1,\dots,m;1,\dots,q-1]} & x_{[p+1,\dots,m;q,\dots,n]} & w_\circ^{m-p} \end{bmatrix}$$

(where the left column, respectively bottom row, is present only if $q > 1$, respectively $p < m$), and so $\overline{x}_{[r,\dots,N;1,\dots,s]}$ has rank $q-1+m-p+\operatorname{rank}(x_{[1,\dots,p;q,\dots,n]})$. On the other hand,

$$\begin{aligned}\operatorname{rank}(w_{[r,\dots,N;1,\dots,s]}) &= s - \operatorname{rank}(w_{[1,\dots,r-1;1,\dots,s]}) \\ &= s-(r-1)+\operatorname{rank}(w_{[1,\dots,r-1;s+1,\dots,N]}) \\ &= s+1-r+\operatorname{rank}\big((w_{12})_{[1,\dots,n+1-q;m+1-p,\dots,m]} \\ &= q-1+m-p+\operatorname{rank}\big((w_\circ^m w_{12}^{\mathrm{tr}} w_\circ^n)_{[1,\dots,p;q,\dots,n]}\big).\end{aligned}$$

Thus, (4.2) holds for $2 \le r \le n+1$ and $n \le s < N$ if and only if (b) holds.

Therefore (4.2) holds for $r, s = 1, \dots, N$ if and only if (a), (b), (c), (d) all hold, and we have established the desired characterization of $\mathcal{P}_w$. The characterization of $\overline{\mathcal{P}}_w$ follows from the information in Cases 1–6 together with Theorem 3.13 and Proposition 4.1(b). $\square$

4.3. Let $x \in M_{m,n}$ and $w = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \in S_N^{\ge(w_\circ^n, w_\circ^m)}$ as in Theorem 4.2. According to Proposition 4.1(a)(c), the first two conditions of the theorem are equivalent to the conditions $x \in B_m^+ w_{21} B_n^+$ and $x \in B_m^- w_\circ^m w_{12}^{\mathrm{tr}} w_\circ^n B_n^-$, respectively. The corollary below follows immediately. As we shall see in Example 4.5, the inclusion (4.3) is typically proper.

Corollary. *Let $w \in S_N^{\ge(w_\circ^n, w_\circ^m)}$, and write $w = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$ as in* (4.1)*. Then*

$$\mathcal{P}_w \subseteq B_m^+ w_{21} B_n^+ \cap B_m^- w_\circ^m w_{12}^{\mathrm{tr}} w_\circ^n B_n^-. \qquad \square \tag{4.3}$$

4.4. Example. Let $m = n$ and $u, v \in S_n$, and set $w = \begin{bmatrix} 0 & u \\ v & 0 \end{bmatrix} \in S_N$. Via Lemma 3.12, it is easily checked that $w \in S_{2n}^{\ge(w_\circ^n, w_\circ^n)}$. Let us use Theorem 4.2 to compute $\mathcal{P}_w$ in this case.

Let $x \in M_n$. As discussed in §4.3, conditions (a) and (b) of the theorem require that $x \in B_n^+ v B_n^+ \cap B_n^- w_\circ^n u^{\mathrm{tr}} w_\circ^n B_n^-$. In particular, x must be invertible. Conditions (c) and (d) of the theorem require

$$\begin{aligned}\operatorname{rank}(x_{[1,\dots,n;p,\dots,q]}) &= q+1-p && (2 \le p \le q \le n) \\ \operatorname{rank}(x_{[p,\dots,q;1,\dots,n]}) &= q+1-p && (1 \le p \le q < n).\end{aligned}$$

These conditions hold automatically for $x \in GL_n$. Therefore, we conclude that

$$\mathcal{P}_{\left[\begin{smallmatrix} 0 & u \\ v & 0 \end{smallmatrix}\right]} = B_n^+ v B_n^+ \cap B_n^- w_\circ^n u^{\mathrm{tr}} w_\circ^n B_n^-,$$

a double Bruhat cell in GL_n. This recovers the previously known description of T-orbits of symplectic leaves in GL_n (cf. [**15**, Appendix A] for the parallel case of SL_n).

4.5. Example. We give an example to show that conditions (c) and (d) of Theorem 4.2 are typically not redundant, i.e., (4.3) is typically a proper inclusion.

Take $m = n = 3$, and consider the permutation matrix

$$w = \begin{bmatrix} 0&0&0&0&0&1 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&1&0&0 \\ 1&0&0&0&0&0 \end{bmatrix} \in M_6.$$

Write $w = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$ as in (4.1), and note that $w_\circ^3 w_{12}^{\mathrm{tr}} w_\circ^3 = w_{12}$. For $x \in M_3$, conditions (a) and (b) of Theorem 4.2 require that

$$\operatorname{rank}(x_{[p,\dots,3;1,\dots,q]}) = \operatorname{rank}\big((w_{21})_{[p,\dots,3;1,\dots,q]}\big) = 1$$
$$\operatorname{rank}(x_{[1,\dots,p;q,\dots,3]}) = \operatorname{rank}\big((w_{12})_{[1,\dots,p;q,\dots,3]}\big) = 1$$

for $p, q = 1, 2, 3$. These requirements boil down to $x_{31}, x_{13} \neq 0$ and $\operatorname{rank}(x) = 1$. It follows that $x_{11}, x_{33} \neq 0$. Consequently,

$$B_3^+ w_{21} B_3^+ \cap B_3^- w_{12} B_3^- = \left\{ x \in \begin{bmatrix} \mathbb{C}^\times & \mathbb{C} & \mathbb{C}^\times \\ \mathbb{C} & \mathbb{C} & \mathbb{C} \\ \mathbb{C}^\times & \mathbb{C} & \mathbb{C}^\times \end{bmatrix} \;\middle|\; \operatorname{rank}(x) = 1 \right\}.$$

Next, observe that $w_\circ^3 w_{11} = \begin{bmatrix} 0&0&1 \\ 0&1&0 \\ 0&0&0 \end{bmatrix}$ and $w_{22} w_\circ^3 = \begin{bmatrix} 0&1&0 \\ 0&0&1 \\ 0&0&0 \end{bmatrix}$. Condition (c) of Theorem 4.2 requires that

$$\operatorname{rank}(x_{[1,2,3;2]}) = 1 - \operatorname{rank}\big((w_\circ^3 w_{11})_{[2,3;2]}\big) = 0$$
$$\operatorname{rank}(x_{[1,2,3;2,3]}) = 2 - \operatorname{rank}\big((w_\circ^3 w_{11})_{[2,3;2,3]}\big) = 1$$
$$\operatorname{rank}(x_{[1,2,3;3]}) = 1 - \operatorname{rank}\big((w_\circ^3 w_{11})_{[3;3]}\big) = 1.$$

The first equation means that the middle column of x must be zero; the other equations follow from the previous conditions. Finally, condition (d) of Theorem 4.2 requires that

$$\operatorname{rank}(x_{[1;1,2,3]}) = 1 - \operatorname{rank}\big((w_{22} w_\circ^3)_{[1;1]}\big) = 1$$
$$\operatorname{rank}(x_{[1,2;1,2,3]}) = 2 - \operatorname{rank}\big((w_{22} w_\circ^3)_{[1,2;1,2]}\big) = 1$$
$$\operatorname{rank}(x_{[2;1,2,3]}) = 1 - \operatorname{rank}\big((w_{22} w_\circ^3)_{[2;1,2]}\big) = 1.$$

The last equation means that the middle row of x must be nonzero, while the other equations follow from the previous conditions.

We conclude that

$$\mathcal{P}_w = \left\{ x \in \begin{bmatrix} \mathbb{C}^\times & 0 & \mathbb{C}^\times \\ \mathbb{C}^\times & 0 & \mathbb{C}^\times \\ \mathbb{C}^\times & 0 & \mathbb{C}^\times \end{bmatrix} \;\middle|\; \operatorname{rank}(x) = 1 \right\},$$

which is properly contained in $B_3^+ w_{21} B_3^+ \cap B_3^- w_{12} B_3^-$. In fact, one can show that the latter intersection is a disjoint union of four T-orbits of symplectic leaves, corresponding to matrices of rank 1 whose middle row or middle column is zero or nonzero.

5. A second approach to $M_{m,n}$ by rank stratification

As above, fix positive integers m, n, and $N = m + n$. We investigate the T-orbits of symplectic leaves of matrices with a given rank t, which leads to a new description of orbits of leaves, quite different from Theorem 3.9.

5.1. The set of rank t matrices. Fix a nonnegative integer $t \le \min\{m, n\}$, and set

$$\mathcal{O}_t^{m,n} = \{x \in M_{m,n} \mid \operatorname{rank}(x) = t\}. \tag{5.1}$$

If $x \in \mathcal{O}_t^{m,n}$, then $x \in \mathcal{P}_w$ for some $w \in S_N^{\geq(w_\circ^n, w_\circ^m)}$ (Theorem 3.9), and Corollary 4.3 shows that $\mathcal{P}_w \subseteq B_m^+ w_{21} B_n^+$ for some partial permutation matrix $w_{21} \in M_{m,n}$. Clearly $\operatorname{rank}(w_{21}) = \operatorname{rank}(x) = t$, whence $B_m^+ w_{21} B_n^+ \subseteq \mathcal{O}_t^{m,n}$, and so $x \in \mathcal{P}_w \subseteq \mathcal{O}_t^{m,n}$. Therefore, $\mathcal{O}_t^{m,n}$ is a union of T-orbits of symplectic leaves. Note that when $w \in S_N$ is written in the form $\left[\begin{smallmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{smallmatrix}\right]$ as in (4.1), we have

$$\operatorname{rank}(w_{21}) = |w^{-1}(\{n+1, \dots, N\}) \cap \{1, \dots, n\}|.$$

Hence, we define

$$S_N^{\geq(w_\circ^n, w_\circ^m)}[t] = \bigl\{w \in S_N^{\geq(w_\circ^n, w_\circ^m)} \bigm| |w^{-1}(\{n+1, \dots, N\}) \cap \{1, \dots, n\}| = t\bigr\}, \tag{5.2}$$

so that we can state

$$\mathcal{O}_t^{m,n} = \bigsqcup_{w \in S_N^{\geq(w_\circ^n, w_\circ^m)}[t]} \mathcal{P}_w\,. \tag{5.3}$$

This statement invites us to view the matrix affine Poisson space $M_{m,n}$ as stratified by matrix rank, and to analyze the T-orbits $\mathcal{P}_w$ of symplectic leaves with special attention to their matrix ranks. This analysis, carried out in the present section, leads to new descriptions of the orbits $\mathcal{P}_w$.

5.2. $\mathcal{O}_t^{m,n}$ as a Poisson homogeneous space. Under the natural action of the group $G = GL_m \times GL_n$ on $M_{m,n}$, given by $(a, b).x = axb^{-1}$, the set $\mathcal{O}_t^{m,n}$ is the G-orbit of the matrix

$$I_t^{m,n} = \begin{bmatrix} I_t & 0_{t,n-t} \\ 0_{m-t,t} & 0_{m-t,n-t} \end{bmatrix}. \tag{5.4}$$

Thus, $\mathcal{O}_t^{m,n}$ is a homogeneous G-space. However, the action of G on $M_{m,n}$ is not a Poisson action for the standard Poisson structure on G. To remedy this, we take

$$G = GL_m \times GL_n^\bullet = (GL_m, \pi^m) \times (GL_n, -\pi^n),$$

where π^m and π^n denote the standard Poisson structures on GL_m and GL_n (recall §1.4). With this change, the action $G \times M_{m,n} \to M_{m,n}$ is a Poisson action, and therefore $\mathcal{O}_t^{m,n}$ is a Poisson homogeneous G-space.

Since the Poisson bivectorfield $\pi^{m,n}$ vanishes at $I_t^{m,n}$, Theorem 1.8(a) shows that the Poisson homogeneous G-space $\mathcal{O}_t^{m,n}$ is isomorphic to $(G/Q_t^{m,n},\, \pi^{G/Q_t^{m,n}})$, where

$$Q_t^{m,n} = \mathrm{Stab}_G(I_t^{m,n}) \\ = \left\{ \left(\begin{bmatrix} a & b \\ 0 & d_1 \end{bmatrix}, \begin{bmatrix} a & 0 \\ c & d_2 \end{bmatrix} \right) \;\middle|\; \begin{matrix} a \in GL_t,\ b \in M_{t,m-t},\ c \in M_{n-t,t}, \\ d_1 \in GL_{m-t},\ d_2 \in GL_{n-t} \end{matrix} \right\}. \tag{5.5}$$

(Note that $\mathfrak{q}_t^{m,n} = \mathrm{Lie}(Q_t^{m,n})$ can be described in the same manner as (5.5).) Thus, we can apply Theorem 1.10 to compute the T-orbits of symplectic leaves within $\mathcal{O}_t^{m,n}$. We sketch the steps in this subsection, leaving details to the reader. When we compare the results with those of Section 3 (see Theorem 5.11), we will obtain an independent derivation, as a corollary of Theorem 3.9.

Write $\mathfrak{g} = \mathfrak{gl}_m \times \mathfrak{gl}_n^\bullet$ for the Lie bialgebra of G. Because of the appearance of $\mathfrak{gl}_n^\bullet$ in the second factor of $\mathfrak{g}$, we use the negative of the Killing form $\langle -, - \rangle$ on that factor. Thus, the bilinear form to be used in $\mathfrak{g}$ is given by

$$\langle (x_1, x_2), (y_1, y_2) \rangle = \langle x_1, y_1 \rangle - \langle x_2, y_2 \rangle,$$

and the corresponding form on the double $D(\mathfrak{g}) \cong \mathfrak{g} \oplus \mathfrak{g}$ (recall (1.6)) is given by

$$\langle (x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \rangle = \langle x_1, y_1 \rangle - \langle x_2, y_2 \rangle - \langle x_3, y_3 \rangle + \langle x_4, y_4 \rangle.$$

The duals appearing in the Manin triples $(D(G), \Delta(G), F)$ and $(D(\mathfrak{g}), \Delta(\mathfrak{g}), \mathfrak{g}^*)$ (recall (1.4) and (1.5)) take the forms

$$F = \{(a, b, a^{-1}, b^{-1}) \mid a \in T_m,\, b \in T_n\}(N_m^+ \times N_n^+ \times N_m^- \times N_n^-) \tag{5.6}$$

and

$$\mathfrak{g}^* = \{(x, y, -x, -y) \mid x \in \mathfrak{h}_m,\, y \in \mathfrak{h}_n\} + (\mathfrak{n}_m^+ \times \mathfrak{n}_n^+ \times \mathfrak{n}_m^- \times \mathfrak{n}_n^-), \tag{5.7}$$

where we have written $N_l^\pm$ for the unipotent radical of $B_l^\pm$ to avoid conflict with the notation (3.3).

In view of Theorem 1.8(b), the Drinfeld Langrangian subalgebra corresponding to the base point $I_t^{m,n}$ in the present situation has the form $\mathfrak{l}_t^{m,n} = \mathrm{diag}(\mathfrak{q}_t^{m,n}) \oplus (\mathfrak{q}_t^{m,n})^\perp$. As is easily computed, $\mathfrak{l}_t^{m,n}$ consists of those 4-tuples

$$\left(\begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix}, \begin{bmatrix} a_2 & 0 \\ c_2 & d_2 \end{bmatrix}, \begin{bmatrix} a_3 & b_3 \\ 0 & d_3 \end{bmatrix}, \begin{bmatrix} a_4 & 0 \\ c_4 & d_4 \end{bmatrix} \right) \\ \in \begin{bmatrix} \mathfrak{gl}_t & M_{t,m-t} \\ 0 & \mathfrak{gl}_{m-t} \end{bmatrix} \times \begin{bmatrix} \mathfrak{gl}_t & 0 \\ M_{n-t,t} & \mathfrak{gl}_{n-t} \end{bmatrix} \times \begin{bmatrix} \mathfrak{gl}_t & M_{t,m-t} \\ 0 & \mathfrak{gl}_{m-t} \end{bmatrix} \times \begin{bmatrix} \mathfrak{gl}_t & 0 \\ M_{n-t,t} & \mathfrak{gl}_{n-t} \end{bmatrix}$$

such that $a_1 = a_2$, $a_3 = a_4$, $d_1 = d_3$, and $d_2 = d_4$. Now $\mathfrak{l}_t^{m,n} = \mathrm{Lie}(L_t^{m,n})$ where the algebraic subgroup $L_t^{m,n} \subseteq D(G)$ can be described in the same manner; we write it as follows:

$$L_t^{m,n} = \left\{ \left(\begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix}, \begin{bmatrix} a_1 & 0 \\ c_1 & d_2 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ 0 & d_1 \end{bmatrix}, \begin{bmatrix} a_2 & 0 \\ c_2 & d_2 \end{bmatrix} \right) \middle| \; a_1, a_2 \in GL_t, \right. \\ \left. b_1, b_2 \in M_{t,m-t},\ c_1, c_2 \in M_{n-t,t},\ d_1 \in GL_{m-t},\ d_2 \in GL_{n-t} \right\}. \tag{5.8}$$

We now apply Theorem 1.10, and conclude that the T-orbits of symplectic leaves in $\mathcal{O}_t^{m,n}$ are the irreducible components of the sets

$$\mathcal{P}_\sigma^t = \{ r_1 I_t^{m,n} r_2^{-1} \mid (r_1, r_2, r_1, r_2) \in (B_m^+ \times B_n^+ \times B_m^- \times B_n^-) \sigma L_t^{m,n} \}, \tag{5.9}$$

for $\sigma \in G \times G$. In fact, as we shall see later (Corollary 5.12), each $\mathcal{P}_\sigma^t$ is a single T-orbit of symplectic leaves. Thus, each $\mathcal{P}_\sigma^t$ is irreducible; we leave it to the reader to seek a direct proof for this fact.

Next, an application of Theorem A.1 shows that a complete, irredundant set of representatives for the $(B_m^+ \times B_n^+ \times B_m^- \times B_n^-)$, $L_t^{m,n}$ double cosets in $G \times G$ is given by

$$S_m^{S_{m-t}^2} \times S_n^{S_t^1} \times S_m^{S_t^1} \times S_n^{S_{n-t}^2}. \tag{5.10}$$

Thus, we analyze the $\mathcal{P}_\sigma^t$ for σ in the set (5.10). In particular, we shall find a criterion for $\mathcal{P}_\sigma^t$ to be nonempty (see Proposition 5.5).

5.3. Lemma. *Let $\sigma = (y, v, z, u) \in S_m^{S_{m-t}^2} \times S_n^{S_t^1} \times S_m^{S_t^1} \times S_n^{S_{n-t}^2}$. Then $\mathcal{P}_\sigma^t$ consists of all matrices $r_1 I_t^{m,n} r_2^{-1}$ for $r_1 \in GL_m$ and $r_2 \in GL_n$ such that*

$$\begin{aligned} r_1 &= b_1^+ y = b_3^- z \begin{bmatrix} a & b \\ 0 & I_{m-t} \end{bmatrix} & \quad & (b_1^+ \in B_m^+,\ b_2^+ \in B_n^+,\ b_3^- \in B_m^-,\ b_4^- \in B_n^-, \\ r_2 &= b_2^+ v = b_4^- u \begin{bmatrix} a & 0 \\ c & I_{n-t} \end{bmatrix} & & a \in GL_t,\ b \in M_{t,m-t},\ c \in M_{n-t,t}). \end{aligned} \tag{5.11}$$

Proof. First, consider a matrix $x = r_1 I_t^{m,n} r_2^{-1}$, where $r_1 \in GL_m$ and $r_2 \in GL_n$ satisfy (5.11). Then

$$\begin{aligned} (r_1, r_2, r_1, r_2) &= (b_1^+, b_2^+, b_3^-, b_4^-)(y, v, z, u) \left(I_m,\ I_n,\ \begin{bmatrix} a & b \\ 0 & I_{m-t} \end{bmatrix}, \begin{bmatrix} a & 0 \\ c & I_{n-t} \end{bmatrix} \right) \\ &\in (B_m^+ \times B_n^+ \times B_m^- \times B_n^-) \sigma L_t^{m,n}, \end{aligned}$$

whence $x \in \mathcal{P}_\sigma^t$.

Conversely, if $x \in \mathcal{P}_\sigma^t$, then $x = r_1 I_t^{m,n} r_2^{-1}$ for some $r_1 \in GL_m$ and $r_2 \in GL_n$ such that

$$(r_1, r_2, r_1, r_2) = \left(b_1^+ y \begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix}, b_2^+ v \begin{bmatrix} a_1 & 0 \\ c_1 & d_2 \end{bmatrix}, b_3^- z \begin{bmatrix} a_2 & b_2 \\ 0 & d_1 \end{bmatrix}, b_4^- u \begin{bmatrix} a_2 & 0 \\ c_2 & d_2 \end{bmatrix} \right),$$

where $b_1^+ \in B_m^+$, $b_2^+ \in B_n^+$, $b_3^- \in B_m^-$, $b_4^- \in B_n^-$, and the a_i, b_i, c_i, d_i satisfy the conditions of (5.8). Set $s_1 = \begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix}^{-1}$ and $s_2 = \begin{bmatrix} a_1 & 0 \\ c_1 & d_2 \end{bmatrix}^{-1}$, and observe that $(s_1, s_2, s_1, s_2) \in L_t^{m,n}$. Hence, the 4-tuple $(r_1s_1, r_2s_2, r_1s_1, r_2s_2)$ lies in $(B_m^+ \times B_n^+ \times B_m^- \times B_n^-)\sigma L_t^{m,n}$. Since $s_1 I_t^{m,n} s_2^{-1} = I_t^{m,n}$, we have $x = (r_1s_1)I_t^{m,n}(r_2s_2)^{-1}$, and so we may replace (r_1, r_2, r_1, r_2) by $(r_1s_1, r_2s_2, r_1s_1, r_2s_2)$. Thus, there is no loss of generality in assuming that

$$(r_1, r_2, r_1, r_2) = \left(b_1^+ y,\, b_2^+ v,\, b_3^- z \begin{bmatrix} a & b \\ 0 & I_{m-t} \end{bmatrix},\, b_4^- u \begin{bmatrix} a & 0 \\ c & I_{n-t} \end{bmatrix} \right)$$

for some $a \in GL_t$, $b \in M_{t,m-t}$, $c \in M_{n-t,t}$. Now r_1 and r_2 satisfy (5.11), and the proof is complete. □

5.4. Recall that the sets $S_n^{S_t^1}$ and $S_n^{S_{n-t}^2}$ of minimal length coset representatives for the subgroups S_t^1 and S_{n-t}^2 of S_n can be described as follows:

$$S_n^{S_t^1} = \{u \in S_n \mid u(1) < \cdots < u(t)\}$$
$$S_n^{S_{n-t}^2} = \{v \in S_n \mid v(t+1) < \cdots < v(n)\}.$$

Lemma. (a) *If $v \in S_n^{S_t^1}$, then $v \begin{bmatrix} B_t^\pm & 0 \\ 0 & I_{n-t} \end{bmatrix} \subseteq B_n^\pm v$.*

(b) *If $u \in S_n^{S_{n-t}^2}$, then $u \begin{bmatrix} I_t & 0 \\ 0 & B_{n-t}^\pm \end{bmatrix} \subseteq B_n^\pm u$.*

Proof. The lemma follows at once from the fact that for given a Weyl group W and a subgroup W_I generated by simple reflections for a subset I of simple roots, an element $w \in W$ belongs to the set W^{W_I} of minimal length representatives of the cosets in W/W_I if and only if $w(\alpha)$ is a positive root for any $\alpha \in I$, cf. [**3**, Proposition 2.3.3]. □

5.5. Proposition. *Let $\sigma = (y, v, z, u) \in S_m^{S_{m-t}^2} \times S_n^{S_t^1} \times S_m^{S_t^1} \times S_n^{S_{n-t}^2}$. Then*

$$\mathcal{P}_\sigma^t = \bigcup_{\substack{\tau \in S_t^1 \\ z\tau \le y,\ v\tau^{-1} \le u}} \big(B_m^+ y B_m^+ \cap B_m^- z\tau\big).I_t^{m,n}.\big(\tau^{-1} B_n^- u^{-1} B_n^- \cap v^{-1} B_n^+\big). \tag{5.12}$$

Further, $\mathcal{P}_\sigma^t \neq \varnothing$ if and only if $z \le y$ and $v \le u$.

Proof. By Theorem 2.4 and Proposition 2.2, $B^+yB^+ \cap B^-z\tau$ is nonempty if and only if $z\tau \le y$, and similarly $B^-u^{-1}B^- \cap \tau v^{-1}B^+$ is nonempty if and only if $v\tau^{-1} \le u$. Hence, the union in (5.12) can just as well be taken over all $\tau \in S_t^1$.

Now assume for the moment that (5.12) has been proved. If $z \le y$ and $v \le u$, then the intersections $B^+yB^+ \cap B^-z$ and $B^-u^{-1}B^- \cap v^{-1}B^+$ are both nonempty, and (5.12) yields $\mathcal{P}_\sigma^t \neq \varnothing$. Conversely, if $\mathcal{P}_\sigma^t \neq \varnothing$, then because of (5.12), there is some $\tau \in S_t^1$ such that both $B^+yB^+ \cap B^-z\tau$ and $\tau^{-1}B^-u^{-1}B^- \cap v^{-1}B^+$ are nonempty, whence $z\tau \le y$ and

$v\tau^{-1} \le u$. But since $z \in S_m^{S_t^1}$ and $v \in S_n^{S_t^1}$, we see that $z \le z\tau$ and $v \le v\tau^{-1}$. Therefore $z \le y$ and $v \le u$, and the final statement of the theorem is proved.

It remains to prove (5.12).

If $x \in \mathcal{P}_\sigma^t$, then $x = r_1 I_t^{m,n} r_2^{-1}$ for some $r_1 \in GL_m$ and $r_2 \in GL_n$ satisfying (5.11). By the B_t^-, B_t^+ Bruhat decomposition in GL_t, we have $a = a^-\tau(a^+)^{-1}$ for some $a^\pm \in B_t^\pm$ and $\tau \in S_t$. Set $s_1 = r_1 \begin{bmatrix} a^+ & -a^{-1}b \\ 0 & I_{m-t} \end{bmatrix}$ and $s_2 = r_2 \begin{bmatrix} a^+ & 0 \\ 0 & I_{n-t} \end{bmatrix}$, so that $x = s_1 I_t^{m,n} s_2^{-1}$ and

$$s_1 = b_1^+ y \begin{bmatrix} a^+ & -a^{-1}b \\ 0 & I_{m-t} \end{bmatrix} = b_3^- z \begin{bmatrix} a^-\tau & 0 \\ 0 & I_{m-t} \end{bmatrix}$$
$$s_2 = b_2^+ v \begin{bmatrix} a^+ & 0 \\ 0 & I_{n-t} \end{bmatrix} = b_4^- u \begin{bmatrix} a^-\tau & 0 \\ ca^+ & I_{n-t} \end{bmatrix}.$$

It follows that $s_1 \in B_m^+ y B_m^+$ and $s_2 \in B_n^- u B_n^- \tau$, where we now view $\tau \in S_t^1 \subseteq S_n$. Since $z \in S_m^{S_t^1}$, Lemma 5.4 implies that $z \begin{bmatrix} a^- & 0 \\ 0 & I_{m-t} \end{bmatrix} \in B_m^- z$, whence $s_1 \in B_m^- z\tau$. Similarly, $v \in S_n^{S_t^1}$ implies that $v \begin{bmatrix} a^+ & 0 \\ 0 & I_{n-t} \end{bmatrix} \in B_n^+ v$, whence $s_2 \in B_n^+ v$. Thus,

$$x = s_1 I_t^{m,n} s_2^{-1} \in \big(B_m^+ y B_m^+ \cap B_m^- z\tau\big).I_t^{m,n}.\big(\tau^{-1} B_n^- u^{-1} B_n^- \cap v^{-1} B_n^+\big).$$

Conversely, let $x \in M_{m,n}$ be a matrix such that

$$x \in \big(B_m^+ y B_m^+ \cap B_m^- z\tau\big).I_t^{m,n}.\big(\tau^{-1} B_n^- u^{-1} B_n^- \cap v^{-1} B_n^+\big)$$

for some $\tau \in S_t^1$. Then $x = r_1 I_t^{m,n} r_2^{-1}$ where

$$r_1 = b_1^+ y \begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix} = b_3^- z\tau \qquad\qquad r_2 = b_4^- u \begin{bmatrix} a_2\tau & 0 \\ c_2\tau & d_2 \end{bmatrix} = b_2^+ v$$

where $b_1^+ \in B_m^+$, $b_2^+ \in B_n^+$, $b_3^- \in B_m^-$, $b_4^- \in B_n^-$, while $a_1 \in B_t^+$, $a_2 \in B_t^-$, $b_1 \in M_{t,m-t}$, $c_2 \in M_{n-t,t}$, $d_1 \in B_{m-t}^+$, $d_2 \in B_{n-t}^-$. Since $y \in S_m^{S_{m-t}^2}$, Lemma 5.4(b) implies that $y \begin{bmatrix} I_t & 0 \\ 0 & d_1 \end{bmatrix} \in B_m^+ y$, and so $r_1 = \beta_1^+ y \begin{bmatrix} a_1 & b_1 \\ 0 & I_{m-t} \end{bmatrix}$ for some $\beta_1^+ \in B_m^+$. Since $v \in S_n^{S_t^1}$, Lemma 5.4(a) implies that $v \begin{bmatrix} a_1 & 0 \\ 0 & I_{n-t} \end{bmatrix} \in B_n^+ v$, and so $r_2 = \beta_2^+ v \begin{bmatrix} a_1 & 0 \\ 0 & I_{n-t} \end{bmatrix}$ for some $\beta_2^+ \in B_n^+$. Similarly, $r_1 = \beta_3^- z \begin{bmatrix} a_2\tau & 0 \\ 0 & I_{m-t} \end{bmatrix}$ and $r_2 = \beta_4^- u \begin{bmatrix} a_2\tau & 0 \\ c_2' & I_{n-t} \end{bmatrix}$ for some $\beta_3^- \in B_m^-$, $\beta_4^- \in B_n^-$, and $c_2' \in M_{n-t,t}$. Consequently,

$$\begin{aligned}(r_1, r_2, r_1, r_2) &= (\beta_1^+ y, \beta_2^+ v, \beta_3^- z, \beta_4^- u)\left(\begin{bmatrix} a_1 & b_1 \\ 0 & I_{m-t} \end{bmatrix}, \begin{bmatrix} a_1 & 0 \\ 0 & I_{n-t} \end{bmatrix}, \begin{bmatrix} a_2\tau & 0 \\ 0 & I_{m-t} \end{bmatrix}, \begin{bmatrix} a_2\tau & 0 \\ c_2' & I_{n-t} \end{bmatrix}\right) \\ &\in (B_m^+ \times B_n^+ \times B_m^- \times B_n^-)\sigma L_t,\end{aligned}$$

and so $x \in \mathcal{P}_\sigma^t$. Therefore (5.12) holds. $\square$

5.6. In view of Proposition 5.5, the following set indexes the nonempty $\mathcal{P}^t_\sigma$:

$$\Sigma^{m,n}_t = \big\{ (y,v,z,u) \in S_m^{S^2_{m-t}} \times S_n^{S^1_t} \times S_m^{S^1_t} \times S_n^{S^2_{n-t}} \bigm| z \le y,\ v \le u \big\}. \tag{5.13}$$

In order to match the $\mathcal{P}^t_\sigma$ with appropriate T-orbits $\mathcal{P}_w$ of symplectic leaves, we need a bijection between $\Sigma^{m,n}_t$ and the index set $S_N^{\ge(w_\circ^n,w_\circ^m)}[t]$ defined in (5.2). Recall from (3.24) and Lemma 3.12 that

$$S_N^{\ge(w_\circ^n,w_\circ^m)} = w_\circ^{m+n} S_N^{[-n,m]}.$$

Hence, we define

$$\begin{aligned} S^{[-n,m]}_{m+n}[t] &= w_\circ^{m+n} S_N^{\ge(w_\circ^n,w_\circ^m)}[t] \\ &= \big\{ w \in S_N^{[-n,m]} \bigm| |w^{-1}\big(\{1,\dots,m\}\big) \cap \{1,\dots,n\}| = t \big\}, \end{aligned} \tag{5.14}$$

so that $S_N^{\ge(w_\circ^n,w_\circ^m)}[t] = w_\circ^{m+n} S^{[-n,m]}_{m+n}[t]$. It is convenient to first construct a bijection $\Sigma^{m,n}_t \to S^{[-n,m]}_{m+n}[t]$. To describe that, we will need the matrix $I^{m,n}_t \in M_{m,n}$ and the analogous matrix $I^{n,m}_t \in M_{n,m}$, as well as

$$J^m_t = \begin{bmatrix} 0_t & 0_{t,m-t} \\ 0_{m-t,t} & I_{m-t} \end{bmatrix} \in M_m \qquad\qquad J^n_t = \begin{bmatrix} 0_t & 0_{t,n-t} \\ 0_{n-t,t} & I_{n-t} \end{bmatrix} \in M_n\,. \tag{5.15}$$

5.7. Lemma. *Let $u, v \in S_n$.*

(a) *If $u \in S_n^{S^2_{n-t}}$ and $v \le u$, then $v(j) \ge u(j)$ for $j = t+1, \dots, n$.*

(b) *If $v \in S_n^{S^1_t}$ and $v(j) \ge u(j)$ for $j = t+1, \dots, n$, then $v \le u$.*

Proof. First consider subsets $U, V \subseteq \{1,\dots,n\}$ with $|U| = |V|$, and let $\widetilde{U}$ and $\widetilde{V}$ denote their complements in $\{1,\dots,n\}$. We claim that $V \le U$ if and only if $\widetilde{V} \ge \widetilde{U}$.

Assume first that $\widetilde{V} \ge \widetilde{U}$. Label the elements of the four sets in ascending order:

$$\begin{aligned} U &= \{u_1 < \cdots < u_r\} & V &= \{v_1 < \cdots < v_r\} \\ \widetilde{U} &= \{\tilde{u}_1 < \cdots < \tilde{u}_{n-r}\} & \widetilde{V} &= \{\tilde{v}_1 < \cdots < \tilde{v}_{n-r}\}. \end{aligned}$$

We have $\tilde{v}_i \ge \tilde{u}_i$ for all i, and must show that $v_j \le u_j$ for all j.

Consider the interval $L = \{1, 2, \dots, v_j - 1\}$ for some $j \le r$. Since L contains exactly $j-1$ elements of V, it contains the first $v_j - j$ elements of $\widetilde{V}$. So for $i = 1, \dots, v_j - j$, we have $\tilde{v}_i \in L$ and $\tilde{u}_i \le \tilde{v}_i$, whence $\tilde{u}_i \in L$. Thus, L contains at least $v_j - j$ elements of $\widetilde{U}$, and hence at most $j-1$ elements of U. It follows that $u_j \notin L$, whence $u_j \ge v_j$. Therefore $V \le U$, as desired.

The fact that $V \le U$ implies $\widetilde{V} \ge \widetilde{U}$ follows by reversing the roles of these sets and their complements.

(a) By assumption, $v(\{1,\dots,t\}) \le u(\{1,\dots,t\})$, and so the claim above implies that $v(\{t+1,\dots,n\}) \ge u(\{t+1,\dots,n\})$. Since $u \in S_n^{S^2_{n-t}}$, the least element of $u(\{t+1,\dots,n\})$ is $u(t+1)$, and consequently $u(t+1) \le v(t+1)$. Moreover, $u \in S_n^{S^2_{n-r}}$ for $t \le r < n$, and so the same argument yields $u(r+1) \le v(r+1)$ for $t \le r < n$.

(b) Our assumption implies that $v(\{t+1,\dots,n\}) \ge u(\{t+1,\dots,n\})$, and so the claim above yields $v(\{1,\dots,t\}) \le u(\{1,\dots,t\})$. Since $v(1) < \cdots, v(t)$, it follows that $v(\{1,\dots,r\}) \le u(\{1,\dots,r\})$ for $r = 1,\dots,t$. Moreover, for $r = t,\dots,n-1$, we have $v(\{r+1,\dots,n\}) \ge u(\{r+1,\dots,n\})$ and the claim yields $v(\{1,\dots,r\}) \le u(\{1,\dots,r\})$. Therefore $v \le u$. □

5.8. Partial permutations. Just as with permutations (cf. §3.1), we view any partial permutation matrix w as both a matrix and a function (a bijection from its domain to its range). Write $\operatorname{dom}(w)$ and $\operatorname{rng}(w)$ for the domain and range of w; then the matrix form of w has a 1 in position $w(j), j$ for each $j \in \operatorname{dom}(w)$, and a 0 in all other positions. Observe that w^{tr} is the inverse bijection, from $\operatorname{rng}(w)$ to $\operatorname{dom}(w)$.

5.9. Proposition. *There is a bijection $\phi : \Sigma_t^{m,n} \to S_{m+n}^{[-n,m]}[t]$ given by*

$$\phi(y,v,z,u) = \begin{bmatrix} w_\circ^m y I_t^{m,n} v^{-1} & w_\circ^m y J_t^m z^{-1} w_\circ^m \\ u J_t^n v^{-1} & u I_t^{n,m} z^{-1} w_\circ^m \end{bmatrix}. \tag{5.16}$$

Proof. Let $(y,v,z,u) \in \Sigma_t^{m,n}$, and let

$$w = \phi(y,v,z,u) = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix},$$

where the w_{ij} stand for the blocks shown in (5.16). Since w can be expressed in the form

$$w = \begin{bmatrix} w_\circ^m y & 0 \\ 0 & u \end{bmatrix} \begin{bmatrix} I_t^{m,n} & J_t^m \\ J_t^n & I_t^{n,m} \end{bmatrix} \begin{bmatrix} v^{-1} & 0 \\ 0 & z^{-1} w_\circ^m \end{bmatrix},$$

it is clear that w is a permutation matrix, which we identify with a permutation in S_N in the usual way. Observe that

$$|w^{-1}\big(\{1,\dots,m\}\big) \cap \{1,\dots,n\}| = \operatorname{rank}(w_{11}) = t.$$

By Lemma 5.7(a), $z(j) \ge y(j)$ and $v(j) \ge u(j)$ for $j > t$. Thus, $w_{21}v(j) = u(j) \le v(j)$ for $j > t$, and so $w_{21}(i) \le i$ for all $i \in \operatorname{dom}(w_{21})$. It follows that $w(i) \le i + m$ for all i. Similarly, $w_{12} w_\circ^m z(j) = w_\circ^m y(j) \ge w_\circ^m z(j)$ for all $j > t$ and so $w_{12}(i) \ge i$ for all $i \in \operatorname{dom}(w_{12})$, whence $w(i) \ge i - n$ for all i. Therefore $w \in S_{m+n}^{[-n,m]}[t]$, which shows that the rule (5.16) does define a map ϕ from $\Sigma_t^{m,n}$ to $S_{m+n}^{[-n,m]}[t]$.

Observe that $y(j) = w_\circ^m w_{11} v(j)$ for $j \le t$. Since $v(1) < \cdots < v(t)$ (because $v \in S_n^{S^1_t}$), it follows that the restriction of y to $\{1,\dots,t\}$ is determined by w_{11}. But $y \in S_m^{S^2_{m-t}}$, and thus y is completely determined by w_{11}. Similarly, $u(j) = w_{22} w_\circ^m z(j)$ for $j \leq t$ and $z(1) < \cdots < z(t)$, whence the restriction of u to $\{1, \ldots, t\}$ is determined by w_{22}. Since $u \in S_n^{S^2_{n-t}}$, it follows that u is completely determined by w_{22}.

For $j = t+1, \ldots, n$, we have $u(j) = w_{21} v(j)$ and so $v(j) = w_{21}^{\mathrm{tr}} u(j)$. Since $v \in S_n^{S^1_t}$, it follows that v is completely determined by u and w_{21}. Similarly, for $j = t+1, \ldots, m$, we have $w_\circ^m y(j) = w_{12} w_\circ^m z(j)$ and so $z(j) = w_\circ^m w_{12}^{\mathrm{tr}} w_\circ^m y(j)$. Since $z \in S_m^{S^1_t}$, it follows that z is completely determined by y and w_{12}. Therefore, (y, v, z, u) is completely determined by w, which shows that the map ϕ is injective.

Now consider an arbitrary element $w \in S_{m+n}^{[-n,m]}[t]$, and write

$$w = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}, \qquad\qquad \begin{pmatrix} w_{11} \in M_{m,n} & w_{12} \in M_m \\ w_{21} \in M_n & w_{22} \in M_{n,m} \end{pmatrix}.$$

Each w_{ij} is a partial permutation matrix, and

$$\begin{aligned} \mathrm{dom}(w_{11}) \sqcup \mathrm{dom}(w_{21}) &= \mathrm{rng}(w_{21}) \sqcup \mathrm{rng}(w_{22}) = \{1, \ldots, n\} \\ \mathrm{dom}(w_{12}) \sqcup \mathrm{dom}(w_{22}) &= \mathrm{rng}(w_{11}) \sqcup \mathrm{rng}(w_{12}) = \{1, \ldots, m\}. \end{aligned} \tag{5.17}$$

Further, $\mathrm{rank}(w_{11}) = t$ (recall (5.14)), from which we see that $\mathrm{rank}(w_{12}) = m - t$ and $\mathrm{rank}(w_{21}) = n - t$, and hence $\mathrm{rank}(w_{22}) = t$. Since $i - n \leq w(i) \leq i + m$ for all $i = 1, \ldots, N$, we have $w_{12}(j) \geq j$ for all $j \in \mathrm{dom}(w_{12})$ and $w_{21}(j) \leq j$ for all $j \in \mathrm{dom}(w_{21})$.

Write the elements of $\mathrm{dom}(w_{11})$ in ascending order: $\mathrm{dom}(w_{11}) = \{v_1 < \cdots < v_t\}$. Set $y(j) = w_\circ^m w_{11}(v_j)$ for $j = 1, \ldots, t$, and extend (uniquely) to a permutation $y \in S_m^{S^2_{m-t}}$. Write the elements of $\mathrm{dom}(w_{22})$ in descending order: $\mathrm{dom}(w_{22}) = \{z_1 > \cdots > z_t\}$. Set $u(j) = w_{22}(z_j)$ for $j = 1, \ldots, t$, and extend (uniquely) to a permutation $u \in S_n^{S^2_{n-t}}$. Next, observe using (5.17) that

$$\begin{aligned} u(\{t+1, \ldots, n\}) = \{1, \ldots, n\} \setminus \mathrm{rng}(w_{22}) = \mathrm{rng}(w_{21}) = \mathrm{dom}(w_{21}^{\mathrm{tr}}) \\ \mathrm{rng}(w_{21}^{\mathrm{tr}}) = \mathrm{dom}(w_{21}) = \{1, \ldots, n\} \setminus \mathrm{dom}(w_{11}). \end{aligned}$$

Hence, we can define a permutation $v \in S_n^{S^1_t}$ such that $v(j) = v_j$ for $j = 1, \ldots, t$ and $v(j) = w_{21}^{\mathrm{tr}} u(j)$ for $j = t+1, \ldots, n$. Similarly,

$$\begin{aligned} w_\circ^m y(\{t+1, \ldots, m\}) = \{1, \ldots, m\} \setminus \mathrm{rng}(w_{11}) = \mathrm{rng}(w_{12}) = \mathrm{dom}(w_{12}^{\mathrm{tr}}) \\ w_\circ^m(\mathrm{rng}(w_{12}^{\mathrm{tr}})) = w_\circ^m(\mathrm{dom}(w_{12})) = \{1, \ldots, m\} \setminus w_\circ^m(\mathrm{dom}(w_{22})), \end{aligned}$$

and so we can define a permutation $z \in S_m^{S^1_t}$ such that $z(j) = w_\circ^m(z_j)$ for $j = 1, \ldots, t$ and $z(j) = w_\circ^m w_{12}^{\mathrm{tr}} w_\circ^m y(j)$ for $j = t+1, \ldots, m$.

We have now defined $(y, v, z, u) \in S_m^{S^2_{m-t}} \times S_n^{S^1_t} \times S_m^{S^1_t} \times S_n^{S^2_{n-t}}$. For $j = t+1, \ldots, m$, we have

$$y(j) = w_\circ^m w_{12} w_\circ^m z(j) \leq w_\circ^m w_\circ^m z(j) = z(j),$$

and so $z \leq y$ by Lemma 5.7(b). Similarly, $u(j) = w_{21}v(j) \leq v(j)$ for $j = t+1, \dots, n$, and so $v \leq u$. Thus, $(y, v, z, u) \in \Sigma_t^{m,n}$. Finally, we analyze the domains and actions of the four components of $\phi(y, v, z, u)$, as follows.

$$\begin{aligned}
w_\circ^m y I_t^{m,n} v^{-1} \ : \qquad & \text{domain} = v(\{1, \dots, t\}) = \{v_1, \dots, v_t\} = \text{dom}(w_{11}) \\
& \qquad v_j = v(j) \mapsto w_\circ^m y(j) = w_{11}(v_j) \\
w_\circ^m y J_t^m z^{-1} w_\circ^m \ : \qquad & \text{domain} = w_\circ^m z(\{t+1, \dots, m\}) = \text{rng}(w_{12}^{\text{tr}}) = \text{dom}(w_{12}) \\
& w_\circ^m z(j) \mapsto w_\circ^m y(j) = w_{12} w_\circ^m z(j) \\
u J_t^n v^{-1} \ : \qquad & \text{domain} = v(\{t+1, \dots, n\}) = \text{rng}(w_{21}^{\text{tr}}) = \text{dom}(w_{21}) \\
& \quad v(j) \mapsto u(j) = w_{21} v(j) \\
u I_t^{n,m} z^{-1} w_\circ^m \ : \qquad & \text{domain} = w_\circ^m z(\{1, \dots, t\}) = \{z_1, \dots, z_t\} = \text{dom}(w_{22}) \\
& \qquad z_j = w_\circ^m z(j) \mapsto u(j) = w_{22}(z_j).
\end{aligned}$$

This shows that $\phi(y, v, z, u) = \left[\begin{smallmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{smallmatrix}\right] = w$, and therefore that ϕ is surjective. $\square$

5.10. Corollary. *There is a bijection $\Sigma_t^{m,n} \to S_N^{\geq (w_\circ^n, w_\circ^m)}[t]$ given by*

$$(y, v, z, u) \longmapsto w_\circ^N \phi(y, v, z, u) = \begin{bmatrix} w_\circ^n u J_t^n v^{-1} & w_\circ^n u I_t^{n,m} z^{-1} w_\circ^m \\ y I_t^{m,n} v^{-1} & y J_t^m z^{-1} w_\circ^m \end{bmatrix}. \qquad \square \tag{5.18}$$

We are now ready to state and prove the main theorem of the section. The description it provides of orbits $\mathcal{P}_w$ of symplectic leaves requires a union involving more than one set in general (see Example 5.14). For a class of cases in which only a single term is required, see Theorem 6.1.

5.11. Theorem. *Let $w \in S_N^{\geq (w_\circ^n, w_\circ^m)}[t]$ (recall (5.2)). Then $w = w_\circ^{m+n} \phi(\sigma)$ for a unique 4-tuple $\sigma = (y, v, z, u) \in \Sigma_t^{m,n}$ (recall (5.13)), and*

$$\mathcal{P}_w = \mathcal{P}_\sigma^t = \bigcup_{\substack{\tau \in S_t^1 \\ z\tau \leq y, \ v\tau^{-1} \leq u}} \big(B_m^+ y B_m^+ \cap B_m^- z \tau \big) . I_t^{m,n} . \big(\tau^{-1} B_n^- u^{-1} B_n^- \cap v^{-1} B_n^+ \big). \tag{5.19}$$

Proof. The existence and uniqueness of σ are given by Corollary 5.10, and the second equality in (5.19) by Proposition 5.5. It remains to prove that $\mathcal{P}_w = \mathcal{P}_\sigma^t$, for which we shall use the description of $\mathcal{P}_\sigma^t$ given in Lemma 5.3.

Observe that, in block form, $w = \left[\begin{smallmatrix} w_\circ^n & 0 \\ 0 & I_m \end{smallmatrix}\right] s \left[\begin{smallmatrix} I_n & 0 \\ 0 & w_\circ^m \end{smallmatrix}\right]$, where

$$s = \begin{bmatrix} u J_t^n v^{-1} & u I_t^{n,m} z^{-1} \\ y I_t^{m,n} v^{-1} & y J_t^m z^{-1} \end{bmatrix}.$$

Hence (recall (3.18)), $\mathcal{P}_w$ consists of those matrices $x \in M_{m,n}$ such that

$$\begin{aligned}\begin{bmatrix} I_n & 0 \\ x & I_m \end{bmatrix} &\in \begin{bmatrix} w_\circ^n & 0 \\ 0 & I_m \end{bmatrix} B^+ \begin{bmatrix} w_\circ^n & 0 \\ 0 & I_m \end{bmatrix} s \begin{bmatrix} I_n & 0 \\ 0 & w_\circ^m \end{bmatrix} B^+ \begin{bmatrix} I_n & 0 \\ 0 & w_\circ^m \end{bmatrix} \\ &= \begin{bmatrix} B_n^- & M_{n,m} \\ 0 & B_m^+ \end{bmatrix} s \begin{bmatrix} B_n^+ & M_{n,m} \\ 0 & B_m^- \end{bmatrix}.\end{aligned} \tag{5.20}$$

If $x \in M_{m,n}$ satisfies (5.20), then

$$\begin{gathered}\begin{bmatrix} I_n & 0 \\ x & I_m \end{bmatrix} = \begin{bmatrix} \alpha_1 & \beta_1 \\ 0 & \gamma_1 \end{bmatrix} \begin{bmatrix} uJ_t^n v^{-1} & uI_t^{n,m} z^{-1} \\ yI_t^{m,n} v^{-1} & yJ_t^m z^{-1} \end{bmatrix} \begin{bmatrix} \alpha_2 & \beta_2 \\ 0 & \gamma_2 \end{bmatrix} \\ (\alpha_1 \in B_n^-,\, \alpha_2 \in B_n^+,\, \gamma_1 \in B_m^+,\, \gamma_2 \in B_m^-,\, \beta_1, \beta_2 \in M_{n,m}).\end{gathered} \tag{5.21}$$

Set $b = u^{-1}\alpha_1^{-1}\beta_1 y \in M_{n,m}$, and rewrite (5.21) in the form

$$\begin{aligned} I_n &= \alpha_1 u(J_t^n + bI_t^{m,n})v^{-1}\alpha_2 \\ 0 &= \alpha_1 u(J_t^n v^{-1}\beta_2 + I_t^{n,m} z^{-1}\gamma_2) + \alpha_1 ub(I_t^{m,n} v^{-1}\beta_2 + J_t^m z^{-1}\gamma_2) \\ x &= \gamma_1 y I_t^{m,n} v^{-1}\alpha_2 \\ I_m &= \gamma_1 y(I_t^{m,n} v^{-1}\beta_2 + J_t^m z^{-1}\gamma_2). \end{aligned} \tag{5.22}$$

Multiply the first equation of (5.22) on the right by $\alpha_2^{-1}vJ_t^n$ and by $\alpha_2^{-1}vI_t^{n,n}$, and the fourth on the left by $I_t^{m,m}y^{-1}\gamma_1^{-1}$ and by $J_t^m y^{-1}\gamma_1^{-1}$, to obtain

$$\begin{aligned} \alpha_2^{-1}vJ_t^n &= \alpha_1 uJ_t^n & \qquad \alpha_2^{-1}vI_t^{n,n} &= \alpha_1 ubI_t^{m,n} \\ I_t^{m,m}y^{-1}\gamma_1^{-1} &= I_t^{m,n}v^{-1}\beta_2 & \qquad J_t^m y^{-1}\gamma_1^{-1} &= J_t^m z^{-1}\gamma_2. \end{aligned} \tag{5.23}$$

Adding the two equations in each row of (5.23) yields

$$\alpha_2^{-1}v = \alpha_1 u(J_t^n + bI_t^{m,n}) \qquad y^{-1}\gamma_1^{-1} = I_t^{m,n}v^{-1}\beta_2 + J_t^m z^{-1}\gamma_2. \tag{5.24}$$

Now substitute the second equation of (5.24) into the second equation of (5.22), and multiply on the left by $I_t^{m,n}u^{-1}\alpha_1^{-1}$, to obtain

$$I_t^{m,m}z^{-1}\gamma_2 + I_t^{m,n}by^{-1}\gamma_1^{-1} = 0. \tag{5.25}$$

The last equation of (5.23) combines with (5.25) to yield $z^{-1}\gamma_2 = (J_t^m - I_t^{m,n}b)y^{-1}\gamma_1^{-1}$, and consequently

$$\gamma_1 y = \gamma_2^{-1} z(J_t^m - I_t^{m,n}b). \tag{5.26}$$

Write $b = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \in \begin{bmatrix} M_t & M_{t,m-t} \\ M_{n-t,t} & M_{n-t,m-t} \end{bmatrix}$. Since, as we see from (5.26), the matrix $J_t^m - I_t^{m,n} b = \begin{bmatrix} -b_{11} & -b_{12} \\ 0 & I_{m-t} \end{bmatrix}$ is invertible, $b_{11} \in GL_t$. Now set

$$\begin{aligned} r_1 &= \gamma_1 y = \gamma_2^{-1} z (J_t^m - I_t^{m,n} b) = \gamma_2^{-1} z \begin{bmatrix} -b_{11} & -b_{12} \\ 0 & I_{m-t} \end{bmatrix} \\ r_2 &= \alpha_2^{-1} v = \alpha_1 u (J_t^n + b I_t^{m,n}) = \alpha_1 u \begin{bmatrix} b_{11} & 0 \\ b_{21} & I_{n-t} \end{bmatrix}. \end{aligned} \tag{5.27}$$

Since $z \begin{bmatrix} -I_t & 0 \\ 0 & I_{m-t} \end{bmatrix} \in z T_m = T_m z$, we have $r_1 \in B_m^- z \begin{bmatrix} b_{11} & b_{12} \\ 0 & I_{m-t} \end{bmatrix}$. Thus, r_1 and r_2 satisfy (5.11), and so $x = \gamma_1 y I_t^{m,n} v^{-1} \alpha_2 = r_1 I_t^{m,n} r_2^{-1} \in \mathcal{P}_\sigma^t$.

Conversely, if $x \in \mathcal{P}_\sigma^t$, then, making use of the relation $z \begin{bmatrix} -I_t & 0 \\ 0 & I_{m-t} \end{bmatrix} \in T_m z$ as above, $x = r_1 I_t^{m,n} r_2^{-1}$ where

$$r_1 = \gamma_1 y = \gamma_2^{-1} z \begin{bmatrix} -b_{11} & -b_{12} \\ 0 & I_{m-t} \end{bmatrix} \qquad r_2 = \alpha_2^{-1} v = \alpha_1 u \begin{bmatrix} b_{11} & 0 \\ b_{21} & I_{n-t} \end{bmatrix} \tag{5.28}$$

$$(\gamma_1 \in B_m^+,\ \gamma_2 \in B_m^-,\ \alpha_2 \in B_n^+,\ \alpha_1 \in B_n^-,\ b_{11} \in GL_t,\ b_{12} \in M_{t,m-t},\ b_{21} \in M_{n-t,t}).$$

In particular,

$$x = \gamma_1 y I_t^{m,n} v^{-1} \alpha_2 \,. \tag{5.29}$$

Set $b = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & 0 \end{bmatrix} \in M_{n,m}$; then (5.28) can be rewritten as

$$r_1 = \gamma_1 y = \gamma_2^{-1} z (J_t^m - I_t^{m,n} b) \qquad r_2 = \alpha_2^{-1} v = \alpha_1 u (J_t^n + b I_t^{m,n}). \tag{5.30}$$

The first equation of (5.30) implies that

$$z^{-1} \gamma_2 = (J_t^m - I_t^{m,n} b) y^{-1} \gamma_1^{-1}. \tag{5.31}$$

The second equation of (5.30), together with (5.31), yields

$$\begin{aligned} \alpha_2^{-1} v J_t^n &= \alpha_1 u J_t^n \qquad\qquad \alpha_2^{-1} v I_t^{n,n} = \alpha_1 u b I_t^{m,n} \\ J_t^m y^{-1} \gamma_1^{-1} &= J_t^m z^{-1} \gamma_2. \end{aligned} \tag{5.32}$$

Now set $\beta_1 = \alpha_1 u b y^{-1}$ and $\beta_2 = -\alpha_2 \alpha_1 u (I_t^{n,m} + b J_t^m) z^{-1} \gamma_2$ in $M_{n,m}$. From (5.32) and the definitions of β_1 and β_2, we get

$$(\alpha_1 u J_t^n + \beta_1 y I_t^{m,n}) v^{-1} \alpha_2 = (\alpha_2^{-1} v J_t^n + \alpha_2^{-1} v I_t^{n,n}) v^{-1} \alpha_2 = I_n, \tag{5.33}$$

which implies

$$\alpha_2^{-1} v = \alpha_1 u J_t^n + \beta_1 y I_t^{m,n} = \alpha_1 u (J_t^n + b I_t^{m,n}), \tag{5.34}$$

as well as

$$(5.35)\quad (\alpha_1 u J_t^n + \beta_1 y I_t^{m,n}) v^{-1}\beta_2 + (\alpha_1 u I_t^{n,m} + \beta_1 y J_t^m) z^{-1}\gamma_2 = \alpha_2^{-1}\beta_2 + \alpha_1 u (I_t^{n,m} + bJ_t^m) z^{-1}\gamma_2 = 0.$$

Note that (5.34) implies that $v^{-1}\alpha_2\alpha_1 u = (J_t^n + bI_t^{m,n})^{-1}$, and so, from (5.31) and the definition of β_2, we have

$$(5.36)\quad \begin{aligned} I_t^{m,n} v^{-1}\beta_2 &= -I_t^{m,n}(J_t^n + bI_t^{m,n})^{-1}(I_t^{n,m} + bJ_t^m)(J_t^m - I_t^{m,n}b) y^{-1}\gamma_1^{-1} \\ &= -I_t^{m,n}\begin{bmatrix} b_{11} & 0 \\ b_{21} & I_{n-t}\end{bmatrix}^{-1}\begin{bmatrix} I_t & b_{12} \\ 0 & 0\end{bmatrix}\begin{bmatrix} -b_{11} & -b_{12} \\ 0 & I_{m-t}\end{bmatrix} y^{-1}\gamma_1^{-1} \\ &= -\begin{bmatrix} b_{11}^{-1} & 0 \\ 0 & 0_{m-t,n-t}\end{bmatrix}\begin{bmatrix} -b_{11} & 0 \\ 0 & 0_{n-t,m-t}\end{bmatrix} y^{-1}\gamma_1^{-1} = I_t^{m,m} y^{-1}\gamma_1^{-1}. \end{aligned}$$

Consequently, with the help of (5.32), we get

$$(5.37)\quad \gamma_1 y (I_t^{m,n} v^{-1}\beta_2 + J_t^m z^{-1}\gamma_2) = \gamma_1 y (I_t^{m,m} y^{-1}\gamma_1^{-1} + J_t^m y^{-1}\gamma_1^{-1}) = I_m.$$

Combine (5.33), (5.29), (5.35) and (5.37) to see that (5.21) holds, whence (5.20), and therefore $x \in \mathcal{P}_w$. □

Theorem 5.11 verifies the main conclusions of §5.2, as follows.

5.12. Corollary. *The T-orbits of symplectic leaves within $\mathcal{O}_t^{m,n}$ are precisely the sets $\mathcal{P}_\sigma^t$ (recall (5.9)) for $\sigma \in \Sigma_t^{m,n}$ (recall (5.13)).*

Proof. Equation (5.3), Corollary 5.10, and Theorem 5.11. □

5.13. Example. We recalculate Example 4.5 from the viewpoint of Theorem 5.11. Here $m = n = 3$ and $t = 1$. Via the proof of Proposition 5.9, one finds that the unique 4-tuple $\sigma = (y, v, z, u) \in \Sigma_1^{3,3}$ such that $w_\circ^6\phi(\sigma) = w$ is given by

$$(y, v, z, u) = \left(\begin{bmatrix} 0&1&0\\0&0&1\\1&0&0\end{bmatrix}, \begin{bmatrix} 1&0&0\\0&0&1\\0&1&0\end{bmatrix}, \begin{bmatrix} 1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \begin{bmatrix} 0&1&0\\0&0&1\\1&0&0\end{bmatrix}\right).$$

Since S_1^1 consists only of the identity, Theorem 5.11 yields

$$(5.38)\quad \mathcal{P}_w = \mathcal{P}_\sigma^1 = \big(B_3^+ y B_3^+ \cap B_3^- z\big).I_1^{3,3}.\big(B_3^- u^{-1} B_3^- \cap v^{-1} B_3^+\big).$$

It follows from Proposition 4.1 that

$$B_3^+ y B_3^+ = \{x \in GL_3 \mid x_{31} \neq 0 \text{ and } \operatorname{rank}(x_{[2,3;1,2]}) = 1\},$$

and consequently (since z is the identity)

$$(5.39)\quad B_3^+ y B_3^+ \cap B_3^- z = \left\{ x \in \begin{bmatrix} \mathbb{C}^\times & 0 & 0 \\ \mathbb{C}^\times & \mathbb{C}^\times & 0 \\ \mathbb{C}^\times & \mathbb{C}^\times & \mathbb{C}^\times \end{bmatrix} \;\middle|\; \operatorname{rank}(x_{[2,3;1,2]}) = 1 \right\}.$$

On the other hand,

$$B_3^- u^{-1} B_3^- = \{x \in GL_3 \mid x_{13} \neq 0 \text{ and } \operatorname{rank}(x_{[1,2;2,3]}) = 1\},$$

and so

$$B_3^- u^{-1} B_3^- \cap v^{-1} B_3^+ = B_3^- u^{-1} B_3^- \cap \begin{bmatrix} \mathbb{C}^\times & \mathbb{C} & \mathbb{C} \\ 0 & 0 & \mathbb{C}^\times \\ 0 & \mathbb{C}^\times & \mathbb{C} \end{bmatrix} = \begin{bmatrix} \mathbb{C}^\times & 0 & \mathbb{C}^\times \\ 0 & 0 & \mathbb{C}^\times \\ 0 & \mathbb{C}^\times & \mathbb{C} \end{bmatrix}. \tag{5.40}$$

We conclude from (5.38), (5.39), and (5.40) that

$$\mathcal{P}_w = \begin{bmatrix} \mathbb{C}^\times & 0 & 0 \\ \mathbb{C}^\times & 0 & 0 \\ \mathbb{C}^\times & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbb{C}^\times & 0 & \mathbb{C}^\times \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \left\{ x \in \begin{bmatrix} \mathbb{C}^\times & 0 & \mathbb{C}^\times \\ \mathbb{C}^\times & 0 & \mathbb{C}^\times \\ \mathbb{C}^\times & 0 & \mathbb{C}^\times \end{bmatrix} \;\middle|\; \operatorname{rank}(x) = 1 \right\},$$

as calculated in Example 4.5.

Next, we offer an example in which the union in (5.19) runs over two disjoint nonempty sets.

5.14. Example. Define $\sigma = (y, v, z, u) \in \Sigma_2^{3,3}$ as follows:

$$(y, v, z, u) = \left(\begin{bmatrix} 0&0&1 \\ 1&0&0 \\ 0&1&0 \end{bmatrix}, \begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix}, \begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix}, \begin{bmatrix} 0&0&1 \\ 1&0&0 \\ 0&1&0 \end{bmatrix} \right).$$

The nontrivial element of S_2^1 can be given as $\tau = \begin{bmatrix} 0&1&0 \\ 1&0&0 \\ 0&0&1 \end{bmatrix}$, and we observe that $z\tau \leq y$ and $v\tau^{-1} \leq u$. Next, we calculate that

$$B_3^+ y B_3^+ = \{x \in GL_3 \mid x_{31} = 0;\ x_{21}, x_{32} \neq 0\}$$
$$B_3^- u^{-1} B_3^- = \{x \in GL_3 \mid x_{13} = 0;\ x_{12}, x_{23} \neq 0\},$$

and consequently

$$B_3^+ y B_3^+ \cap B_3^- z = \begin{bmatrix} \mathbb{C}^\times & 0 & 0 \\ \mathbb{C}^\times & \mathbb{C}^\times & 0 \\ 0 & \mathbb{C}^\times & \mathbb{C}^\times \end{bmatrix} \qquad B_3^- u^{-1} B_3^- \cap v^{-1} B_3^+ = \begin{bmatrix} \mathbb{C}^\times & \mathbb{C}^\times & 0 \\ 0 & \mathbb{C}^\times & \mathbb{C}^\times \\ 0 & 0 & \mathbb{C}^\times \end{bmatrix}$$
$$B_3^+ y B_3^+ \cap B_3^- z\tau = \begin{bmatrix} 0 & \mathbb{C}^\times & 0 \\ \mathbb{C}^\times & \mathbb{C} & 0 \\ 0 & \mathbb{C}^\times & \mathbb{C}^\times \end{bmatrix} \quad \tau^{-1} B_3^- u^{-1} B_3^- \cap v^{-1} B_3^+ = \begin{bmatrix} \mathbb{C}^\times & \mathbb{C} & \mathbb{C}^\times \\ 0 & \mathbb{C}^\times & 0 \\ 0 & 0 & \mathbb{C}^\times \end{bmatrix}.$$

Thus, we find that

$$\big(B_3^+ y B_3^+ \cap B_3^- z\big).I_2^{3,3}.\big(B_3^- u^{-1} B_3^- \cap v^{-1} B_3^+\big)$$
$$= \begin{bmatrix} \mathbb{C}^\times & 0 & 0 \\ \mathbb{C}^\times & \mathbb{C}^\times & 0 \\ 0 & \mathbb{C}^\times & 0 \end{bmatrix} \begin{bmatrix} \mathbb{C}^\times & \mathbb{C}^\times & 0 \\ 0 & \mathbb{C}^\times & \mathbb{C}^\times \\ 0 & 0 & 0 \end{bmatrix} = \left\{ x \in \begin{bmatrix} \mathbb{C}^\times & \mathbb{C}^\times & 0 \\ \mathbb{C}^\times & \mathbb{C} & \mathbb{C}^\times \\ 0 & \mathbb{C}^\times & \mathbb{C}^\times \end{bmatrix} \;\middle|\; \operatorname{rank}(x) = 2 \right\}$$

$$\big(B_3^+ y B_3^+ \cap B_3^- z\tau\big).I_2^{3,3}.\big(\tau^{-1} B_3^- u^{-1} B_3^- \cap v^{-1} B_3^+\big)$$
$$= \begin{bmatrix} 0 & \mathbb{C}^\times & 0 \\ \mathbb{C}^\times & \mathbb{C} & 0 \\ 0 & \mathbb{C}^\times & 0 \end{bmatrix} \begin{bmatrix} \mathbb{C}^\times & \mathbb{C} & \mathbb{C}^\times \\ 0 & \mathbb{C}^\times & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mathbb{C}^\times & 0 \\ \mathbb{C}^\times & \mathbb{C} & \mathbb{C}^\times \\ 0 & \mathbb{C}^\times & 0 \end{bmatrix}.$$

The union of these two disjoint sets equals $\mathcal{P}_\sigma^t$.

6. Row- and column-echelon forms

We show that, up to Zariski closure, the T-orbits of symplectic leaves in $M_{m,n}$ are matrix products of orbits with specific row- and column-echelon forms. Further, the quasi-affine varieties of matrices with fixed row-echelon (or column-echelon) forms are unions of orbits of symplectic leaves of a particularly nice form. Throughout the section, overbars will denote Zariski closures within matrix varieties. As in Section 5, we fix the positive integers m and n as well as a nonnegative integer $t \le \min\{m,n\}$, and we concentrate on T-orbits of symplectic leaves within $\mathcal{O}_t^{m,n}$ (recall (5.3)).

Recall (§3.9) that the action of T on $M_{m,n}$ is given by viewing $T = T_m \times T_n$ and letting $(a,b).x = axb^{-1}$ for $a \in T_m$, $b \in T_n$, and $x \in M_{m,n}$. We shall use the analogous actions of $T_m \times T_t$ and $T_t \times T_n$ on $M_{m,t}$ and $M_{t,n}$, respectively.

6.1. Theorem. *Let $w \in S_N^{\ge(w_\circ^n, w_\circ^m)}[t]$ (recall (5.2)), write $w = w_\circ^{m+n}\phi(\sigma)$ for a unique $\sigma = (y,v,z,u) \in \Sigma_t^{m,n}$ (recall (5.13)), and set*

$$\text{(6.1)} \qquad \begin{aligned} \mathcal{C}_{y,z} &= \big(B_m^+ y B_m^+ \cap B_m^- z\big).I_t^{m,t} \subseteq M_{m,t} \\ \mathcal{R}_{u,v} &= I_t^{t,n}.\big(B_n^- u^{-1} B_n^- \cap v^{-1} B_n^+\big) \subseteq M_{t,n}. \end{aligned}$$

Then $\mathcal{C}_{y,z}$ (respectively, $\mathcal{R}_{u,v}$) is a $(T_m \times T_t)$-orbit (respectively, $(T_t \times T_n)$-orbit) of symplectic leaves within $M_{m,t}$ (respectively, $M_{t,n}$), and

$$\text{(6.2)} \qquad \mathcal{C}_{y,z}.\mathcal{R}_{u,v} \subseteq \mathcal{P}_w \subseteq \overline{\mathcal{C}_{y,z}.\mathcal{R}_{u,v}}.$$

In particular, $\overline{\mathcal{P}}_w = \overline{\mathcal{C}_{y,z}.\mathcal{R}_{u,v}}$.

Proof. We have $\mathcal{C}_{y,z}.\mathcal{R}_{u,v} \subseteq \mathcal{P}_w$ by Theorem 5.11 (take $\tau = 1$ in (5.19)).

Next, viewing $(y,1,z,1)$ as an element of $\Sigma_t^{m,t}$, we see by Theorem 5.11 that

$$\text{(6.3)} \qquad \mathcal{P}^t_{(y,1,z,1)} = \big(B_m^+ y B_m^+ \cap B_m^- z\big).I_t^{m,t}.(B_t^- \cap B_t^+) = \big(B_m^+ y B_m^+ \cap B_m^- z\big).I_t^{m,t} = \mathcal{C}_{y,z}.$$

Thus, $\mathcal{C}_{y,z}$ is a $(T_m \times T_t)$-orbit of symplectic leaves in $M_{m,t}$. Similarly, $\mathcal{R}_{u,v}$ is a $(T_t \times T_n)$-orbit of symplectic leaves in $M_{t,n}$. In particular, it follows that their closures $\overline{\mathcal{C}}_{y,z}$ and $\overline{\mathcal{R}}_{u,v}$ are Poisson subvarieties of $M_{m,t}$ and $M_{t,n}$, stable under the respective tori $T_m \times T_t$ and $T_t \times T_n$.

Let $\mu : M_{m,t} \times M_{t,n} \to M_{m,n}$ denote the morphism given by matrix multiplication, and observe that μ is a Poisson map. Since $\overline{\mathcal{C}_{y,z} \times \mathcal{R}_{u,v}} = \overline{\mathcal{C}}_{y,z} \times \overline{\mathcal{R}}_{u,v}$ (e.g., [**25**, Corollary to Theorem 28, p. 45]), we have $\mu\big(\overline{\mathcal{C}}_{y,z} \times \overline{\mathcal{R}}_{u,v}\big) \subseteq \overline{\mathcal{C}_{y,z}.\mathcal{R}_{u,v}}$. Moreover, as $\overline{\mathcal{C}}_{y,z} \times \overline{\mathcal{R}}_{u,v}$ is a closed Poisson subvariety of $M_{m,t} \times M_{t,n}$, the closure Z of $\mu\big(\overline{\mathcal{C}}_{y,z} \times \overline{\mathcal{R}}_{u,v}\big)$ is a Poisson subvariety of $M_{m,n}$, and $Z \subseteq \overline{\mathcal{C}_{y,z}.\mathcal{R}_{u,v}}$. Note also that if the action of $T_m \times T_t \times T_t \times T_n$ on $M_{m,t} \times M_{t,n}$ is restricted to $T_m \times \langle 1 \rangle \times \langle 1 \rangle \times T_n \cong T$, then μ is T-equivariant. Since $\mathcal{C}_{y,z} \times \mathcal{R}_{u,v}$ is T-stable, it follows that $\mu\big(\overline{\mathcal{C}}_{y,z} \times \overline{\mathcal{R}}_{u,v}\big)$ is T-stable, and thus Z is a T-stable subvariety of $M_{m,n}$.

Now $\mathcal{C}_{y,z}.\mathcal{R}_{u,v} \subseteq \mathcal{P}_w \cap Z$, so that $\mathcal{P}_w \cap Z$ is nonempty. Choose $a \in \mathcal{P}_w \cap Z$ and let $\mathcal{L}$ denote the symplectic leaf containing a; then $\mathcal{P}_w = T_m.\mathcal{L}.T_n$. On the other hand, as Z is a T-stable closed Poisson subvariety of $M_{m,n}$, it is a union of T-orbits of symplectic leaves. Consequently, $T_m.\mathcal{L}.T_n \subseteq Z$, and therefore $\mathcal{P}_w \subseteq Z \subseteq \overline{\mathcal{C}_{y,z}.\mathcal{R}_{u,v}}$. □

6.2. Remark. Theorem 6.1 can be interpreted as a tensor product result concerning prime Poisson ideals in coordinate rings, as follows. First, note that the ideal P_w defining the T-stable closed Poisson subvariety $\overline{\mathcal{P}}_w \subseteq M_{m,n}$ is a T-stable Poisson ideal in $\mathcal{O}(M_{m,n})$, where the action of T on $\mathcal{O}(M_{m,n})$ by automorphisms is induced from the T-action on $M_{m,n}$ in the usual way. It can be shown that P_w is a prime ideal, and that all T-stable prime Poisson ideals of $\mathcal{O}(M_{m,n})$ have this form. Similarly, the defining ideal of $\overline{\mathcal{C}}_{y,z}$ (respectively, $\overline{\mathcal{R}}_{u,v}$) is a $(T_m \times T_t)$-stable (respectively, $(T_t \times T_n)$-stable) prime Poisson ideal $P_{y,z} \subseteq \mathcal{O}(M_{m,t})$ (respectively, $P_{u,v} \subseteq \mathcal{O}(M_{t,n})$). The statement that $\overline{\mathcal{P}}_w = \overline{\mathcal{C}_{y,z}.\mathcal{R}_{u,v}}$ is equivalent to the statement that P_w equals the kernel of the homomorphism

$$\mathcal{O}(M_{m,n}) \xrightarrow{\mu^*} \mathcal{O}(M_{m,t}) \otimes \mathcal{O}(M_{t,n}) \xrightarrow{\text{quo}\otimes\text{quo}} \big(\mathcal{O}(M_{m,t})/P_{y,z}\big) \otimes \big(\mathcal{O}(M_{t,n})/P_{u,v}\big),$$

where μ^* is the comorphism of the matrix multiplication map from $M_{m,t} \times M_{t,n}$ to $M_{m,n}$. Consequently,

$$P_w = (\mu^*)^{-1}\big((P_{y,z} \otimes \mathcal{O}(M_{t,n})) + (\mathcal{O}(M_{m,t}) \otimes P_{u,v})\big).$$

Such tensor product decompositions were proved to hold for T-stable prime ideals in the generic quantized coordinate ring of $n \times n$ matrices, $\mathcal{O}_q(M_n)$, by Goodearl and Lenagan [**13**, Theorem 3.5]. Their development can be used, *mutatis mutandis* (e.g., by replacing additive commutators with Poisson brackets), to prove results of the type above. (While that route only gives information about closures of T-orbits of symplectic leaves in $M_{m,n}$, it does have the advantage of working over an arbitrary base field of characteristic zero.)

6.3. Column-echelon and row-echelon forms. We next wish to observe that the sets $\mathcal{C}_{y,z}$ and $\mathcal{R}_{u,v}$ in (6.1) consist of matrices with a single column-echelon (respectively, row-echelon) form. Note that to specify a particular column-echelon form for rank t matrices in $M_{m,t}$, we just need to specify the rows in which the highest nonzero entries of columns $1, \dots, t$ occur; column-echelon form requires that the list of these row indices is strictly increasing.

Let Inc_t^m denote the set of all strictly increasing sequences in $\{1, \dots, m\}$ of length t, that is,

$$\mathrm{Inc}_t^m = \{\boldsymbol{e} = (e_1, \dots, e_t) \in \{1, \dots, m\}^t \mid e_1 < \cdots < e_t\},$$

and define Inc_t^n analogously. For $\boldsymbol{r} \in \mathrm{Inc}_t^m$ and $\boldsymbol{c} \in \mathrm{Inc}_t^n$, define

$$\begin{aligned} \mathcal{C}_{\boldsymbol{r}}^m &= \{a \in M_{m,t} \mid a_{r_j j} \neq 0 \text{ for } j = 1, \dots, t \text{ and } a_{ij} = 0 \text{ when } i < r_j\} \\ \mathcal{R}_{\boldsymbol{c}}^n &= \{a \in M_{t,n} \mid a_{i c_i} \neq 0 \text{ for } i = 1, \dots, t \text{ and } a_{ij} = 0 \text{ when } j < c_i\}. \end{aligned} \tag{6.4}$$

For example,

$$\mathcal{R}_{(2,4,5)}^6 = \begin{bmatrix} 0 & \mathbb{C}^\times & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} \\ 0 & 0 & 0 & \mathbb{C}^\times & \mathbb{C} & \mathbb{C} \\ 0 & 0 & 0 & 0 & \mathbb{C}^\times & \mathbb{C} \end{bmatrix},$$

the variety of 3×6 matrices in row-echelon form with pivot columns 2, 4, and 5.

Consider a permutation $z \in S_m^{S_t^1}$. Then $z(1) < \cdots < z(t)$, whence $\boldsymbol{r} = (z(1), \ldots, z(t))$ lies in Inc_t^m. Given an accompanying $y \in S_m^{S_{m-t}^2}$ with $z \leq y$, we thus see that

$$\mathcal{C}_{y,z} \subseteq B_m^- z.I_t^{m,t} = \mathcal{C}_{\boldsymbol{r}}^m. \tag{6.5}$$

Similarly, if $v \in S_n^{S_t^1}$ and $u \in S_n^{S_{n-t}^2}$ with $v \leq u$, then $\boldsymbol{c} = (v(1), \ldots, v(t)) \in \mathrm{Inc}_t^n$ and

$$\mathcal{R}_{u,v} \subseteq I_t^{t,n}.v^{-1} B_n^+ = \mathcal{R}_{\boldsymbol{c}}^n. \tag{6.6}$$

The inclusions (6.5) and (6.6) exhibit orbits of symplectic leaves contained within $\mathcal{C}_{\boldsymbol{r}}^m$ and $\mathcal{R}_{\boldsymbol{c}}^n$, indexed by the following sets.

For $\boldsymbol{r} \in \mathrm{Inc}_t^m$ and $\boldsymbol{c} \in \mathrm{Inc}_t^n$, define

$$\begin{aligned} \Sigma_{\boldsymbol{r}}^{m,t} &= \{(y,z) \in S_m^{S_{m-t}^2} \times S_m^{S_t^1} \mid z \leq y \text{ and } z(j) = r_j \text{ for } j = 1, \ldots, t\} \\ \Sigma_{\boldsymbol{c}}^{t,n} &= \{(u,v) \in S_n^{S_{n-t}^2} \times S_n^{S_t^1} \mid v \leq u \text{ and } v(i) = c_i \text{ for } i = 1, \ldots, t\}. \end{aligned} \tag{6.7}$$

(There is no ambiguity in this notation in the one overlapping case, namely when $m = t = n$ and $\boldsymbol{r} = \boldsymbol{c}$, since then $\boldsymbol{r} = \boldsymbol{c} = (1, 2, \ldots, t)$ and so $z = v = 1$.) We now show that the orbits of symplectic leaves indexed by $\Sigma_{\boldsymbol{r}}^{m,t}$ and $\Sigma_{\boldsymbol{c}}^{t,n}$ cover $\mathcal{C}_{\boldsymbol{r}}^m$ and $\mathcal{R}_{\boldsymbol{c}}^n$, as follows.

6.4. Theorem. *If $\boldsymbol{r} \in \mathrm{Inc}_t^m$, then $\mathcal{C}_{\boldsymbol{r}}^m$ is a disjoint union of $(T_m \times T_t)$-orbits of symplectic leaves of $M_{m,t}$, indexed by $\Sigma_{\boldsymbol{r}}^{m,t}$, as follows:*

$$\mathcal{C}_{\boldsymbol{r}}^m = \bigsqcup_{(y,z) \in \Sigma_{\boldsymbol{r}}^{m,t}} \big(B_m^+ y B_m^+ \cap B_m^- z\big).I_t^{m,t}. \tag{6.8}$$

Proof. Recall from (6.3) that $\mathcal{C}_{y,z} = \mathcal{P}_{(y,1,z,1)}^t$ for $(y,z) \in \Sigma_{\boldsymbol{r}}^{m,t}$, where each $(y,1,z,1)$ is viewed as an element of $\Sigma_t^{m,t}$. Hence, the sets $\mathcal{C}_{y,z}$ are $(T_m \times T_t)$-orbits of symplectic leaves of $M_{m,t}$, and they are pairwise disjoint. Further, (6.5) shows that each such $\mathcal{C}_{y,z}$ is contained in $\mathcal{C}_{\boldsymbol{r}}^m$. Thus, $\mathcal{C}_{\boldsymbol{r}}^m$ contains the disjoint union displayed in (6.8), and it only remains to prove equality.

Given $a \in \mathcal{C}_{\boldsymbol{r}}^m$, note that $\mathrm{rank}(a) = t$. By Theorem 3.9 and equation (5.3), $a \in \mathcal{P}_w$ for some $w \in S_{m+t}^{\geq (w_\circ^t, w_\circ^m)}[t]$. Now apply Corollary 5.10 and Theorem 5.11 (with $n = t$), to get $w = w_\circ^{m+t} \phi(\sigma)$ for some $\sigma = (y,v,z,u) \in \Sigma_t^{m,t}$ and $\mathcal{P}_w = \mathcal{P}_\sigma^t$. Note that since $v \in S_t^{S_t^1}$, it must be the identity. Write $w = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$ as in (4.1) (with $n = t$), and observe from (5.18) that $w_\circ^m w_{12}^{\mathrm{tr}} w_\circ^t = z I_t^{m,t} u^{-1}$. Hence, Corollary 4.3 implies that

$$\mathcal{P}_w \subseteq B_m^- z I_t^{m,t} u^{-1} B_t^-. \tag{6.9}$$

Let $s \in M_{m,t}$ be the (unique) partial permutation matrix such that $s(j) = r_j$ for $j = 1, \ldots, t$. Then

$$\mathcal{C}_{\boldsymbol{r}}^m = B_m^- s \subseteq B_m^- s B_t^-. \tag{6.10}$$

From (6.9) and (6.10), we obtain $B_m^- z I_t^{m,t} u^{-1} B_t^- \cap B_m^- s B_t^- \neq \varnothing$. Since $zI_t^{m,t}u^{-1}$ and s are partial permutation matrices, it follows that $zI_t^{m,t}u^{-1} = s$. (See §7.1 below for more detail.) In particular, $su(j) = z(j)$ for $j = 1, \dots, t$. Since $s(1) < \cdots < s(t)$ and $z(1) < \cdots < z(t)$, it follows that $u(1) < \cdots < u(t)$. But u is a permutation in S_t, and so $u = 1$. Thus, $\sigma = (y, 1, z, 1)$, whence $\mathcal{P}_w = \mathcal{P}_\sigma^t = \mathcal{C}_{y,z}$ by (6.3). Moreover, $z(j) = s(j) = r_j$ for $j = 1, \dots, t$, whence $(y, z) \in \Sigma_{\boldsymbol{r}}^{m,t}$.

Therefore $a \in \mathcal{C}_{y,z} \subseteq \mathcal{C}_{\boldsymbol{r}}^m$, and the proof is complete. □

6.5. Corollary. *If $\boldsymbol{c} \in \mathrm{Inc}_t^n$, then $\mathcal{R}_{\boldsymbol{c}}^n$ is a disjoint union of $(T_t \times T_n)$-orbits of symplectic leaves of $M_{t,n}$, indexed by $\Sigma_{\boldsymbol{c}}^{t,n}$, as follows:*

$$\mathcal{R}_{\boldsymbol{c}}^n = \bigsqcup_{(u,v) \in \Sigma_{\boldsymbol{c}}^{t,n}} I_t^{t,n}.\big(B_n^- u^{-1} B_n^- \cap v^{-1} B_n^+\big). \tag{6.11}$$

Proof. Note that matrix transposition provides a Poisson isomorphism from $\mathcal{C}_{\boldsymbol{c}}^n$ onto $\mathcal{R}_{\boldsymbol{c}}^n$. Moreover, this map sends $(T_n \times T_t)$-orbits to $(T_t \times T_n)$-orbits. Note also that the transpose of a permutation matrix is its inverse. Therefore, (6.11) follows from (6.8). □

7. Generalized double Bruhat cells

7.1. Bruhat decompositions in $M_{m,n}$. In the theory of reductive algebraic monoids (cf. [**26**]), the role of the Weyl group is taken over by what is now called the *Renner monoid*. In the case of the algebraic monoid M_n, the Renner monoid is naturally identified with the monoid of all $n \times n$ *partial permutation matrices*, that is, 0, 1-matrices with at most one nonzero entry in each row or column [**26**, pp. 326-7]. The Bruhat decomposition of a reductive algebraic monoid M corresponding to any Borel subgroup B of the group of invertible elements of M partitions M into Bruhat cells BwB where w runs through the Renner monoid [**26**, Corollary 5.8]. Thus, for any Borel subgroup B of GL_n, the monoid M_n is a disjoint union of Bruhat cells BwB, where w runs through the partial permutation matrices in M_n.

As is well known and easily checked, the above Bruhat decomposition of M_n holds for the rectangular matrix variety $M_{m,n}$ as well. Namely, if $\widetilde{S}_{m,n}$ denotes the set of partial permutations in $M_{m,n}$, then

$$M_{m,n} = \bigsqcup_{w \in \widetilde{S}_{m,n}} B_m^+ w B_n^+ = \bigsqcup_{w \in \widetilde{S}_{m,n}} B_m^- w B_n^-. \tag{7.1}$$

Consequently, $M_{m,n}$ is also the disjoint union of the *generalized double Bruhat cells*

$$\mathcal{B}^{w_1,w_2} = B_m^+ w_1 B_n^+ \cap B_m^- w_2 B_n^- \tag{7.2}$$

for $w_1, w_2 \in \widetilde{S}_{m,n}$. The latter generalize the standard double Bruhat cells for GL_m, which are obtained when $n = m$ and $w_1, w_2 \in S_m \subset \widetilde{S}^{m,m}$.

Each double Bruhat cell $\mathcal{B}^{w_1,w_2}$ is a locally closed subset of $M_{m,n}$ because it is an intersection of two orbits of algebraic groups. As is surely well known, $\mathcal{B}^{w_1,w_2}$ is also smooth and irreducible, but we could not locate a reference in the literature. We indicate in Proposition 7.2 and Theorem 7.4 how these properties follow from our results.

7.2. Proposition. *Let $w_1, w_2 \in \widetilde{S}_{m,n}$.*

(a) *The generalized double Bruhat cell $\mathcal{B}^{w_1,w_2} = B_m^+ w_1 B_n^+ \cap B_m^- w_2 B_n^-$ is nonempty if and only if there exists some $w \in S_N^{\geq (w_\circ^n, w_\circ^m)}$ of the form $w = \begin{bmatrix} * & w_\circ^n w_2^{\mathrm{tr}} w_\circ^m \\ w_1 & * \end{bmatrix}$.*

(b) *When $\mathcal{B}^{w_1,w_2}$ is nonempty, it is a smooth locally closed subvariety of $M_{m,n}$ which is in addition a T-stable complete Poisson subvariety. In fact,*

$$\mathcal{B}^{w_1,w_2} = \bigsqcup \left\{ \mathcal{P}_w \;\middle|\; w \in \begin{bmatrix} M_n & w_\circ^n w_2^{\mathrm{tr}} w_\circ^m \\ w_1 & M_m \end{bmatrix} \cap S_N^{\geq (w_\circ^n, w_\circ^m)} \right\}. \tag{7.3}$$

Proof. The smoothness of $\mathcal{B}^{w_1,w_2}$ in the case when it is nonempty can be obtained as follows. First, note that the Bruhat cells $B_m^+ w_1 B_n^+$ and $B_m^- w_2 B_n^-$ are smooth, because they are orbits of the algebraic groups $B_m^+ \times B_n^+$ and $B_m^- \times B_n^-$. Secondly, $\mathcal{B}^{w_1,w_2}$ lies within a single $GL_m \times GL_n$ orbit $\mathcal{O}_t^{m,n}$ in $M_{m,n}$ for the action $(g_1, g_2).m = g_1 m g_2^{-1}$, cf. §5.2. Now the intersection of $B_m^+ w_1 B_n^+$ and $B_m^- w_2 B_n^-$ in $\mathcal{O}_t^{m,n}$ is transversal because the Lie algebras of B_m^+ and B_m^- span $\mathfrak{gl}_m$, hence $\mathcal{B}^{w_1,w_2}$ is smooth.

The rest of the proposition follows from Corollary 4.3 and Theorem 3.9. □

We will describe the partition (7.3) in terms of the T-orbits of symplectic leaves $\mathcal{P}_\sigma^t$ (recall (5.12)) more explicitly in Theorem 7.4 below. Additional criteria for $\mathcal{B}^{w_1,w_2}$ to be nonempty are given in Theorem 7.4 and Corollary 7.7.

For the remainder of this section,

Fix a nonnegative integer $t \leq \min\{m, n\}$,

and let $\widetilde{S}_{m,n}^t$ denote the subset of $\widetilde{S}_{m,n}$ consisting of partial permutations of rank t.

7.3. Lemma. *Every partial permutation in $\widetilde{S}_{m,n}^t$ can be uniquely represented in the form*

$$y I_t^{m,n} v^{-1} \tag{7.4}$$

for some $y \in S_m^{S_{m-t}^2}$ and $v \in S_n^{S_t^1 S_{n-t}^2}$, and also uniquely in the form

$$z I_t^{m,n} u^{-1} \tag{7.5}$$

for some $z \in S_m^{S_t^1 S_{m-t}^2}$ and $u \in S_n^{S_{n-t}^2}$.

Proof. The second statement follows from the first by noting that $S_m^{S_{m-t}^2} = S_m^{S_t^1 S_{m-t}^2} S_t^1$ and $S_n^{S_{n-t}^2} = S_n^{S_t^1 S_{n-t}^2} S_t^1$, and that $\tau I_t^{m,n} = I_t^{m,n} \tau$ for all $\tau \in S_t^1 \subseteq S_m$, S_n.

To prove the first statement, we first show that each element of $\widetilde{S}_{m,n}^t$ can be represented in the form (7.4). This follows from the facts that

$$\begin{aligned} \widetilde{S}_{m,n}^t &= S_m I_t^{m,n} S_n \\ \tau_1 I_t^{m,n} &= I_t^{m,n} \tau_2 = I_t^{m,n} && \text{for all } \tau_1 \in S_{m-t}^2 \text{ and } \tau_2 \in S_{n-t}^2 \\ \tau I_t^{m,n} &= I_t^{m,n} \tau && \text{for all } \tau \in S_t^1 \subseteq S_m,\ S_n. \end{aligned}$$

The lemma will now follow if we prove that the sets $\widetilde{S}^t_{m,n}$ and $S_m^{S^2_{m-t}} \times S_n^{S^1_t S^2_{n-t}}$ have the same number of elements. The cardinality of the second set is $\frac{m!}{(m-t)!}\frac{n!}{t!(n-t)!} = t!\binom{m}{t}\binom{n}{t}$ because each coset in S_m/S^2_{m-t} or $S_n/S^1_t S^2_{n-t}$ has a unique minimal length representative. Observe that a partial permutation $w \in \widetilde{S}^t_{m,n}$ is uniquely defined by prescribing its domain $\operatorname{dom} w$, range $\operatorname{rng} w$ (both of cardinality t), and a bijective mapping from $\operatorname{dom} w$ to $\operatorname{rng} w$. Therefore the cardinality of $\widetilde{S}^t_{m,n}$ is $\binom{m}{t}\binom{n}{t}t!$. $\square$

7.4. Theorem. *Fix two partial permutations $w_1, w_2 \in \widetilde{S}^t_{m,n}$ with (unique) decompositions*

$$w_1 = yI_t^{m,n}v^{-1} \qquad\qquad w_2 = zI_t^{m,n}u^{-1} \tag{7.6}$$

for some $y \in S_m^{S^2_{m-t}}$, $v \in S_n^{S^1_t S^2_{n-t}}$, $z \in S_m^{S^1_t S^2_{m-t}}$, and $u \in S_n^{S^2_{n-t}}$ (cf. Lemma 7.3). Then the following hold.

(a) *The generalized double Bruhat cell $\mathcal{B}^{w_1,w_2} = B^+_m w_1 B^+_n \cap B^-_m w_2 B^-_n$ is nonempty if and only if $z \le y$ and $v \le u$.*

If $z \le y$ and $v \le u$, then:

(b) *The partition of $\mathcal{B}^{w_1,w_2}$ into T-orbits of symplectic leaves is given by*

$$\mathcal{B}^{w_1,w_2} = \bigsqcup \left\{ \mathcal{P}^t_{(y,v\tau_2,z\tau_1,u)} \;\middle|\; \begin{array}{l} \tau_1 \in S^2_{m-t} \subseteq S_m,\ z\tau_1 \le y \\ \tau_2 \in S^2_{n-t} \subseteq S_n,\ v\tau_2 \le u \end{array} \right\}. \tag{7.7}$$

(c) *The T-orbit of symplectic leaves $\mathcal{P}^t_{(y,v,z,u)}$ is an open and dense subset of $\mathcal{B}^{w_1,w_2}$.*

(d) *$\mathcal{B}^{w_1,w_2}$ is a smooth irreducible locally closed subvariety of $M_{m,n}$.*

For the proof of Theorem 7.4 we will need two lemmas. Recall the set $\Sigma^{m,n}_t$ from (5.13).

7.5. Lemma. *For any $\sigma = (y, v, z, u) \in \Sigma^{m,n}_t$, we have*

$$\mathcal{P}^t_\sigma \subseteq B^+ yI_t^{m,n}v^{-1}B^+ \cap B^- zI_t^{m,n}u^{-1}B^-.$$

Proof. The lemma follows from Lemma 5.3 because, for r_1, r_2 as in (5.11),

$$r_1 I_t^{m,n} r_2^{-1} = b_1^+ yI_t^{m,n}v^{-1}(b_2^+)^{-1} \in B^+ yI_t^{m,n}v^{-1}B^+$$

and

$$\begin{aligned} r_1 I_t^{m,n} r_2^{-1} &= b_3^- z \begin{bmatrix} a & b \\ 0 & I_{m-t} \end{bmatrix} I_t^{m,n} \begin{bmatrix} a & 0 \\ c & I_{n-t} \end{bmatrix}^{-1} u^{-1}(b_4^-)^{-1} \\ &= b_3^- zI_t^{m,n}u^{-1}(b_4^-)^{-1} \in B^- zI_t^{m,n}u^{-1}B^-. \qquad \square \end{aligned}$$

7.6. Lemma. *Set*

$$\widetilde{\Sigma}_t^{m,n} = \{(y, v_0, z_0, u) \in S_m^{S^2_{m-t}} \times S_n^{S^1_t S^2_{n-t}} \times S_m^{S^1_t S^2_{m-t}} \times S_n^{S^2_{n-t}} \mid z_0 \le y,\ v_0 \le u\}.$$

Then

$$\Sigma_t^{m,n} = \left\{ (y, v_0\tau_2, z_0\tau_1, u) \;\middle|\; \begin{array}{c} (y, v_0, z_0, u) \in \widetilde{\Sigma}_t^{m,n},\ \tau_1 \in S^2_{m-t} \subseteq S_m, \\ \tau_2 \in S^2_{n-t} \subseteq S_n,\ z_0\tau_1 \le y,\ v_0\tau_2 \le u \end{array} \right\}. \tag{7.8}$$

Proof. It is clear that every element of $\Sigma_t^{m,n}$ has the form $(y, v_0\tau_2, z_0\tau_1, u)$ for some

$$(y, v_0, z_0, u) \in S_m^{S^2_{m-t}} \times S_n^{S^1_t S^2_{n-t}} \times S_m^{S^1_t S^2_{m-t}} \times S_n^{S^2_{n-t}}$$

and some

$$\tau_1 \in S^2_{m-t} \subseteq S_m \qquad\qquad \tau_2 \in S^2_{n-t} \subseteq S_n$$

such that

$$z_0\tau_1 \le y \qquad\qquad v_0\tau_2 \le u.$$

But $z_0 \in S_m^{S^1_t S^2_{m-t}}$ and $\tau_1 \in S^2_{m-t}$ imply that $z_0 \le z_0\tau_1$ and therefore $z_0 \le y$. Analogously, one obtains that $v_0 \le v_0\tau_2$ and as a consequence of it $v_0 \le u$. Therefore

$$(y, v_0, z_0, u) \in \widetilde{\Sigma}_t^{m,n}.$$

This proves that $\Sigma_t^{m,n}$ is contained in the set on the right hand side of (7.8). The opposite inclusion is straightforward. □

Proof of Theorem 7.4. Combining Lemma 7.6 and Corollary 5.12, one obtains

$$\mathcal{O}_t^{m,n} = \bigsqcup_{(y,v_0,z_0,u) \in \widetilde{\Sigma}_t^{m,n}} \bigsqcup \left\{ \mathcal{P}^t_{(y, v_0\tau_2, z_0\tau_1, u)} \;\middle|\; \begin{array}{c} \tau_1 \in S^2_{m-t} \subseteq S_m,\ z_0\tau_1 \le y \\ \tau_2 \in S^2_{n-t} \subseteq S_n,\ v_0\tau_2 \le u \end{array} \right\}. \tag{7.9}$$

At the same time,

$$\mathcal{O}_t^{m,n} = \bigsqcup_{w_1, w_2 \in \widetilde{S}^t_{m,n}} B^+ w_1 B^+ \cap B^- w_2 B^-. \tag{7.10}$$

From Lemma 7.5, for each T-orbit of leaves on the right hand side of (7.9) one derives:

$$\mathcal{P}^t_{(y, v_0\tau_2, z_0\tau_1, u)} \subseteq B^+ y I_t^{m,n} v_0^{-1} B^+ \cap B^- z_0 I_t^{m,n} u^{-1} B^-.$$

Comparing (7.9) and (7.10) now proves at once parts (a) and (b).

(c) Because of (7.7), it suffices to show that

$$\mathcal{P}^t_{(y,v\tau_2,z\tau_1,u)} \subseteq \overline{\mathcal{P}^t_{(y,v,z,u)}} \tag{7.11}$$

for $\tau_1 \in S^2_{m-t}$ and $\tau_2 \in S^2_{n-t}$ such that $z\tau_1 \le y$ and $v\tau_2 \le u$. Fix such τ_1, τ_2, recall the bijection $\Sigma^{m,n}_t \to S_N^{\ge(w^n_\circ, w^m_\circ)}[t]$ given in Corollary 5.10, and set

$$\overline{w} = w^N_\circ \phi(y,v,z,u) = \begin{bmatrix} w^n_\circ u J^n_t v^{-1} & w^n_\circ u I^{n,m}_t z^{-1} w^m_\circ \\ y I^{m,n}_t v^{-1} & y J^m_t z^{-1} w^m_\circ \end{bmatrix}$$
$$w = w^N_\circ \phi(y,v\tau_2,z\tau_1,u) = \begin{bmatrix} w^n_\circ u J^n_t \tau_2^{-1} v^{-1} & w^n_\circ u I^{n,m}_t z^{-1} w^m_\circ \\ y I^{m,n}_t v^{-1} & y J^m_t \tau_1^{-1} z^{-1} w^m_\circ \end{bmatrix}.$$

By Theorems 5.11 and 3.13, (7.11) is equivalent to $w \le \overline{w}$.

First, note that $w(j) = \overline{w}(j) = n + w_1(j)$ for $j \in v(\{1,\dots,t\})$. Now w and $\overline{w}$ both map $v(\{t+1,\dots,n\})$ bijectively onto $w^n_\circ u(\{t+1,\dots,n\})$, and for $\overline{w}$ this restriction is order-reversing because $u, v \in S_n^{S^2_{n-t}}$. It follows that $w(\{1,\dots,j\}) \le \overline{w}(\{1,\dots,j\})$ for $j = 1,\dots,n$. Similarly, w and $\overline{w}$ agree on $n + w^m_\circ z(\{1,\dots,t\})$, and the restriction of $\overline{w}$ to $n + w^m_\circ z(\{t+1,\dots,m\})$ is order-reversing, from which we conclude that $w(\{1,\dots,j\}) \le \overline{w}(\{1,\dots,j\})$ for $j = n+1,\dots,N$. Therefore $w \le \overline{w}$, as required.

(d) The irreducibility of $\mathcal{B}^{w_1,w_2}$ follows from part (c) since $\mathcal{P}^t_{(y,v,z,u)}$ is irreducible by Theorem 3.9. □

7.7. Corollary. *For partial permutations $w_1, w_2 \in \widetilde{S}^t_{m,n}$, the generalized double Bruhat cell $\mathcal{B}^{w_1,w_2} = B^+_m w_1 B^+_n \cap B^-_m w_2 B^-_n$ is nonempty if and only if*

$$\operatorname{dom}(w_1) \le \operatorname{dom}(w_2) \qquad \textit{and} \qquad \operatorname{rng}(w_1) \ge \operatorname{rng}(w_2) \tag{7.12}$$

(*recall* §3.11).

Proof. Let $w_1 = y I^{m,n}_t v^{-1}$ and $w_2 = z I^{m,n}_t u^{-1}$ for y, v, z, and u as in Theorem 7.4.

If $\mathcal{B}^{w_1,w_2}$ is nonempty, then by the theorem, $z \le y$ and $v \le u$. Hence,

$$\begin{aligned} \operatorname{dom}(w_1) = v(\{1,\dots,t\}) \le u(\{1,\dots,t\}) = \operatorname{dom}(w_2) \\ \operatorname{rng}(w_1) = y(\{1,\dots,t\}) \ge z(\{1,\dots,t\}) = \operatorname{rng}(w_2). \end{aligned} \tag{7.13}$$

Conversely, assume that $\operatorname{dom}(w_1) \le \operatorname{dom}(w_2)$ and $\operatorname{rng}(w_1) \ge \operatorname{rng}(w_2)$, so that (7.13) holds. It follows, as shown in the proof of Lemma 5.7, that $v(\{t+1,\dots,n\}) \ge u(\{t+1,\dots,n\})$. Since $u, v \in S_n^{S^2_{n-t}}$, we obtain $v(j) \ge u(j)$ for $j = t+1,\dots,n$. But then, since $v \in S_n^{S^1_t}$, Lemma 5.7(b) implies that $v \le u$. Similarly, $z \le y$. □

Appendix A. Double coset representatives

A.1. Let G be a complex reductive algebraic group with fixed positive/negative Borel subgroups $B^{\pm}$ and maximal torus $T = B^+ \cap B^-$. Fix a parabolic subgroup P of G, containing a Borel subgroup $B \supset T$ of G with the property that for each simple factor F of G, either $B \cap F = B^+ \cap F$ or $B \cap F = B^- \cap F$.

Denote by L_0 the Levi factor of P containing T and by N the unipotent radical of P. So, we have the the Levi decomposition $P \cong L_0 \ltimes N$. Denote by $\overline{N}$ the unipotent subgroup of G dual to N.

We will assume that L_0 is decomposed as a product of two reductive subgroups

$$L_0 = L_1 \times L_2 \tag{A.1}$$

such that there is an isomorphism

$$\Theta : L_1 \xrightarrow{\cong} L_2 \tag{A.2}$$

with the property that for every simple factor F_1 of L_1,

$$\Theta(F_1 \cap B^{\pm}) = F_2 \cap B^+ \tag{A.3}$$

for some simple factor F_2 of L_2 and an appropriate choice of the sign.

Denote the Weyl group of G by W and the Weyl groups of L_i ($i = 0, 1, 2$) by W_i, considered as subgroups of W. Clearly $W_0 = W_1 \times W_2$. Denote the composition of the projections $P \longrightarrow P/N \cong L_0$ and $L_0 \longrightarrow L_i$ ($i = 1, 2$) by $\pi_i : P \longrightarrow L_i$.

Finally, define the following subgroup of P:

$$R = \{p \in P \mid \Theta\pi_1(p) = \pi_2(p)\}. \tag{A.4}$$

In this Appendix we give a classification of all (B^+, R) double cosets of G. Recall that W^{W_i} denotes the set of (unique) minimal length representatives of cosets from W/W_i, see [**3**, Proposition 2.3.3] for details. For an element $w \in W$, we will denote by $\dot{w}$ a representative of it in the normalizer of T in G.

Theorem. *In the above setting, every (B^+, R) double coset of G is of the form*

$$B^+ \dot{w} R, \quad \textit{for some} \quad w \in W^{W_2}.$$

For distinct $w \in W^{W_2}$, the above double cosets are distinct.

Let us note that in the case when L_1 and L_2 have more than one simple factor, it is possible to obtain R as a subgroup of P in several different ways by changing L_1 and L_2. In such a case, Theorem A.1 produces different sets of representatives for the (B^+, R) double cosets of G. As is clear from Lemma 3.8, sometimes one of these sets has better properties than the others.

For the proof of Theorem A.1, we will need the following lemma.

A.2. Lemma. (a) (Bruhat Lemma) *All* (B^+, P) *double cosets in* G *are uniquely parametrized by* W^{W_0}*, by* $v \in W^{W_0} \mapsto B^+ \dot{v} P$.

(b) *For any* $v \in W^{W_0}$,

$$B^+ \dot{v} = \dot{v} \overline{N}_v B_0^+ N_v$$

where $B_0^+ = B^+ \cap L_0$ *and*

$$N_v = N \cap Ad_{\dot{v}}^{-1}(B^+) \qquad \overline{N}_v = \overline{N} \cap Ad_{\dot{v}}^{-1}(B^+).$$

(c) *There is a bijection of sets*

$$W^{W_0} \times W_1 \to W^{W_2}, \qquad (v, u) \longmapsto vy.$$

(d) *Set* $Q = R \cap L_0 = \{l_1 \Theta(l_1) \mid l_1 \in L_1\}$. *All* (B_0^+, Q) *double cosets of* L_0 *are uniquely parametrized by* W_1*, by* $w_1 \mapsto B_0^+ \dot{w}_1 Q$.

Proof. Part (a) is well known.

Part (b) follows from the well known description of minimal length representatives:

$$W^{W_0} = \{w \in W \mid w(\alpha) \text{ is a positive root for any positive root } \alpha \text{ of } L_0\}.$$

See, e.g., [**3**, Proposition 2.3.3]. Part (c) is a consequence of $W_0 = W_1 \times W_2$.

To prove part (d), we first show that it suffices to establish (d) in the case that

$$\Theta(L_1 \cap B^+) = L_2 \cap B^+. \tag{A.5}$$

For each simple factor F_1 of L_1, the assumption (A.3) can be written in the form

$$\Theta \operatorname{Ad}_{\dot{u}}(F_1 \cap B^+) = F_2 \cap B^+,$$

where u is either the identity or the longest element of the Weyl group of F_1. Hence, there exists an element $u_1 \in W_1$ such that $u_1^2 = 1$ and

$$\Theta \operatorname{Ad}_{\dot{u}_1}(L_1 \cap B^+) = L_2 \cap B^+.$$

The map $\widetilde{\Theta} = \Theta \circ \operatorname{Ad}_{\dot{u}_1}|_{L_1}$ is an isomorphism of L_1 onto L_2, and the subgroup $\widetilde{Q}$ of G obtained by changing Θ to $\widetilde{\Theta}$ in the definition of Q can be written as

$$\widetilde{Q} = \{\tilde{l}_1 \widetilde{\Theta}(\tilde{l}_1) \mid \tilde{l}_1 \in \operatorname{Ad}_{\dot{u}_1}(L_1)\} = \{\operatorname{Ad}_{\dot{u}_1}(l_1)\Theta(l_1) \mid l_1 \in L_1\} = \operatorname{Ad}_{\dot{u}_1}(Q).$$

If (d) holds for $\widetilde{\Theta}$, then, since $W_1 = W_1 u_1$, we may express the result as

$$L_0 = \bigsqcup_{w_1 \in W_1} B_0^+ \dot{w}_1 \dot{u}_1 \widetilde{Q},$$

and consequently

$$L_0 = L_0\dot{u}_1 = \bigsqcup_{w_1 \in W_1} B_0^+ \dot{w}_1 Q,$$

as desired. Thus, we may assume (A.5), as claimed.

Recall the fact that if F_1, F_2 are subgroups of a group C, then the set of $(F_1 \times F_2,\, \Delta(C))$ double cosets of $C \times C$ (where $\Delta(C) \subseteq C \times C$ denotes the diagonal copy of C) is in one to one correspondence with the set of (F_1, F_2) double cosets of C, by $(F_1 \times F_2)(y_1, y_2)\Delta(C) \mapsto F_1 y_1 y_2^{-1} F_2$. If we identify L_0 with $L_1 \times L_1$ via Θ, then Q is identified with $\Delta(L_1)$, and because of (A.5), B_0^+ is identified with $B_1^+ \times B_1^+$, where $B_1^+ = L_1 \cap B^+$. Since the (B_1^+, B_1^+) double cosets of L_1 are uniquely parametrized by W_1, the $(B_1^+ \times B_1^+,\, \Delta(L_1))$ double cosets of $L_1 \times L_1$ are uniquely parametrized by $W_1 \times \{1\}$, and part (d) follows. $\square$

Proof of Theorem A.1. Since $P = L_0 R$, the Bruhat lemma implies that every (B^+, R) double coset of G is of the form $B^+ \dot{v} l_0 R$ for some $v \in W^{W_0}$ and $l_0 \in L_0$. In addition, the Bruhat lemma also implies that if $B^+ \dot{v} l_0 R = B^+ \dot{v}' l_0' R$ for some $v, v' \in W^{W_0}$ and $l_0, l_0' \in L_0$, then $v' = v$.

From the facts that $R = QN = NQ$, that L_0 normalizes N, and $N_v \subseteq N$, we get

$$N_v l_0 R = N_v l_0 Q N = l_0 Q N.$$

Thus, from part (b) of the above lemma, we have

$$B^+ \dot{v} l_0 R = \dot{v} \overline{N}_v (B_0^+ l_0 Q) N.$$

Since $\overline{N}_v \subseteq \overline{N}$ and $\overline{N} L_0 N$ is the Cartesian product of the subsets $\overline{N}$, L_0 and N of G, we get that for $v \in W^{W_0}$ and $l_0, l_0' \in L_0$,

$$B^+ \dot{v} l_0 R = B^+ \dot{v} l_0' R \iff B_0^+ l_0 Q = B_0^+ l_0' Q.$$

Part (d) of the lemma now implies that all (B^+, R) double cosets of G are uniquely parametrized by $W^{W_0} \times W_1$, by $(v, u) \mapsto B^+ \dot{v} \dot{u} R$. The theorem, finally, follows from part (c) of the lemma. $\square$

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