AFFINE JACQUET FUNCTORS AND HARISH-CHANDRA CATEGORIES

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Abstract. We define an affine Jacquet functor and use it to describe the structure of induced affine Harish-Chandra modules at noncritical levels, extending the theorem of Kac and Kazhdan [10] on the structure of Verma modules in the Bernstein–Gelfand–Gelfand categories $\mathcal{O}$ for Kac–Moody algebras. This is combined with a vanishing result for certain extension groups to construct a block decomposition of the categories of affine Harish-Chandra modules of Lian and Zuckerman [13]. The latter provides an extension of the works of Rocha-Caridi, Wallach [15] and Deodhar, Gabber, Kac [5] on block decompositions of BGG categories for Kac-Moody algebras. We also prove a compatibility relation between the affine Jacquet functor and the Kazhdan–Lusztig tensor product. A modification of this is used to prove that the affine Harish-Chandra category is stable under fusion tensoring with the Kazhdan–Lusztig category (a case of our finiteness result [17]) and will be further applied in studying translation functors for Kac–Moody algebras, based on the fusion tensor product.

1. Introduction

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\tilde{\mathfrak{g}}$ be the corresponding affine Kac–Moody algebra.

In this paper we define an affine version of the Jacquet functor [4, 16], introduced by Casselman and Wallach, and use it to deduce properties of the affine Harish-Chandra categories and the Kazhdan–Lusztig fusion tensor product.

The standard Harish-Chandra category, associated to a real form $\mathfrak{g}_0$ of $\mathfrak{g}$ (consisting of finite length, admissible $(\mathfrak{g}, \mathfrak{k})$-modules) will be denoted by $\mathcal{H}$. The Jacquet module of $V \in H$ is given by

$$j(V) = \varinjlim \mathrm{Ann}_{\mathfrak{n}_0^k} V^* \subset V^*$$

where $\mathfrak{n}_0$ is the nil radical of a minimal parabolic subalgebra $\mathfrak{q}_0$ of $\mathfrak{g}_0$. It is a faithful and exact (contravariant) functor from $\mathcal{H}$ to a generalized Bernstein–Gelfand–Gelfand category, to be denoted by $\mathcal{O}'$. (On the latter $(\mathfrak{q}_0/\mathfrak{n}_0)_{\mathbb{C}}$ only acts locally finitely, but in general not semisimply.)

The affine Harish-Chandra category $\widehat{\mathcal{H}}_\kappa$ consists of finitely generated smooth $\tilde{\mathfrak{g}}$-modules of central charge $\kappa - h^\vee$ on which the Sugawara operator L_0 acts locally finitely and the corresponding generalized eigenspace decomposition of L_0 has the

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form

$$V = \bigoplus_{\xi \colon \xi - \xi_i \in \mathbb{Z}_{\geq 0}} V^\xi \text{ for some } \xi_1, \ldots, \xi_n \in \mathbb{C} \tag{1.1}$$

where $V^\xi \in \mathcal{H}$, considered as $\mathfrak{g}$-modules. These categories were introduced by Lian and Zuckerman [13, 14] in a slightly different way. See section 2.3 for details. (As usual $h^\vee$ denotes the dual Coxeter number of $\mathfrak{g}$.)

The generalized affine Bernstein–Gelfand–Gelfand category $\widehat{\mathcal{O}}'_\kappa$ is the category of smooth $\tilde{\mathfrak{g}}$-modules of central charge $\kappa - h^\vee$ with the above property (1.1) and $V^\xi \in \mathcal{O}'$.

The affine Jacquet functor is a faithful and exact (contravariant) functor $\hat{j} \colon \widehat{\mathcal{H}}_\kappa \to \widehat{\mathcal{O}}'_\kappa$, given by

$$\hat{j}(V) = \left[j(V)^\sharp(\infty) \right]^{\mathbb{C}[L_0]-fin}. \tag{1.2}$$

Here $(.)^\sharp$ denotes the twist of a $\tilde{\mathfrak{g}}$-module by the automorphism of $\mathfrak{g}$: $(xt^n)^\sharp = x(-t)^n$, $x \in \mathfrak{g}$, $K^\sharp = -K$. By $(.)(\infty)$ we denote the strictly smooth part of a $\tilde{\mathfrak{g}}$-module, see section 2.3. The notation $W^{\mathbb{C}[L_0]-fin}$ is for the $\mathbb{C}[L_0]$ locally finite part of W. *This can be omitted in the case* $\kappa \notin \mathbb{Q}_{\geq 0}$.

If the $\tilde{\mathfrak{g}}$-module $V \in \widehat{\mathcal{H}}_\kappa$ has the L_0 generalized eigenspace decomposition (1.1), then

$$\hat{j}(V) = \bigoplus_{\xi \colon \xi - \xi_i \in \mathbb{Z}_{\geq 0}} \lim_{\to} \mathrm{Ann}_{\mathfrak{n}_0^k} (V^\xi)^*$$

as $\tilde{\mathfrak{g}}$-submodules of $(V^*)^\sharp$ where $(V^\xi)^*$ is identified with the subspace of V^*, consisting of functionals vanishing on $\bigoplus_{\xi' \neq \xi} V^{\xi'}$.

For every irreducible $\mathfrak{g}$-module M the induced $\tilde{\mathfrak{g}}$-module

$$\mathrm{Ind}(M) = U(\tilde{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}K)} M$$

and its unique irreducible quotient $\mathrm{Irr}(M)_\kappa$ belong to $\widehat{\mathcal{H}}_\kappa$. Here K acts on M by $\kappa - h^\vee$ and $\mathfrak{g}[t]$ acts on M through the quotient $\mathfrak{g}[t] \twoheadrightarrow \mathfrak{g}[t]/t\mathfrak{g}[t] \cong \mathfrak{g}$.

We prove that

$$\hat{j}(\mathrm{Ind}(M)_\kappa) = D(\mathrm{Ind}(j(M)^d)_\kappa) \tag{1.3}$$

where $(.)^d$ and $D(.)$ refer to certain natural duality functors in $\mathcal{H}$ and $\widehat{\mathcal{H}}_\kappa$, respectively, see section 2.1 and 2.3.

The idea to use the Jacquet functor to study the structure of induced modules in the affine Harish-Chandra categories belongs to Lian and Zuckerman [14]. They formulated a version of (1.3) but with the use of the standard Jacquet functor which makes it slightly incorrect.

From (1.3) we prove a generalization to the affine Harish-Chandra categories of the result of Kac and Kazhdan on the structure of Verma modules for Kac–Moody algebras:

If M is an irreducible Harish-Chandra module with infinitesimal character $\chi_\lambda \in \mathfrak{h}^*/W$ for some $\lambda \in \mathfrak{h}^*$, then all irreducible subquotients of $\mathrm{Ind}(M)_\kappa$ *are isomorphic to* $\mathrm{Ind}(M_2)_\kappa$ *for some irreducible Harish-Chandra modules M_2 with infinitesimal character χ_{λ_2} such that there exists $w \in W$ for which the pair $[w\lambda, \lambda_2] \in \mathfrak{h}^* \times \mathfrak{h}^*$ satisfies the condition $(*)$ of Kac-Kazhdan, reviewed in Definition 2.7.*

We further obtain in section 4.3 a block decomposition of the categories $\widehat{\mathcal{H}}_\kappa$, extending the works of Rocha-Caridi, Wallach [15] and Deodhad, Gabber, Kac [5] on block decomposition of the categories $\mathcal{O}$ for Kac–Moody algebras. *Define the equivalence relation* $\sim$ *in* $\mathfrak{h}$, *induced by* $\lambda \sim \mu$ *if* $\mu \in W\lambda$ *or the pair* $[\lambda, \mu] \in \mathfrak{h}^* \times \mathfrak{h}^*$ *satisfies the condition* $(*)$ *in Definition 2.7. Then the categories of affine Harish-Chandra modules* $\widehat{\mathcal{H}}_\kappa$ *posses the block decompositions*

$$\widehat{\mathcal{H}}_\kappa = \bigoplus_{\widehat{\chi} \in (\mathfrak{h}^*/\sim)} \widehat{\mathcal{H}}_\kappa^{\widehat{\chi}}.$$

The subcategories $\widehat{\mathcal{H}}_\kappa^{\widehat{\chi}}$ *consist of modules* $V \in \widehat{\mathcal{H}}_\kappa$ *with a filtration by* $\tilde{\mathfrak{g}}$*-submodules*

$$0 = W_0 \subset W_1 \subset \ldots \subset W_N = V$$

for which the subquotients W_i/W_{i-1} *are isomorphic to quotients of* $\mathrm{Ind}(M)_\kappa$ *for irreducible Harish-Chandra modules with infinitesimal characters in the class* $\widehat{\chi} \in (\mathfrak{h}^*/\sim)$. Similar result is proved for infinitely generated affine Harish-Chandra modules in analogy with [5].

We also show vanishing of Ext *groups between different blocks in a much larger category* in the spirit of the results of Rocha-Caridi and Wallach [15], and the vanishing of Ext's between blocks of $\mathcal{H}$ in the larger category of all $(\mathfrak{g}, \mathfrak{k})$-modules (neither finitely generated, nor admissible, in general), see [3, Theorem 4.1, Ch 1] and Proposition 2.1 below. This is done in section 4.2.

Our approach is very similar to the one of Rocha-Caridi, Wallach [15] and Deodhar, Gabber, Kac [5]. (The paper [5] discusses only the case of vanishing of Ext^1 groups in the the $\mathcal{O}$ category for an arbitrary Kac–Moody algebra and [15] the general case.) The proof of the vanishing of Ext groups in the category $\mathcal{O}$ in [15, 5] uses the fact that modules from this category, restricted to the Cartan subalgebra of the extended affine Kac–Moody algebra belong to a semisimple category. On the contrary modules from the category $\widehat{\mathcal{H}}_\kappa$, which we treat, restricted to $\mathfrak{g} \hookrightarrow \hat{\mathfrak{g}}$ form essentially the Harish-Chandra category (which is non-semisimple). The use of semisimplicity in [5, 15] can be avoided. When written in terms of spectral sequences the arguments of [15, 5] simplify a lot, as it is always the case in similar situations.

In section 5 we derive the following compatibility property between the affine Jacquet functor and the Kazhdan–Lusztig tensor product:

Let $\kappa \notin \mathbb{Q}_{\geq 0}$. *For any module* U *in the Kazhdan–Lusztig category and* $V \in \widehat{\mathcal{H}}_\kappa$ *the following isomorphism holds:*

$$\hat{j}(U \dot{\otimes} V) \cong D\left[U \dot{\otimes} D\hat{j}(V)\right]. \tag{1.4}$$

In [17] we showed that for any subalgebra $\mathfrak{f}$ of $\mathfrak{g}$ which is reductive in $\mathfrak{g}$ the affine analogs of the categories of finite length, admissible $(\mathfrak{g}, \mathfrak{f})$-modules are stable under the fusion tensoring with the Kazhdan–Lusztig category for $\kappa \notin \mathbb{R}_{\geq 0}$. This is an affine version Kostant's theorem [12].

A modification of (1.4) provides another proof of a case of our result [17], namely that the affine Harish-Chandra category is stable under fusion tensoring with modules from the Kazhdan–Lusztig category.

The above compatibility of the fusion tensor product and the affine Jacquet functor will be further used in studying fusion translation functors in the spirit of Zuckerman [18] and Jantzen [8].

After the work of Beilinson and Bernstein [1] a very general nonvanishing of the Jacquet modules of a $\mathfrak{g}$-module with an infinitesimal character (associated to the nilradicals of Borel subalgebras in a dense subset of the flag variety) is known. It is interesting to understand if this can be used to define block decomposition of the category of all smooth (L_0-locally finite) $\tilde{\mathfrak{g}}$-modules for which V^ξ are finitely generated $U(\mathfrak{g})$-modules on which the center of $U(\mathfrak{g})$ acts locally finitely. In this way the affine Jacquet functor would substitute very efficiently the triviality of the center of an affine Kac–Moody algebra.

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2. Preliminaries on real semisimple Lie algebras and affine Kac-Moody algebras

2.1. **Real semisimple Lie algebras.** Let $\mathfrak{g}_0$ be a real semisimple Lie algebra with complexification $\mathfrak{g} = (\mathfrak{g}_0)_{\mathbb{C}}$. Fix a Cartan decomposition of $\mathfrak{g}_0$

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0.$$

Let $\mathfrak{a}_0 \subset \mathfrak{p}_0$ be a maximal commutative subalgebra. Denote by Σ_+ a fixed set of positive restricted roots of $\mathfrak{g}_0$ with respect to $\mathfrak{a}_0$ and set

$$\mathfrak{n}_0 = \bigoplus_{\lambda \in \Sigma_+} \mathfrak{g}_0^\lambda.$$

Define the minimal parabolic subalgebra of $\mathfrak{g}_0$

$$\mathfrak{q}_0 = \mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$$

where $\mathfrak{m}_0 = Z_{\mathfrak{k}_0}(\mathfrak{a}_0)$.

Fix a maximal commutative subalgebra $\mathfrak{t}_0 \subset \mathfrak{m}_0$ and consider the related maximally noncompact Cartan subalgebra

$$\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$$

of $\mathfrak{g}_0$. Then the complexification $\mathfrak{h} = (\mathfrak{h}_0)_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}$ and one can find a set of positive roots Δ_+ for $\mathfrak{g}$ relative to $\mathfrak{h}$ which extends Σ_+, i.e.

$$\Delta|_{\mathfrak{a}_0} = \Sigma_+ \cup \{0\}.$$

Denote the nil radical

$$\mathfrak{n}_+ = \bigoplus_{\lambda \in \Delta_+} \mathfrak{g}^\lambda$$

of the related positive Borel subalgebra. Then

$$\mathfrak{n}_+ \supset (\mathfrak{n}_0)_{\mathbb{C}}.$$

Recall that the category $\mathcal{H}$ of Harish-Chandra modules for the pair $(\mathfrak{g}, \mathfrak{k})$ is the category of finitely generated $\mathfrak{g}$-modules V such that

$$V|_{\mathfrak{k}} = \bigoplus_{\mu} (V^\mu)^{\oplus m_\mu}$$

where V^μ are representatives of the equivalence classes of all irreducible finite dimensional $\mathfrak{k}$-modules and the multiplicities m_μ are finite (admissibility condition). All Harish-Chandra modules have finite length.

The generalized Bernstein–Gelfand–Gelfand category $\mathcal{O}'$ for $\mathfrak{g}$ related to the choice of Borel subalgebra $\mathfrak{b}$ and Cartan subalgebra $\mathfrak{h}$ as above, consists of finitely generated $\mathfrak{g}$-modules V on which $\mathfrak{n}_+$ acts locally nilpotently and

$$V|_{\mathfrak{h}} = \bigoplus_{\lambda\in\mathfrak{h}^*} V_\lambda, \quad \dim V_\lambda < \infty$$

where V_λ denote the generalized eigenspaces of $\mathfrak{h}$:

$$V_\lambda = \{v \in V \mid (h-\lambda(h))^k v = 0,\, \forall h \in \mathfrak{h} \text{ for some } k \in \mathbb{Z}_{>0}\}.$$

The usual BGG category for $\mathfrak{g}$ for the choices of Cartan and Borel subalgebras made will be denoted by $\mathcal{O}$.

The center of $\mathfrak{g}$ will be denoted by $Z(\mathfrak{g})$. By the Harish-Chandra isomorphism $Z(\mathfrak{g}) \cong S(\mathfrak{h})^W$ and the characters of $Z(\mathfrak{g})$ are parametrized by $\mathfrak{h}^*/W$. The character corresponding to $\lambda \in \mathfrak{h}^*$ will be denoted by $\chi_\lambda \colon Z(\mathfrak{g}) \to \mathbb{C}$. Recall that for the Casimir element of $U(\mathfrak{g})$

$$\chi_\lambda(\Omega) = |\lambda|^2 - |\rho|^2. \tag{2.1}$$

where ρ is the half-sum of the positive roots of the Borel subalgebra of $\mathfrak{g}$ used to define the Harish-Chandra isomorphism.

For a $\mathfrak{g}$-module V possessing an infinitesimal character, by $\chi(V)$ we will denote the latter.

The categories $\mathcal{H}$ and $\mathcal{O}'$ posses block decompositions

$$\mathcal{H} = \bigoplus_{\chi\in\mathfrak{h}^*/W} \mathcal{H}^\chi, \quad \mathcal{O}' = \bigoplus_{\chi\in\mathfrak{h}^*/W} \mathcal{O}'^{\chi}. \tag{2.2}$$

where $\mathcal{H}^\chi$ and $\mathcal{O}'^{\chi}$ are the full subcategories of $\mathcal{H}$ and $\mathcal{O}'$, respectively, consisting of modules with filtrations by $\mathfrak{g}$-modules with infinitesimal character χ.

Later we will need an important result on vanishing of Ext groups between Harish-Chandra modules modules from different blocks of $\mathcal{H}$, see [3, Chapter I, Theorem 4.1]. Denote by $\mathcal{C}_{(\mathfrak{g},\mathfrak{k})}$ the category of $(\mathfrak{g},\mathfrak{k})$ modules, i.e. $\mathfrak{g}$-modules which are locally $\mathfrak{k}$-finite and $\mathfrak{k}$-semisimple. It is well known that these categories have enough projectives and injectives, see e.g. [3, Chapter I, 2.4].

Proposition 2.1. *Assume that* $V_i \in \mathcal{H}^{\chi_i}$, $i = 1, 2$ *and* $\chi_1 \neq \chi_2$. *Then*

$$\mathrm{Ext}^n_{\mathfrak{g},\mathfrak{k}}(V_1, V_2) = 0,\ n \in \mathbb{Z}_{\geq 0}$$

where $\mathrm{Ext}^n_{\mathfrak{g},\mathfrak{k}}$ *refer to the* Ext *groups in the category* $\mathcal{C}_{(\mathfrak{g},\mathfrak{k})}$.

The categories $\mathcal{H}$ and $\mathcal{O}'$ have natural duality functors (involutive antiequivalences). Both of them will be denoted by $V \mapsto V^d$. They are given by

$$V^d = (V^*)^{U(\mathfrak{k})-fin}, \quad V \in \mathcal{H}, \tag{2.3}$$

$$V^d = (V^*)^{U(\mathfrak{h})-fin}, \quad V \in \mathcal{H}. \tag{2.4}$$

Here and later for any module W over a Lie algebra by W^* we denote the (full) dual module. For a $\mathbb{C}$-algebra A and an A-module W we denote by W^{A-fin} the A-submodule of W consisting of A-finite vectors, i.e.

$$W^{A-fin} := \{w \in W \mid \dim Aw < \infty\}.$$

Clearly $(.)^d$ restricts to a functor from $\mathcal{H}^{\chi_\lambda}$ to $\mathcal{H}^{\chi_{-\lambda}}$ and from $\mathcal{O}'^{\chi_\lambda}$ to $\mathcal{O}'^{\chi_{-\lambda}}$.

2.2. **The Jacquet functor.** Fix $V \in \mathcal{H}$. The natural increasing subsequence of $\mathfrak{m}_0 \oplus \mathfrak{a}_0$-submodules of V

$$\mathfrak{n}_0 V \supset \mathfrak{n}_0^2 V \supset \dots$$

gives rise to the increasing subsequence of $\mathfrak{m}_0 \oplus \mathfrak{a}_0$-submodules of V^*

$$(V/\mathfrak{n}_0 V)^* \hookrightarrow (V/\mathfrak{n}_0^2 V)^* \hookrightarrow \dots.$$

Here for a subspace V_1 of V, $(V/V_1)^*$ is naturally identified with the subspace of V^* that consists of $\eta \in V^*$ such that $\eta|_{V_1} = 0$. Note that

$$(V/\mathfrak{n}_0^k V)^* = \mathrm{Ann}_{\mathfrak{n}_0^k} V^*.$$

Finally the Jacquet module of V is defined by

$$j(V) = \lim_{\rightarrow} \mathrm{Ann}_{\mathfrak{n}_0^k} V^* = \lim_{\rightarrow} (V/\mathfrak{n}_0^k V)^*.$$

Since the adjoint action of $\mathfrak{n}_0$ on $\mathfrak{g}_0$ is nilpotent the space $j(V)$ is a $\mathfrak{g}$-submodule of V^*.

Theorem 2.2. *(i) For any $V \in \mathcal{H}$, $\dim(V/\mathfrak{n}_0^k V) < \infty$ and $j(V) \in \mathcal{O}'$.*
(ii) The contravariant functor $j \colon \mathcal{H} \to \mathcal{O}'$ is faithful and exact.

Because of part (i) of the above theorem, the Jacquet functor is also given by

$$j(V) = \lim_{\rightarrow} \mathrm{Ann}_{\mathfrak{n}_+^k} V^* = \lim_{\rightarrow} (V/\mathfrak{n}_+^k V)^*,$$

see section 2.1 for the definition of $\mathfrak{n}_+$.

In fact the Jacquet functor takes values in the subcategory of $\mathcal{O}'$ consisting of $\mathfrak{g}$-modules on which the Levi factor $(\mathfrak{m}_0 \oplus \mathfrak{a}_0)_{\mathbb{C}}$ of the parabolic subalgebra $\mathfrak{q} = (\mathfrak{q}_0)_{\mathbb{C}}$ of $\mathfrak{g}$ acts locally finitely.

It is also clear that $j \colon \mathcal{H}^{\chi_\lambda} \to {\mathcal{O}'}^{\chi_{-\lambda}}$.

Recall that the Harish-Chandra category $\mathcal{H}$ is stable under tensoring with finite dimensional modules. We have the following property, relating this tensoring with the Jacquet functor.

Proposition 2.3. *For any finite dimensional $\mathfrak{g}$-module U and a Harish-Chandra module V the Jacquet module of $U \otimes V \in \mathcal{H}$ is given by*

$$j(U \otimes V) \cong U^* \otimes j(V).$$

2.3. **Untwisted Affine Kac–Moody algebras.** Consider the loop algebra $\mathfrak{g}[t, t^{-1}] = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$. The untwisted affine Kac-Moody algebra $\tilde{\mathfrak{g}}$ associated to $\mathfrak{g}$ is the central extension of this loop algebra by

$$[xt^n, yt^m] = [x, y]t^{n+m} + n\delta_{n,-m}(x, y)K, \ x, y \in \mathfrak{g} \tag{2.5}$$

where $(.,.)$ is an invariant nondegenerate bilinear form on $\mathfrak{g}$ normalized by $(\alpha, \alpha) = 2$ for long roots α of $(\mathfrak{g}, \mathfrak{h})$. For a full exposition of Kac–Moody algebras we refer to Kac's book [9].

A $\tilde{\mathfrak{g}}$-module V is called smooth if any $v \in V$ is annihilated by xt^n for all $x \in \mathfrak{g}$ and $n \gg 0$.

On any smooth $\tilde{\mathfrak{g}}$-module of central charge $\kappa - h^\vee$ ($\kappa \neq 0$) one has a natural representation of the Virasoro algebra [9], given by the Sugawara operators

$$L_k = \frac{1}{2\kappa} \sum_p \sum_{j \in \mathbb{Z}} : (x_p t^{-j})(x_p t^{j+k}) : . \tag{2.6}$$

Here $\{x_p\}$ is an orthonormal basis of $\mathfrak{g}$ with respect to the bilinear form $(.,.)$. The normal ordering in (2.6) prescribes pulling to the right the term xt^n with larger n. Here and later $h^\vee$ denotes the dual Coxeter number of $\mathfrak{g}$. The following commutation relations hold

$$[L_k, xt^n] = -n(xt^{n+k}). \tag{2.7}$$

For any smooth $\tilde{\mathfrak{g}}$-module V define the generalized eigenspaces of L_0

$$V^\xi = \{v \in V \mid (L_0 - \xi)^n v = 0, \text{ for some integer } n\},\ \xi \in \mathbb{C}.$$

Because $\mathfrak{g} \hookrightarrow \tilde{\mathfrak{g}}$ commutes with L_0 every $V^\xi \subset V$ is naturally a $\mathfrak{g}$-module.

Definition 2.4. Let $\mathcal{C}$ be a category of finitely generated $\mathfrak{g}$-modules that is closed under tensoring with finite dimensional $\mathfrak{g}$-modules. Define the category $\mathcal{AFF}(\mathcal{C})_\kappa$ to be the representation category of finitely generated smooth $\tilde{\mathfrak{g}}$-modules of central charge $\kappa - h^\vee$ which are L_0 locally finite and satisfy

$$V = \bigoplus_{\xi:\ \xi-\xi_i\in\mathbb{Z}_+} V^\xi \quad \text{for some} \quad \xi_1, \xi_2, \ldots, \xi_n \in \mathbb{C} \tag{2.8}$$

with $V^\xi \in \mathcal{C}$, considered as $\mathfrak{g}$-modules.

If $\mathcal{F}in$ is the category of finite dimensional $\mathfrak{g}$-modules then the Kazhdan–Lusztig categories [11] for $\tilde{\mathfrak{g}}$ are $\mathcal{KL}_\kappa = \mathcal{AFF}(\mathcal{F}in)_\kappa$ in the case $\kappa \notin \mathbb{Q}_{\geq 0}$.

Denote

$$\widehat{\mathcal{O}}_\kappa = \mathcal{AFF}(\mathcal{O})_\kappa, \quad \widehat{\mathcal{O}}'_\kappa = \mathcal{AFF}(\mathcal{O}')_\kappa, \quad \kappa \neq 0.$$

Recall that the extended Kac-Moody algebra $\hat{\mathfrak{g}}$ associated to $\mathfrak{g}$ is the Lie algebra $\tilde{\mathfrak{g}} \oplus \mathbb{C}d$ where

$$[d, xt^n] = nxt^n, \quad [d, K] = 0,$$

see [9] for details. Consider its Cartan subalgebra $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d$ and Borel subalgebra $\mathfrak{b}_+ \oplus \mathfrak{g}[t] \oplus \mathbb{C}K \oplus \mathbb{C}d$. The associated affine BGG category of noncritical central charge $\kappa - h^\vee$ is essentially the category $\widehat{O}_\kappa$ with the difference that the generator d can act on any module $V \in \widehat{O}_\kappa$ by $const - L_0$ for any choice of the constant involved.

The categories of affine Harish-Chandra modules defined by Lian and Zuckerman [13, 14] are $\mathcal{AFF}(\mathcal{H})_\kappa$. They will be denoted by $\widehat{\mathcal{H}}_\kappa$.

The Lie algebra $\tilde{\mathfrak{g}}$ is $\mathbb{Z}$-graded by

$$\deg xt^n = -n, n \in \mathbb{Z}, x \in \mathfrak{g};\ \deg K = 0. \tag{2.9}$$

Each module in the categories $\mathcal{AFF}(\mathcal{C})_\kappa$ is naturally $\mathbb{C}$-graded with respect to the grading (2.9) of $\tilde{\mathfrak{g}}$ using the generalized eigenvalue decomposition (2.8) of L_0. This is the reason for the choice of the negative sign in (2.9). Moreover each morphism in $\mathcal{AFF}(\mathcal{C})_\kappa$ preserves the grading (2.8).

For simplicity we will denote the maximal parabolic subalgebra of $\tilde{\mathfrak{g}}$

$$\tilde{\mathfrak{p}}_+ = \mathfrak{g}[t] \oplus \mathbb{C}K \tag{2.10}$$

and its ideal

$$\tilde{\mathfrak{n}}_+ = t\mathfrak{g}[t]. \tag{2.11}$$

Clearly $\tilde{\mathfrak{p}}_+/\tilde{\mathfrak{n}}_+ \cong \mathfrak{g} \oplus \mathbb{C}K$. Later we will also need "the opposite to $\tilde{\mathfrak{n}}_+$ subalgebra" of $\tilde{\mathfrak{g}}$

$$\tilde{\mathfrak{n}}_- = t^{-1}\mathfrak{g}[t^{-1}]. \tag{2.12}$$

For any subalgebra $\mathfrak{f}$ of $\tilde{\mathfrak{g}}$ the component of degree N of $U(\mathfrak{f})$ will be denoted by $U(\mathfrak{f})^N$. Recall:

Definition 2.5. (Kazhdan–Lusztig) For a given $\tilde{\mathfrak{g}}$-module V set

$$V(N) = \mathrm{Ann}_{U(\tilde{\mathfrak{n}}_+)^{-N}} V, \ N \in \mathbb{Z}_{>0}.$$

A $\tilde{\mathfrak{g}}$-module V is called strictly smooth if $\cup_N V(N) = V$.

Note that $V(N) \subset V$ is naturally a $\mathfrak{g}$-module for the embedding $\mathfrak{g} \hookrightarrow \tilde{\mathfrak{g}}$, $x \mapsto xt^0$.

For any $\tilde{\mathfrak{g}}$-module W the strictly smooth part of it

$$W(\infty) = \cup_N W(N) \tag{2.13}$$

is a $\tilde{\mathfrak{g}}$-submodule of W. The functor $W \mapsto W(\infty)$ in the category of $\tilde{\mathfrak{g}}$-modules is left exact.

The commutation relation (2.7) implies that any $\tilde{\mathfrak{g}}$-module in $\mathcal{AFF}(\mathcal{C})_\kappa$ is strictly smooth. The following equivalent characterization of the categories $\mathcal{AFF}(\mathcal{C})_\kappa$ for $\kappa \notin \mathbb{R}_{\geq 0}$ was obtained in [17] and in the case $\mathcal{C} = \mathcal{F}in$ ($\kappa \notin \mathbb{Q}_{\geq 0}$) previously in [11].

Proposition 2.6. *Assume that $\mathcal{C}$ is a category of finite length $\mathfrak{g}$-modules and is closed under tensoring with finite dimensional $\mathfrak{g}$-modules. If $\kappa \notin \mathbb{R}_{\geq 0}$ then all modules in $\mathcal{AFF}(\mathcal{C})_\kappa$ have finite length. The category $\mathcal{AFF}(\mathcal{C})_\kappa$ coincides with the category of finitely generated, strictly smooth $\tilde{\mathfrak{g}}$ modules of central charge $\kappa - h^\vee$ for which*

$$V(N) \in \mathcal{C},$$

considered as a $\mathfrak{g}$-module, for all $N \in \mathbb{Z}_{>0}$.

For a given $\mathfrak{g}$-module M consider the induced $\tilde{\mathfrak{g}}$-modules

$$\mathrm{Ind}(M)_\kappa = U(\tilde{\mathfrak{g}}) \otimes_{U(\tilde{\mathfrak{p}}_+)} M \tag{2.14}$$

where $\tilde{\mathfrak{p}}_+$ acts on M through the quotient $\tilde{\mathfrak{p}}_+ \twoheadrightarrow \tilde{\mathfrak{p}}_+/\tilde{\mathfrak{n}}_+ \cong \mathfrak{g} \oplus \mathbb{C}K$ and the central element K acts by $\kappa - h^\vee$.

If M is an irreducible $\mathfrak{g}$-module and $\kappa \neq 0$ then $\mathrm{Ind}(M)_\kappa$ has a unique irreducible quotient, to be denoted by $\mathrm{Irr}(M)_\kappa$. Both modules belong to $\mathcal{AFF}(\mathcal{C})_\kappa$ if $M \in \mathcal{C}$ and the modules of the latter type exhaust all irreducible modules in $\mathcal{AFF}(\mathcal{C})_\kappa$.

For central charge $\kappa - h^\vee$, $\kappa \notin \mathbb{R}_{\geq 0}$ the categories $\widehat{\mathcal{H}}_\kappa$, $\widehat{\mathcal{O}}'_\kappa$ have natural duality functors, to be denoted by $D(.)$:

$$D(V) = (V^d)^\sharp(\infty), \quad V \in \widehat{\mathcal{H}}_\kappa \text{ or } \widehat{\mathcal{O}}'_\kappa. \tag{2.15}$$

Given a $\tilde{\mathfrak{g}}$-module W the notation $W^\sharp$ stays for its twist by the automorphism of $\tilde{\mathfrak{g}}$

$$(xt^k)^\sharp = x(-t)^k, \ x \in \mathfrak{g}, \ k \in \mathbb{Z}; \ K^\sharp = -K.$$

Assume that the $\tilde{\mathfrak{g}}$-module $V \in \widehat{\mathcal{H}}_\kappa$ or $\widehat{\mathcal{O}}'_\kappa$ is given by (2.8) with $V^\xi \in \mathcal{H}$ or $\mathcal{O}'$, respectively. Denote

$$(V^d)^\xi = \{\eta \in V^d \mid \eta|_{V^{\xi'}} = 0 \text{ for } \xi' \neq \xi\}.$$

The underlining space of the dual module $D(V)$ is the following subspace of of V^d

$$D(V) = \bigoplus_{\xi:\ \xi - \xi_i \in \mathbb{Z}_{\geq 0}} (V^d)^\xi \tag{2.16}$$

and the generalized L_0-eigenspaces of $D(V)$ are

$$D(V)^\xi = (V^d)^\xi, \tag{2.17}$$

see [11, 17]. The exactness of the functor D on $\widehat{\mathcal{H}}_\kappa$ and $\widehat{O}'_\kappa$ follows from the first equality.

The functors D on $\widehat{\mathcal{H}}_\kappa$ and $\widehat{O}'_\kappa$ restrict further to a duality functor in $\mathcal{KL}_\kappa$ which in a more simple way is given by

$$D(V) = (V^*)^\sharp(\infty), \quad V \in \mathcal{KL}_\kappa. \tag{2.18}$$

2.4. **Structure of the categories $\widehat{O}_\kappa$.** Denote by M_λ the highest weight module for $\mathfrak{g}$ corresponding to highest weight $\lambda \in \mathfrak{h}^*$, with respect to to the choice of Cartan and Borel subalgebras, made in Section 2.1. Denote by L_λ its unique irreducible quotient. Consider the Cartan and Borel subalgebras

$$\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d,\ \widehat{\mathfrak{b}}_+ = \widehat{\mathfrak{h}} \oplus \mathfrak{n}_+ \oplus t\mathfrak{g}[t]$$

of the extended affine Kac–Moody algebra $\hat{\mathfrak{g}}$, defined in Section 2.3. Recall from [9] the notation

$$\delta \in \widehat{\mathfrak{h}}^*,\ \delta|_{\mathfrak{h}} = 0,\ \delta(K) = 0,\ \delta(d) = 1,$$
$$\Lambda_0 \in \widehat{\mathfrak{h}}^*,\ \Lambda_0|_{\mathfrak{h}} = 0,\ \Lambda_0(K) = 1,\ \Lambda_0(d) = 0.$$

The induced module $\mathrm{Ind}(M_\lambda)_\kappa$, viewed as a module over the extended Kac–Moody algebra by setting $d = -L_0$ is isomorphic to the highest weight module with highest weight

$$\lambda + (\kappa - h^\vee)\Lambda_0 - \frac{(\lambda, \lambda + 2\rho)}{2\kappa}\delta. \tag{2.19}$$

Computing the determinant of the Shapovalov form on a highest weight module and using the Jantzen filtration, Kac and Kazhdan described the structure of Verma modules for all Kac–Moody algebras by an extension of Jantzen's argument [8]. We will state the result for affine Kac–Moody algebras in terms of the modules $\mathrm{Ind}(M_\lambda)_\kappa$ rather than the affine weights (2.19).

Definition 2.7. One says [10] that pair $[\lambda, \mu] \in \mathfrak{h}^* \times \mathfrak{h}^*$ satisfies the condition $(*)$ if there exists a sequence of roots $\{\beta_i\}_{i=1}^k \subset \Delta$ and a sequence of nonnegative integers $\{m_i\}_{i=1}^k \subset \mathbb{Z}_{\geq 0}$ ($\beta_i \in \Delta_+$ if $m_i = 0$) such that $\{\lambda_i = \lambda - \beta_1 - \ldots - \beta_i\}_{i=0}^k$ satisfies

$$\lambda_{i+1} = r_{\beta_i}(\lambda_i) + \kappa m_i \beta_i^\vee \text{ and } (\lambda, \beta_i^\vee) + 2\kappa m_i/|\beta_i|^2 \in \mathbb{Z}_{>0},\ i = 1, \ldots, k, \tag{2.20}$$

$\lambda_k = \mu$.

The second condition in (2.20) means that $\lambda_{i+1} - \lambda_i \in Q$ which can only happen if $\lambda_{i+1} - \lambda_i = n_i\beta_i$ for some positive integer n_i.

Theorem 2.8. *(Kac–Kazhdan) The module* $\mathrm{Irr}(L_\lambda)_\kappa$ *is a subquotient of* $\mathrm{Ind}(M_\mu)_\kappa$ *if only if the pair* $[\lambda + \rho, \mu + \rho]$ *satisfies the condition* $(*)$.

Denote by $\widehat{\mathcal{O}}_{\kappa,inf}$ the category of $\tilde{\mathfrak{g}}$-modules, satisfying the conditions for $\widehat{\mathcal{O}}_\kappa$ except the condition to be finitely generated. Each $\tilde{\mathfrak{g}}$-module $V \in \widehat{\mathcal{O}}_{\kappa,inf}$ has a filtration by $\tilde{\mathfrak{g}}$-submodules

$$0 = W_0 \subset W_1 \subset \ldots \subset V \tag{2.21}$$

such that each subquotient W_n/W_{n-1} is isomorphic to a quotient of $\mathrm{Ind}(L_{\lambda_n - \rho})_\kappa$ for some $\lambda_n \in \mathfrak{h}^*$, see [5]. The modules in $\widehat{\mathcal{O}}_\kappa$ have finite filtrations with this property.

Consider the equivalence relation on $\mathfrak{h}^*$ induced by $\lambda \sim \mu$ if the pair $[\lambda, \mu]$ satisfies the condition $(*)$. The equivalence class of an element $\lambda \in \mathfrak{h}^*$ will be denoted by $\widehat{\chi}_\lambda$.

For a given $\widehat{\chi} \in (\mathfrak{h}/\sim)$ define $\widehat{\mathcal{O}}^{\widehat{\chi}}_{\kappa,inf}$ to be the subcategory of $\widehat{\mathcal{O}}_{\kappa,inf}$, consisting of $\tilde{\mathfrak{g}}$-modules having a filtration of the type (2.21) with the property that for all subquotients W_n/W_{n-1}, $\lambda_n \in \widehat{\chi}$.

This construction gives rise to the subcategories $\widehat{\mathcal{O}}^{\widehat{\chi}}_{\kappa} = \widehat{\mathcal{O}}_\kappa \cap \widehat{\mathcal{O}}^{\widehat{\chi}}_{\kappa,inf}$ of $\widehat{\mathcal{O}}_\kappa$. In the case $\kappa \in \mathbb{Q}_{<0}$ all modules in $\widehat{\mathcal{O}}_\kappa$ have finite length and $\widehat{\mathcal{O}}^{\widehat{\chi}}_{\kappa}$ consists of those modules having a composition series with subquotients isomorphic to $\mathrm{Irr}(L_\lambda)_\kappa$ for $\lambda + \rho \in \widehat{\chi}$.

Theorem 2.9. *(Rocha-Caridi–Wallach* [15] *and Deodhar–Gabber–Kac* [5]*)* *The category $\widehat{\mathcal{O}}_{\kappa,inf}$ has the block decomposition*

$$\widehat{\mathcal{O}}_{\kappa,inf} = \bigoplus_{\widehat{\chi}\in\mathfrak{h}/\sim} \widehat{\mathcal{O}}^{\widehat{\chi}}_{\kappa,inf}.$$

Moreover

$$\mathrm{Ext}^n_{\widehat{\mathfrak{g}},\widehat{\mathfrak{h}}}(V_1, V_2) = 0, \ n \in \mathbb{Z}_{\geq 0}$$

if $V_i \in \widehat{\mathcal{O}}^{\widehat{\chi}_i}_{\kappa}$, $i = 1, 2$ and $\widehat{\chi}_1 \neq \widehat{\chi}_2$. Here $\mathrm{Ext}^n_{\widehat{\mathfrak{g}},\widehat{\mathfrak{h}}}$ refer to the Ext *groups in the category of $\widehat{\mathfrak{g}}$-modules of central charge $\kappa - h^\vee$ which are $\widehat{\mathfrak{h}}$ locally finite and semisimple.*

3. Affine Jacquet functors and vanishing of Hom groups for induced affine Harish-Chandra modules

Throughout this section we will assume

$$\kappa \notin \mathbb{R}_{\geq 0},$$

except in subsection 3.3 where we will generalize most of the results to the case of arbitrary noncritical central charge.

3.1. **Affine Jacquet functors.** Let $V \in \widehat{\mathcal{H}}_\kappa$. Define the affine Jacquet functor by

$$\hat{j}(V) = j(V)^\sharp(\infty). \tag{3.1}$$

Here $j(V)$ refers to the Jacquet module of V considered as a $\mathfrak{g}$-module under the natural inclusion $\mathfrak{g} \hookrightarrow \tilde{\mathfrak{g}}$.

Since $(\mathfrak{n}_0)_\mathbb{C} \subset \mathfrak{g}$ acts locally nilpotently on $\tilde{\mathfrak{g}}$ the subspace $j(V) \subset V^*$ is a $\tilde{\mathfrak{g}}$-submodule and thus $j(V)$ is naturally a $\tilde{\mathfrak{g}}$-module of central charge $\kappa - h^\vee$.

Assume that the module V has the form (2.8) with $V^\xi \in \mathcal{H}$. The $\mathfrak{g}$-submodule

$$(V^*)^\xi := \{\eta \in V^* \mid \eta(V^{\xi'}) = 0 \text{ if } \xi' \neq \xi\}$$

of V^* is naturally identified with $(V^\xi)^*$. Under this identification the $\mathfrak{g}$-module

$$\{\eta \in V^* \mid \eta(V^{\xi'}) = 0 \text{ if } \xi' \neq \xi, \ \eta \in \cup_N \mathrm{Ann}_{\mathfrak{n}_0^N} V^*\} \tag{3.2}$$

is naturally identified with $j(V^\xi)$. By abuse of notation we will denote the $\mathfrak{g}$-submodule (3.2) of V^* by $j(V^\xi)$. Then as a $\mathfrak{g}$-submodule of $(V^*)^\sharp(\infty)$, the affine

Jacquet module (3.1) $\hat{j}(V)$ is

$$\hat{j}(V) = \bigoplus_{\xi:\ \xi-\xi_i\in\mathbb{Z}_+} j(V^\xi) \subset \bigoplus_{\xi:\ \xi-\xi_i\in\mathbb{Z}_+} (V^\xi)^* = (V^*)^\sharp(\infty). \tag{3.3}$$

The fact that the rhs of (3.3) is a subset of the lhs is obvious. The opposite inclusion follows from the following lemma proved analogously to [11, Proposition 2.21].

Lemma 3.1. *Let $V \in \widehat{\mathcal{H}}_\kappa$ then:*

(i) There exists a real number ζ such that the map $\mathfrak{g}\otimes V^\xi \to V^{\xi+1}$ given by $x\otimes v \mapsto (xt^{-1})v$ is an isomorphism for Re$\xi \geq \zeta$.

(ii) For any positive integer N there exists $\zeta\in\mathbb{R}$ such that $\oplus_{\mathrm{Re}\xi\geq\zeta}V^\xi \subset U(\tilde{\mathfrak{n}}_-)^{-N}V$.

Eq. (3.3) implies that the affine Jacquet functor is exact and faithful. It can also be used as a definition of the affine Jacquet functor. In this definition of $\hat{j}$ the (standard) Jacquet functor j is applied only to the finitely generated, admissible $(\mathfrak{g},\mathfrak{k})$-modules V^ξ as opposite to (3.1) where the Jacquet functor is applied to the $(\mathfrak{g},\mathfrak{k})$-module $V\in\mathcal{C}_{(\mathfrak{g},\mathfrak{k})}$ which is neither admissible, nor finitely generated.

It is straightforward to show that

$$(\hat{j}(V))^\xi = j(V^\xi). \tag{3.4}$$

The fact that $\hat{j}(V)$ has finite length is not obvious. It will be deduced from the following Proposition.

Proposition 3.2. *If $M\in\mathcal{H}$ then*

$$\hat{j}(\mathrm{Ind}(M)_\kappa) \cong D(\mathrm{Ind}(j(M)^d)_\kappa), \tag{3.5}$$

$$\hat{j}(D(\mathrm{Ind}(M)_\kappa)) \cong \mathrm{Ind}(j(M^d)_\kappa) \tag{3.6}$$

where in the left and right hand sides D refers to the duality functor in the categories $\widehat{\mathcal{H}}_\kappa$ and $\widehat{\mathcal{O}}'_\kappa$, respectively.

Proposition 3.2, the exactness, and the faithfulness of $\hat{j}$ imply that $\hat{j}(V)$ is a $\tilde{\mathfrak{g}}$-module of finite length for any $V\in\widehat{\mathcal{H}}_\kappa$. This coupled with (3.4) shows that $j:\widehat{\mathcal{H}}_\kappa\to\widehat{\mathcal{O}}'_\kappa$. Thus we obtain:

Proposition 3.3. *The affine Jacquet functor $\hat{j}$ given by* (3.1) *or* (3.3) *is faithful and exact (contravariant) functor from $\widehat{\mathcal{H}}_\kappa$ to $\widehat{\mathcal{O}}'_\kappa$.*

Proof of Proposition 3.2 We will show (3.5). Eq. (3.6) can be proved analogously. Since the Jacquet functor j and the induction functor $\mathrm{Ind}(.)_\kappa : \mathcal{H}\to\widehat{\mathcal{H}}_\kappa$ commute with finite direct sums, it is sufficient to prove the statement of the lemma for a module $M\in\mathcal{H}^{\chi_\lambda}$, $\lambda\in\mathfrak{h}^*$. Because of (2.7)

$$\mathrm{Ind}(M)_\kappa = \bigoplus_{N\in\mathbb{Z}_{\geq 0}} \mathrm{Ind}(M)_\kappa^{\phi(\lambda)+N} \tag{3.7}$$

where

$$\phi(\lambda) = (|\lambda|^2 - |\rho|^2)/\kappa, \tag{3.8}$$

recall (2.1). Under the natural isomorphism of $\mathfrak{g}$-modules

$$\mathrm{Ind}(M)_\kappa \cong U(\tilde{\mathfrak{n}}_-)\otimes_{\mathbb{C}} M,$$

$\mathrm{Ind}(M)_\kappa^{\phi(\lambda)+N} \subset \mathrm{Ind}(M)_\kappa$ is mapped to $U(\tilde{\mathfrak{n}}_-)^N \otimes_{\mathbb{C}} M \subset U(\tilde{\mathfrak{n}}_-) \otimes_{\mathbb{C}} M$. Here $\mathfrak{g}$ acts on $U(\tilde{\mathfrak{n}}_-)$ by the adjoint action. We obtain the chain of natural isomorphisms of $\mathfrak{g}$-modules

$$\hat{j}(\mathrm{Ind}(M)_\kappa) \cong \bigoplus_{N\in\mathbb{Z}_{\geq 0}} j(U(\tilde{\mathfrak{n}}_-)^N \otimes_{\mathbb{C}} M) \cong \bigoplus_{N\in\mathbb{Z}_{\geq 0}} (U(\tilde{\mathfrak{n}}_-)^N)^* \otimes_{\mathbb{C}} j(M) \cong \cong \bigoplus_{N\in\mathbb{Z}_{\geq 0}} (U(\tilde{\mathfrak{n}}_-)^N \otimes_{\mathbb{C}} j(M)^d)^d \cong D(j(M)^d). \tag{3.9}$$

The first isomorphism comes from (3.3), the second is an application of Proposition 2.3, and the forth one is from (2.16).

It is straightforward to check that (3.9) is an isomorphism of $\tilde{\mathfrak{g}}$-modules. □

Corollary 3.4. *If $M \in \mathcal{H}^{\chi_\lambda}$, $\lambda \in \mathfrak{h}^*$ then $\hat{j}(\mathrm{Ind}(M)_\kappa)$ has a filtration by Weyl modules $\mathrm{Ind}(L_{-\lambda'+\rho})_\kappa$ for $\lambda'_i \in W\lambda$.*

3.2. **Structure of induced Harish-Chandra modules.** The main result of this subsection is the following analog of Kac–Kazhdan's Theorem 2.8 for the affine Harish-Chandra categories.

Theorem 3.5. *If $M_i \in \mathcal{H}$, $i = 1, 2$ are two irreducible $\tilde{\mathfrak{g}}$-modules then $\mathrm{Irr}(M_2)_\kappa$ is a subquotient of $\mathrm{Ind}(M_1)_\kappa$ only if M_1, M_2 have infinitesimal characters χ_{λ_1} and χ_{λ_2} for a pair $[\lambda_1, \lambda_2] \in \mathfrak{h}^* \times \mathfrak{h}^*$ satisfying the condition $(*)$ from Definition 2.7.*

Proof. Assume that $\mathrm{Irr}(M_2)_\kappa$ is a subquotient of $\mathrm{Ind}(M_1)_\kappa$. Then

$$\mathrm{Hom}(\mathrm{Ind}(M_2)_\kappa, \mathrm{Ind}(M_1)_\kappa/V_1) \neq 0$$

for some $\tilde{\mathfrak{g}}$-submodule V_1 of $\mathrm{Ind}(M_1)_\kappa$. The exactness and the faithfulness of the affine Jacquet functor imply that

$$\mathrm{Hom}(\hat{j}D(\mathrm{Ind}(M_2)_\kappa), \hat{j}D(\mathrm{Ind}(M_1)_\kappa)/\hat{j}D(V_1)) \neq 0. \tag{3.10}$$

Applying Proposition 3.2, we see that $\mathrm{Irr}(j(M_2^d))_\kappa$ is a subquotient of $\mathrm{Ind}(j(M_1^d))_\kappa$. Thus $\mathrm{Irr}(L_{\lambda_2})_\kappa$ is a subquotient of $\mathrm{Ind}(L_{\lambda_1})_\kappa$ for some $\lambda_i \in \chi(M_i)$, $i = 1, 2$. Then Corollary 3.4 and Kac–Kazhdan's Theorem 2.8 imply the statement of the proposition. □

3.3. **The general case of noncritical central charge.** The affine Jacquet functor (3.1) and most of its properties can be extended to the case $\kappa \neq 0$. This in particular gives extensions of Theorem 3.5 to all noncritical levels $\kappa \neq 0$, and of Proposition 2.6 to the case $\kappa \notin \mathbb{Q}_{\geq 0}$.

In this section we will assume that the central charge $\kappa - h^\vee$ is noncritical, i.e. that

$$\kappa \neq 0.$$

Introduce the categories $\widehat{\mathcal{H}}_{\kappa,inf}$ of $\tilde{\mathfrak{g}}$-modules of central charge $\kappa - h^\vee$, satisfying the same properties as the modules in $\widehat{\mathcal{H}}_\kappa$ except the condition to be finitely generated.

There are natural duality functors in the categories $\widehat{\mathcal{H}}_{\kappa,inf}$ and $\widehat{\mathcal{O}}'_{\kappa,inf}$ (defined in section 2.4), extending the duality functors D in $\widehat{\mathcal{H}}_\kappa$ and $\widehat{\mathcal{O}}'_\kappa$, defined by

$$D(V) = ((V^d)^\sharp(\infty))^{\mathbb{C}[L_0]-fin}, \quad V \in \widehat{\mathcal{H}}_{\kappa,inf} \text{ or } \widehat{\mathcal{O}}'_{\kappa,inf}. \tag{3.11}$$

If the $\tilde{\mathfrak{g}}$-module $V \in \widehat{\mathcal{H}}_{\kappa,inf}$ or $\widehat{\mathcal{O}}'_{\kappa,inf}$ has the decomposition (2.8), then as a submodule of $(V^d)^\sharp$ the dual module $D(V)$ is given by

$$D(V) = \bigoplus_{\xi} (V^d)^\xi \tag{3.12}$$

in the notation (2.17).

One can define an extension of the affine Jacquet functor (3.1) to a functor from the category $\widehat{\mathcal{H}}_{\kappa,inf}$ to the category $\widehat{\mathcal{O}}'_{\kappa,inf}$ by

$$\hat{j}(V) = (j(V)^\sharp(\infty))^{\mathbb{C}[L_0]-fin}. \tag{3.13}$$

If the $\tilde{\mathfrak{g}}$-module $V \in \widehat{\mathcal{H}}_{\kappa,inf}$ has the decomposition (2.8) then as a submodule of V^*, the affine Jacquet module $\hat{j}(V)$ is given by

$$\hat{j}(V) = \bigoplus_{\xi\colon \xi-\xi_i\in\mathbb{Z}_+} j(V^\xi) \subset \bigoplus_{\xi\colon \xi-\xi_i\in\mathbb{Z}_+} (V^\xi)^* = ((V^*)^\sharp(\infty))^{\mathbb{C}[L_0]-fin} \tag{3.14}$$

assuming the identification of $j(V^\xi)$ with the $\mathfrak{g}$-submodule (3.2) of V^*. This implies that

$$\hat{j}(V) \in \widehat{\mathcal{O}}'_{\kappa,inf}, \quad \text{for all } V \in \widehat{\mathcal{H}}_{\kappa,inf}$$

and the validity of an extension of Proposition 3.2 and Proposition 3.3 for the categories $\widehat{\mathcal{H}}_{\kappa,inf}$.

Proposition 3.6. *(i) The affine Jacquet functor* (3.13) $\hat{j}\colon \widehat{\mathcal{H}}_{\kappa,inf} \to \widehat{\mathcal{O}}'_{\kappa,inf}$ *is exact and faithful.*

(ii) For any $M \in \mathcal{H}$ we have the isomorphisms of $\tilde{\mathfrak{g}}$-modules in the category $\widehat{\mathcal{O}}'_{\kappa,inf}$

$$\hat{j}(\mathrm{Ind}(M)_\kappa) \cong D(\mathrm{Ind}(j(M)^d)_\kappa),$$
$$\hat{j}(D(\mathrm{Ind}(M)_\kappa)) \cong \mathrm{Ind}(j(M^d))_\kappa.$$

From part (ii) of Proposition 3.6 one obtains the following extension of Proposition 2.6 and Theorem 3.5.

Proposition 3.7. *(i) The statement of Theorem 3.5 is valid for any noncritical central charge.*

(ii) If $\kappa \notin \mathbb{Q}_{\geq 0}$ then every module in the category $\widehat{\mathcal{H}}_\kappa$ has finite length. In addition the category $\widehat{\mathcal{H}}_\kappa$ consists of all finite length, strictly smooth $\tilde{\mathfrak{g}}$-modules of central charge $\kappa - h^\vee$ for which

$$V(N) \in \mathcal{H},$$

considered as $\mathfrak{g}$-modules, for all $N \in \mathbb{Z}_{>0}$.

According to [17, Section 3], to prove part (ii) of Proposition 3.7 we need to show that $\mathrm{Ind}(M)_\kappa$ has finite length for any irreducible $\mathfrak{g}$-module $M \in \mathcal{H}$. Because of the faithfulness of the affine Jacquet functor and part (ii) of Proposition 3.6 it is sufficient to show that $\mathrm{Ind}(L_{\lambda-\rho})_\kappa$ has finite length for all $\lambda \in \mathfrak{h}^*$, recall the notation from section 2.4. The latter follows from Theorem 2.8 of Kac and Kazhdan. What needs to be proved is that $\mathrm{Irr}(L_{\mu-\rho})_\kappa$ is an irreducible subquotient of $\mathrm{Ind}(L_{\lambda-\rho})_\kappa$ for finitely many $\mu \in \mathfrak{h}^*$. If the pair $[\lambda, \mu] \in \mathfrak{h}^* \times \mathfrak{h}^*$ satisfies the condition $(*)$ from Definition 2.7 then

$$\mu = \lambda + \alpha = w\lambda + \kappa\beta \text{ with } \alpha, \beta \in Q \tag{3.15}$$

for some element $w \in W$. Notice that given $w \in W$ there is at most one weight $\mu \in \mathfrak{h}^*$ that satisfies (3.15) for some α, $\beta \in Q$. Indeed if (3.15) is satisfied then

$$\alpha + \kappa\beta = w\lambda - \lambda$$

which uniquely defines α and β since $Q \cap \kappa Q = 0$.

4. Block decomposition of affine Harish-Chandra categories

4.1. Relation of Ext **groups for $\tilde{\mathfrak{g}}$ and $\mathfrak{g}$-modules.** In this subsection we will assume that the central charge $\kappa - h^\vee$ is noncritical, i.e.

$$\kappa \neq 0.$$

By $\mathcal{C}_{(\tilde{\mathfrak{g}},\mathfrak{k},\kappa)}$ and $\mathcal{C}_{(\tilde{\mathfrak{p}}_+,\mathfrak{k},\kappa)}$ we will denote the categories of $\mathbb{C}$-graded $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{p}}_+$-modules, respectively, (with respect to the grading (2.9)) of central charge $\kappa - h^\vee$ which are locally finite and semisimple when restricted to $\mathfrak{k} \subset \tilde{\mathfrak{g}}$. For each module $V \in \mathcal{C}_{(\tilde{\mathfrak{g}},\mathfrak{k},\kappa)}$ or $\mathcal{C}_{(\tilde{\mathfrak{p}}_+,\mathfrak{k},\kappa)}$ we will denote by V^ξ its degree $\xi (\in \mathbb{C})$ subspace which is naturally a $\mathfrak{g}$-module in $\mathcal{C}_{(\mathfrak{g},\mathfrak{k})}$.

The category $\widehat{\mathcal{H}}_\kappa$ is canonically embedded in $\mathcal{C}_{(\tilde{\mathfrak{g}},\mathfrak{k},\kappa)}$ using the grading (2.8) related to the generalized eigenvalue decomposition of the Sugawara operator L_0 on $V \in \widehat{\mathcal{H}}_\kappa$. Clearly each morphism in $\widehat{\mathcal{H}}_\kappa$ preserves this decomposition.

Similarly to the category $\mathcal{C}_{(\mathfrak{g},\mathfrak{k})}$, the categories $\mathcal{C}_{(\tilde{\mathfrak{g}},\mathfrak{k},\kappa)}$ and $\mathcal{C}_{(\tilde{\mathfrak{p}}_+,\mathfrak{k},\kappa)}$ have enough projectives. Indeed if $V \in \mathcal{C}_{(\tilde{\mathfrak{g}},\mathfrak{k},\kappa)}$ is generated as a $U(\tilde{\mathfrak{g}})$-module by the $\mathfrak{k}$ stable graded subspace E of V then

$$U(\tilde{\mathfrak{g}})/\langle K - \kappa + h^\vee \rangle \otimes_{U(\mathfrak{k})} E$$

is a projective cover of V. Here $\langle K - \kappa + h^\vee \rangle$ denotes the (two-sided) ideal of $U(\tilde{\mathfrak{g}})$ generated by $K - \kappa + h^\vee$. The first factor $U(\tilde{\mathfrak{g}})$ is graded using (2.9). Analogously one shows that $\mathcal{C}_{(\tilde{\mathfrak{p}}_+,\mathfrak{k},\kappa)}$ has enough projectives.

By $\mathrm{Ext}_{\tilde{\mathfrak{g}},\mathfrak{k},\kappa}(.,.)$ and $\mathrm{Ext}_{\tilde{\mathfrak{p}}_+,\mathfrak{k},\kappa}(.,.)$ we will denote the extension groups in the categories $\mathcal{C}_{(\tilde{\mathfrak{g}},\mathfrak{k},\kappa)}$ and $\mathcal{C}_{(\tilde{\mathfrak{p}}_+,\mathfrak{k},\kappa)}$, respectively. For simplicity the Hom groups in $\mathcal{C}_{(\tilde{\mathfrak{g}},\mathfrak{k},\kappa)}$ and $\mathcal{C}_{(\tilde{\mathfrak{p}}_+,\mathfrak{k},\kappa)}$ will be denoted by $\mathrm{Hom}_{\tilde{\mathfrak{g}}}$ and $\mathrm{Hom}_{\tilde{\mathfrak{p}}_+}$.

In this subsection we relate Ext-groups between induced modules in $\mathcal{C}_{(\tilde{\mathfrak{g}},\mathfrak{k},\kappa)}$ to Ext-groups in $\mathcal{C}_{(\mathfrak{g},\mathfrak{k})}$.

Consider the induction functor

$$\mathrm{I}(.) \colon \mathcal{C}_{(\tilde{\mathfrak{p}}_+,\mathfrak{k},\kappa)} \to \mathcal{C}_{(\tilde{\mathfrak{g}},\mathfrak{k},\kappa)}$$

defined by

$$\mathrm{I}(P) = U(\tilde{\mathfrak{g}}) \otimes_{U(\tilde{\mathfrak{p}}_+)} P, \quad P \in \mathcal{C}_{(\tilde{\mathfrak{p}}_+,\mathfrak{k},\kappa)}.$$

Since it is exact, the canonical isomorphism

$$\mathrm{Hom}_{\tilde{\mathfrak{g}}}(\mathrm{I}(P)_\kappa, V) \cong \mathrm{Hom}_{\tilde{\mathfrak{p}}_+}(P, V), \quad P \in \mathcal{C}_{(\tilde{\mathfrak{p}}_+,\mathfrak{k},\kappa)},\ V \in \mathcal{C}_{(\tilde{\mathfrak{g}},\mathfrak{k},\kappa)}$$

gives rise to the isomorphisms of the corresponding Ext groups

$$\mathrm{Ext}_{\tilde{\mathfrak{g}},\mathfrak{k},\kappa}(\mathrm{I}(P)_\kappa, V) \cong \mathrm{Ext}_{\tilde{\mathfrak{p}}_+,\mathfrak{k},\kappa}(P, V), \quad P \in \mathcal{C}_{(\tilde{\mathfrak{p}}_+,\mathfrak{k},\kappa)},\ V \in \mathcal{C}_{(\tilde{\mathfrak{g}},\mathfrak{k},\kappa)}. \tag{4.1}$$

In particular we obtain the isomorphisms

$$\mathrm{Ext}^n_{\tilde{\mathfrak{g}},\mathfrak{k},\kappa}(\mathrm{Ind}(M)_\kappa, V) \cong \mathrm{Ext}^n_{\tilde{\mathfrak{p}}_+,\mathfrak{k},\kappa}(M, V), \quad M \in \mathcal{H},\ V \in \mathcal{C}_{(\tilde{\mathfrak{g}},\mathfrak{k},\kappa)} \tag{4.2}$$

where in the rhs $M \in \mathcal{H}$ is considered as a $\tilde{\mathfrak{p}}_+$-module in the category $\mathcal{C}_{(\tilde{\mathfrak{p}}_+,\mathfrak{k},\kappa)}$ by setting $\tilde{\mathfrak{p}}_+$ to act on M through the quotient $\tilde{\mathfrak{p}}_+ \to \tilde{\mathfrak{p}}_+/\tilde{\mathfrak{n}}_+ \cong \mathfrak{g} \oplus \mathbb{C}K$ and K to act on M by $(\kappa - h^\vee)$.id. If $M = \bigoplus_{\chi_\lambda \in \mathfrak{h}^*/W} M^{\chi_\lambda}$ is the decomposition of M with respect to (2.2) then $M^{\chi_\lambda} \subset M$ is put in degree $\phi(\lambda)$, recall (3.8). (Recall also the definition (2.14) of the induction functor $\mathrm{Ind}(.)_\kappa : \mathcal{C}_{(\mathfrak{g},\mathfrak{k})} \to \mathcal{C}_{(\tilde{\mathfrak{g}},\mathfrak{k},\kappa)}$.)

Assume for simplicity that $M \in \mathcal{H}^{\chi_\lambda}$ for some $\lambda \in \mathfrak{h}^*$. Consider the natural isomorphism

$$\mathrm{Hom}_{\tilde{\mathfrak{p}}_+}(M, V) \cong \mathrm{Hom}_{\mathfrak{g}}(M, (V^{\phi(\lambda)})^{\tilde{\mathfrak{n}}_+}), \quad M \in \mathcal{H},\ V \in \mathcal{C}_{(\tilde{\mathfrak{g}},\mathfrak{k},\kappa)}, \tag{4.3}$$

where M is thought of as a $\tilde{\mathfrak{p}}_+$ module in $\mathcal{C}_{(\tilde{\mathfrak{p}}_+,\mathfrak{k},\kappa)}$ as described above.

Given a complex number ξ, denote by $H^q(\tilde{\mathfrak{n}}_+, .)^\xi$ the q-th derived functor of the left exact functor $V \mapsto (V^\xi)^{\tilde{\mathfrak{n}}_+}$ from $\mathcal{C}_{(\tilde{\mathfrak{p}}_+,\mathfrak{k},\kappa)}$ to $\mathcal{C}_{(\mathfrak{g},\mathfrak{k})}$, for any $\xi \in \mathbb{C}$. A version of the Hochschild–Serre spectral sequence relates the Ext groups, corresponding to Hom's in (4.3):

$$E^2_{pq} = \mathrm{Ext}^p_{\mathfrak{g},\mathfrak{k}}(M, H^q(\tilde{\mathfrak{n}}_+, V)^{\phi(\lambda)}) \Rightarrow \mathrm{Ext}^n_{\tilde{\mathfrak{p}}_+,\mathfrak{k},\kappa}(M, V), \quad M \in \mathcal{C}_{(\mathfrak{g},\mathfrak{k})},\ V \in \mathcal{C}_{(\tilde{\mathfrak{p}}_+,\mathfrak{k},\kappa)}. \tag{4.4}$$

Since

$$V^{\tilde{\mathfrak{n}}_+} = \bigoplus_{\xi \in \mathbb{C}} (V^\xi)^{\tilde{\mathfrak{n}}_+},\ \forall V \in \mathcal{C}_{(\tilde{\mathfrak{p}}_+,\mathfrak{k},\kappa)},$$

if $H^q(\tilde{\mathfrak{n}}_+, .)$ denotes the q-th derived functor of the functor $V \mapsto V^{\tilde{\mathfrak{n}}_+}$ from $\mathcal{C}_{(\tilde{\mathfrak{p}}_+,\mathfrak{k},\kappa)}$ to $\mathcal{C}_{(\mathfrak{g},\mathfrak{k})}$, then

$$H^q(\tilde{\mathfrak{n}}_+, .) = \bigoplus_{\xi \in \mathbb{C}} H^q(\tilde{\mathfrak{n}}_+, .)^\xi. \tag{4.5}$$

Combining (4.2) and (4.4) we obtain:

Proposition 4.1. *For all modules $M \in \mathcal{H}^{\chi_\lambda}$ and $V \in \mathcal{C}_{(\tilde{\mathfrak{g}},\mathfrak{k},\kappa)}$ one has the spectral sequence*

$$E^2_{pq} = \mathrm{Ext}^p_{\mathfrak{g},\mathfrak{k}}(M, H^q(\tilde{\mathfrak{n}}_+, V)^{\phi(\lambda)}) \Rightarrow \mathrm{Ext}^n_{\tilde{\mathfrak{g}},\mathfrak{k},\kappa}(\mathrm{Ind}(M)_\kappa, V).$$

Consider the equivalence relation $\sim$ in $\mathfrak{h}^*$ induced by the relation $\lambda \sim \mu$ if $\mu \in W\lambda$ or the pair $[\lambda, \mu]$ satisfies the condition $(*)$ from Definition 2.7.

Proposition 4.2. *If $\lambda_i \in \mathfrak{h}^*$, $i = 1, 2$ and $\lambda_1 \not\sim \lambda_2$, then*

$$\mathrm{Ext}^n_{\tilde{\mathfrak{g}},\mathfrak{k},\kappa}(\mathrm{Ind}(M_1)_\kappa, D(\mathrm{Ind}(M_2^d)_\kappa)) = 0, \tag{4.6}$$

for all $\mathfrak{g}$-modules $M_1 \in \mathcal{H}^{\chi_{\lambda_1}}$ and $M_2 \in \mathcal{H}^{\chi_{\lambda_2}}$.

We will deduce Proposition 4.2 from the following lemma to be proved at the end of the subsection.

Lemma 4.3. *If $M \in \mathcal{H}$*

$$H^n(\tilde{\mathfrak{n}}_+, D(\mathrm{Ind}(M)_\kappa)) = 0$$

for $n > 0$.

Proof of Proposition 4.2. Lemma 4.3 shows that the spectral sequence from Proposition 4.1 degenerates at the E^2 term in the case $V_2 = D(\mathrm{Ind}(M_2)_\kappa)$. Therefore

$$\mathrm{Ext}^n_{\tilde{\mathfrak{g}},\mathfrak{k},\kappa}(\mathrm{Ind}(M_1)_\kappa, D(\mathrm{Ind}(M_2)_\kappa)) \cong \mathrm{Ext}^n_{\mathfrak{g},\mathfrak{k}}(M_1, D(\mathrm{Ind}(M_2)_\kappa)^{\tilde{\mathfrak{n}}_+}), \tag{4.7}$$

for all $M_1 \in \mathcal{H}^{\chi_{\lambda_1}}$ $M_2 \in \mathcal{H}^{\chi_{\lambda_2}}$.

Theorem 3.5 and part (i) of Proposition 3.7 imply that any irreducible subquotient of $\mathrm{Ind}(M_2^d)_\kappa$ is isomorphic to $\mathrm{Irr}(M)_\kappa$ for some irreducible $\mathfrak{g}$-module $M \in \mathcal{H}^{\chi_\lambda}$, $-\lambda \sim \lambda_2$. Recall that

$$D(\mathrm{Ind}(M^d)_\kappa) \cong \mathrm{Ind}(M)_\kappa, \quad \text{for all irreducible } M \in \mathcal{H} \tag{4.8}$$

according to [17, Proposition 4.1, part 3]. Because of this all irreducible subquotients of $D(\mathrm{Ind}(M_2^d)_\kappa$ are of the type $\mathrm{Irr}(M)_\kappa$ for irreducible $\mathfrak{g}$-modules $M \in \mathcal{H}^{\chi_\lambda}$, $\lambda \sim \lambda_2$. Therefore

$$(D(\mathrm{Ind}(M_2^d)_\kappa))^{\tilde{\mathfrak{n}}_+} = D(\mathrm{Ind}(M_2^d)_\kappa)(0) \in \prod_{\lambda\sim\lambda_2} \mathcal{H}^{\chi_\lambda}.$$

Eq. (4.7) and Proposition 2.1 imply (4.6). □

Proof of Lemma 4.3. We follow the strategy of the proof of [5, Proposition 4.7] of Deodhar, Gabber, and Kac.

Because of (4.5) we need to show

$$H^q(\tilde{\mathfrak{n}}_+, D(\mathrm{Ind}(M)_\kappa))^\xi = 0 \tag{4.9}$$

for all $\xi \in \mathbb{C}$.

Denote by $\mathbb{C}$ the trivial $\tilde{\mathfrak{p}}_+$-module in degree 0. We have the $U(\tilde{\mathfrak{n}}_+)$-free resolution of $\mathbb{C}[-\xi]$ in $\mathcal{C}_{\tilde{\mathfrak{p}}_+,\mathfrak{k},\kappa}$:

$$\ldots \to R^1 \to R^0 \to \mathbb{C}[-\xi] \to 0$$

where

$$R^q = U(\tilde{\mathfrak{n}}_+) \otimes_{\mathbb{C}} \wedge^q \tilde{\mathfrak{n}}_+[-\xi]$$

and $\tilde{\mathfrak{p}}_+$ is defined to act on R^j as follows. The ideal $\tilde{\mathfrak{n}}_+$ acts by left multiplication on the first term, $\mathfrak{g}$ acts by adjoint transformation on all factors, and K acts by $(\kappa - h^\vee)$.id. The grading (2.9) is used in the definition of the graded components of R^q. By $(.)[-\xi]$ we denote the standard operator of shift of the grading by $\xi \in \mathbb{C}$.

The group $H^q(\tilde{\mathfrak{n}}_+, D(\mathrm{Ind}(M)_\kappa))^\xi$ is the q-th homology group of the associated complex $\{\mathrm{Hom}(R^\bullet, D(\mathrm{Ind}(M)_\kappa)))\}$. Note that the latter consists of finite dimensional vector spaces.

One has the isomorphism of vector spaces

$$\mathrm{Hom}(R^\bullet, D(\mathrm{Ind}(M)_\kappa))) \cong \mathrm{Hom}_{gr}(\wedge^\bullet \tilde{\mathfrak{n}}_+, D(\mathrm{Ind}(M)_\kappa)))$$

where Hom_{gr} denotes the spaces of degree 0 homomorphism of graded vector spaces.

Consider the complex defining the homology groups $H_q(\tilde{\mathfrak{n}}_-, \mathrm{Ind}(M)_\kappa)$:

$$0 \to \mathrm{Ind}(M)_\kappa \to C_0 \to C_0 \to \ldots$$

where

$$C_q = \wedge^q \tilde{\mathfrak{n}}_- \otimes_{\mathbb{C}} \mathrm{Ind}(M)_\kappa.$$

It is a complex of $\mathbb{C}$ graded vector spaces using the grading (2.9) of $\tilde{\mathfrak{n}}_-$ and the grading (2.8) of $\mathrm{Ind}(M)_\kappa$. The differential preserves this grading. Denote the component of degree $\xi \in \mathbb{C}$ of C_q by C_q^ξ. Note that it is a finite dimensional vector space.

There is a natural isomorphism of complexes

$$\{\mathrm{Hom}_{gr}(\wedge^\bullet \tilde{\mathfrak{n}}_+, D(\mathrm{Ind}(M)_\kappa)))\} \cong \{\left[(\wedge^\bullet \tilde{\mathfrak{n}}_- \otimes_{\mathbb{C}} \mathrm{Ind}(M)_\kappa)^\xi\right]^*\}$$

by $\varphi \mapsto \left[\omega \otimes v \mapsto \varphi(\omega^\sharp)(v)\right]$, $\omega \in \wedge^\bullet \tilde{\mathfrak{n}}_+$, $v \in \mathrm{Ind}(M)_\kappa$.

Since $H_q(\tilde{\mathfrak{n}}_-, \mathrm{Ind}(M)_\kappa) = 0$ for $q > 0$ ($\mathrm{Ind}(M)_\kappa$ is a free $\tilde{\mathfrak{n}}_-$-module) we obtain (4.9). □

4.2. **Block decomposition of $\widehat{\mathcal{H}}_\kappa$ for $\kappa \notin \mathbb{Q}_{\geq 0}$.** In this subsection we will assume that

$$\kappa \notin \mathbb{Q}_{\geq 0}.$$

For any $\widehat{\chi}_\lambda \in (\mathfrak{h}^*/\sim)$ denote by $\widehat{\mathcal{H}}_\kappa^{\widehat{\chi}_\lambda}$ the full subcategory of $\widehat{\mathcal{H}}_\kappa$ consisting of $\tilde{\mathfrak{g}}$-modules having composition series with subquotients $\mathrm{Irr}(M)_\kappa$ for irreducible $\mathfrak{g}$-modules with infinitesimal character $\chi_{\lambda'}$ for some $\lambda' \sim \lambda$.

Theorem 3.5 implies

$$\mathrm{Ind}(M)_\kappa \in \widehat{\mathcal{H}}_\kappa^{\widehat{\chi}_\lambda}, \quad \text{if } \chi(M) = \chi_\lambda. \tag{4.10}$$

The main result in this section is the following generalization of the important Proposition 2.1 on vanishing of $\mathrm{Ext}_{\mathfrak{g},\mathfrak{k}}$-groups between different blocks in the standard Harish-Chandra category.

Theorem 4.4. *If $\widehat{\chi}_1, \widehat{\chi}_2 \in (\mathfrak{h}^*/\sim)$ are such that*

$$\widehat{\chi}_1 \neq \widehat{\chi}_2 \tag{4.11}$$

then

$$\mathrm{Ext}^n_{\tilde{\mathfrak{g}},\mathfrak{k},\kappa}(V_1, V_2) = 0 \tag{4.12}$$

for all $V_1 \in \widehat{\mathcal{H}}_\kappa^{\widehat{\chi}_1}$ and $V_2 \in \widehat{\mathcal{H}}_\kappa^{\widehat{\chi}_2}$, $n \in \mathbb{Z}_{\geq 0}$.

The special case of Theorem 4.4 for vanishing of Ext^1 groups implies a block decomposition of the categories $\widehat{\mathcal{H}}_\kappa$ for $\kappa \notin \mathbb{Q}_{\geq 0}$ which is a generalization of the block decomposition Theorem 2.9 of Deodhar–Gabber–Kac and Rocha-Caridi–Wallach for the BGG category $\mathcal{O}$ for Kac–Moody algebras.

Proposition 4.5. *For $\kappa \notin \mathbb{Q}_{\geq 0}$ the category $\widehat{\mathcal{H}}_\kappa$ is a direct sum of its subcategories $\widehat{\mathcal{H}}_\kappa^{\widehat{\chi}}$:*

$$\widehat{\mathcal{H}}_\kappa \cong \bigoplus_{\widehat{\chi} \in (\mathfrak{h}^*/\sim)} \widehat{\mathcal{H}}_\kappa^{\widehat{\chi}}.$$

We will prove Theorem 4.4 in two steps. First we will prove an auxiliary lemma.

Lemma 4.6. *Assume that $\lambda_1, \lambda_2 \in \mathfrak{h}^*$ and*

$$\lambda_1 \not\sim \lambda_2 \tag{4.13}$$

If $M_1 \in \mathcal{H}^{\chi_{\lambda_1}}$ and $V_2 \in \widehat{\mathcal{H}}^{\widehat{\chi}_{\lambda_2}}$ then

$$\mathrm{Ext}^n_{\tilde{\mathfrak{g}},\mathfrak{k},\kappa}(\mathrm{Ind}(M_1)_\kappa, V_2) = 0 \tag{4.14}$$

for all $n \in \mathbb{Z}_{\geq 0}$.

Proof. We will show (4.14) by induction on n. The case $n = 0$ follows from Theorem 3.5. Assume the validity of (4.14) for some $n \in \mathbb{Z}_{\geq 0}$. Every $\tilde{\mathfrak{g}}$-module in $\widehat{\mathcal{H}}^{\widehat{\chi}_{\lambda_2}}$ has a finite composition series with subquotients of the type $\mathrm{Irr}(M)_\kappa$ for irreducible $\mathfrak{g}$-modules $M \in \mathcal{H}$ with infinitesimal character $\chi(M) = \chi_{\lambda'}$ such that $\lambda' \sim \lambda_2$. Using the long exact sequence for the Ext groups one see that to prove (4.14) for $n+1$ one only needs to show that

$$\mathrm{Ext}^n_{\tilde{\mathfrak{g}},\mathfrak{k},\kappa}(\mathrm{Ind}(M_1)_\kappa, \mathrm{Irr}(M_2)_\kappa) = 0$$

for $M_1 \in \mathcal{H}^{\chi_{\lambda_1}}$ and $M_2 \in \mathcal{H}^{\chi_{\lambda_2}}$, assuming (4.13).

Denote the maximal, nontrivial $\tilde{\mathfrak{g}}$-submodule of $\mathrm{Ind}(M_2)_\kappa$ by $X(M_2^d)_\kappa$. Then one has the exact sequence of $\tilde{\mathfrak{g}}$-modules

$$0 \to X(M_2^d)_\kappa \to \mathrm{Ind}(M_2^d)_\kappa \to \mathrm{Irr}(M_2^d)_\kappa \to 0.$$

Applying the exact duality functor D and using the fact that $D(\mathrm{Irr}(M_2^d)_\kappa) \cong \mathrm{Irr}(M_2)_\kappa$, see (4.8), we get the exact sequence

$$0 \to \mathrm{Irr}(M_2)_\kappa \to D(\mathrm{Ind}(M_2^d)_\kappa) \to D(X(M_2^d)_\kappa) \to 0. \tag{4.15}$$

This finally gives the exact sequence of Ext groups

$$\begin{aligned}\mathrm{Ext}^n_{\tilde{\mathfrak{g}},\mathfrak{k},\kappa}(\mathrm{Ind}(M_1)_\kappa, D(X(M_2^d)_\kappa)) \to \mathrm{Ext}^{n+1}_{\tilde{\mathfrak{g}},\mathfrak{k},\kappa}(\mathrm{Ind}(M_1)_\kappa, \mathrm{Irr}(M_2)_\kappa) \to \\ \to \mathrm{Ext}^{n+1}_{\tilde{\mathfrak{g}},\mathfrak{k},\kappa}(\mathrm{Ind}(M_1)_\kappa, D(\mathrm{Ind}(M_2^d)_\kappa)).\end{aligned} \tag{4.16}$$

The induction implies that the first group vanishes because $D(X(M_2^d)_\kappa)$ is a quotient of $D(\mathrm{Ind}(M_2^d)_\kappa) \in \widehat{\mathcal{H}}^{\widehat{\chi}_{\lambda_2}}$. The third group vanishes, as proved in Proposition 4.2. This implies that the middle group vanishes and completes the induction argument. □

Proof of Theorem 4.4. Again we use induction on n. The case $n = 0$ is trivial. Assume the validity of the statement in Theorem 4.4 for an integer n. Since every module in the categories $\widehat{\mathcal{H}}_\kappa$ has finite length for $\kappa \notin \mathbb{Q}_{\geq 0}$, to show that (4.12) holds for $n + 1$, we only need to prove

$$\mathrm{Ext}^n_{\tilde{\mathfrak{g}},\mathfrak{k},\kappa}(\mathrm{Irr}(M_1)_\kappa, V_2) = 0 \tag{4.17}$$

for an irreducible $\mathfrak{g}$-module $M_1 \in \mathcal{H}$ with infinitesimal character $\chi(M_1) = \chi_{\lambda_1} \in \mathfrak{h}^*/W$ such that $\widehat{\chi}_{\lambda_1} \neq \widehat{\chi}_2$. Using the exact sequence

$$0 \to X(M_1)_\kappa \to \mathrm{Ind}(M_1)_\kappa \to \mathrm{Irr}(M_1)_\kappa \to 0$$

we obtain the exact sequence

$$\begin{aligned}\mathrm{Ext}^n_{\tilde{\mathfrak{g}},\mathfrak{k},\kappa}(X(M_1)_\kappa, V_2) \to \mathrm{Ext}^{n+1}_{\tilde{\mathfrak{g}},\mathfrak{k},\kappa}(\mathrm{Irr}(M_1)_\kappa, V_2) \to \\ \to \mathrm{Ext}^{n+1}_{\tilde{\mathfrak{g}},\mathfrak{k},\kappa}(\mathrm{Ind}(M_1)_\kappa, V_2)\end{aligned}$$

The first term vanishes due to the induction hypothesis. The third term vanishes because of Lemma 4.6. This implies (4.17) and completes the proof of Theorem 4.4. □

4.3. **A block decomposition of $\widehat{\mathcal{H}}_{\kappa,inf}$ for arbitrary noncritical central charge.** In this section we extend the block decomposition of the categories $\widehat{\mathcal{H}}_\kappa$ from section 4.2 to the case of arbitrary noncritical central charge. Even more generally we provide such a decomposition for the categories $\widehat{\mathcal{H}}_\kappa$ of not necessarily finitely generated affine Harish-Chandra modules. Note that they are naturally embedded in the categories $\mathcal{C}_{(\tilde{\mathfrak{g}},\mathfrak{k},\kappa)}$, defined in section 4.1.

It is easy to show by induction that:

(•) *Every module V in $\widehat{\mathcal{H}}_{\kappa,inf}$ has an increasing filtration*

$$0 = W_0 \subset W_1 \subset \ldots \subset V$$

for which the subquotients W_i/W_{i-1} are isomorphic to quotients of induced modules $\mathrm{Ind}(M_i)_\kappa$ *for some irreducible $\mathfrak{g}$-modules $M_i \in \mathcal{H}$.*

The subcategory $\widehat{\mathcal{H}}_\kappa$ of $\widehat{\mathcal{H}}_{\kappa,inf}$ consists of those $\tilde{\mathfrak{g}}$-modules in $\widehat{\mathcal{H}}_{\kappa,inf}$ that have a finite filtration with the property (∗). If $\kappa \in \mathbb{Q}_{>0}$ the modules in $\widehat{\mathcal{H}}_\kappa$ have in general infinite length.

Given any $\widehat{\chi}_\lambda \in (\mathfrak{h}^*/\sim)$ for some $\lambda \in \mathfrak{h}^*$ denote by $\widehat{\mathcal{H}}^{\widehat{\chi}_\lambda}_{\kappa,inf}$ the subcategory of $\widehat{\mathcal{H}}_{\kappa,inf}$ consisting of $\tilde{\mathfrak{g}}$-modules possessing a filtration with the property $(\bullet)$ such that

$$\chi(M_i) = \chi_{\lambda'} \text{ for some } \lambda' \sim \lambda. \tag{4.18}$$

Theorem 3.5 and part (i) of Proposition 3.7 imply that $\widehat{\mathcal{H}}^{\widehat{\chi}_\lambda}_{\kappa,inf}$ can be characterized as the subcategory of $\widehat{\mathcal{H}}_\kappa$ consisting of $\tilde{\mathfrak{g}}$-module for which all irreducible subquotients are isomorphic to $\mathrm{Irr}(M)_\kappa$ for some irreducible $\mathfrak{g}$-module $M \in \mathcal{H}$ satisfying (4.18).

The subcategories $\widehat{\mathcal{H}}^{\widehat{\chi}_\lambda}_\kappa = \widehat{\mathcal{H}}_\kappa \cap \widehat{\mathcal{H}}^{\widehat{\chi}_\lambda}_{\kappa,inf}$ of $\widehat{\mathcal{H}}_\kappa$ consist of those $\tilde{\mathfrak{g}}$-modules $V \in \widehat{\mathcal{H}}_\kappa$, having a finite filtration by $\tilde{\mathfrak{g}}$-modules with subquotients that are isomorphic to quotients of induced modules $\mathrm{Irr}(M)_\kappa$ for irreducible $\mathfrak{g}$-modules $M \in \mathcal{H}$ with infinitesimal character as in (4.18). In the case $\kappa \notin \mathbb{Q}_{\geq 0}$ the definition of $\widehat{\mathcal{H}}^{\widehat{\chi}_\lambda}_\kappa$ is consistent with the one from section 4.2.

The main result of this section is the following generalization of the basic decomposition theorem of Deodhar–Gabber–Kac and Rocha-Caridi–Wallach for the BGG category $\mathcal{O}$ for Kac-Moody algebras.

Theorem 4.7. *For any noncritical central charge the category $\widehat{\mathcal{H}}_{\kappa,inf}$ is a direct sum of its subcategories $\widehat{\mathcal{H}}^{\widehat{\chi}}_{\kappa,inf}$:*

$$\widehat{\mathcal{H}}_{\kappa,inf} = \bigoplus_{\widehat{\chi} \in (\mathfrak{h}^*/\sim)} \widehat{\mathcal{H}}^{\widehat{\chi}}_{\kappa,inf}.$$

Theorem 4.7 follows from Proposition 4.11 below as [5, Theorem 4.2] of Deodhar, Gabber, and Kac.

First we state two corollaries of Theorem 4.7. There exists a coarser block decomposition of the affine Harish-Chandra categories $\widehat{\mathcal{H}}_\kappa$ which more closely resembles the finite dimensional case. To state it, consider the equivalence relation on $\mathfrak{h}^*$, defined by

$$\lambda \approx \mu \text{ if } \mu \in W\lambda + Q^\vee \text{ and } \mu - \lambda \in Q.$$

Corollary 4.8. *For any noncritical central charge the affine Harish-Chandra category has the block decomposition*

$$\widehat{\mathcal{H}}_\kappa = \bigoplus_{\widehat{\chi} \in (\mathfrak{h}^*/\approx)} \widehat{\mathcal{H}}^{\widehat{\chi}}_\kappa$$

where for $\widehat{\chi} \in (\mathfrak{h}^*/\approx)$, $\widehat{\mathcal{H}}^{\widehat{\chi}}_\kappa$ is the subcategory of $\widehat{\mathcal{H}}_\kappa$ consisting of affine Harish-Chandra modules possessing a finite filtration with subquotients isomorphic to quotients of induced modules $\mathrm{Ind}(M)_\kappa$ for $\chi(M) = \chi_\lambda$, $\lambda \in \widehat{\chi}$.

This block decomposition is even more explicit in the case of rational central charge. Define $\widehat{\mathcal{H}}^{int}_\kappa$ to be the subcategory of $\widehat{\mathcal{H}}_\kappa$ consisting of $\tilde{\mathfrak{g}}$-modules V for which all $\mathfrak{g}$-submodules V^ξ have composition series with subquotients with infinitesimal characters in P/W where P is the weight lattice of $\mathfrak{g}$. In other words $\widehat{\mathcal{H}}^{int}_\kappa$ is the subcategory of $\widehat{\mathcal{H}}_\kappa$ consisting of $\tilde{\mathfrak{g}}$-modules possessing a finite filtration with subquotients isomorphic to quotients of induced modules $\mathrm{Ind}(M)_\kappa$ for $\chi(M) = \chi_\lambda$, $\lambda \in P$.

Corollary 4.9. *Assume that $\kappa = p/q$ for two nonzero, relatively prime integers p and q. Then the subcategory $\widehat{\mathcal{H}}^{int}_\kappa$ of $\widehat{\mathcal{H}}_\kappa$ has the block decomposition*

$$\widehat{\mathcal{H}}^{int}_\kappa = \bigoplus_{\widehat{\chi}_\lambda \in \mathfrak{h}^*/(W \ltimes pQ^\vee)} \widehat{\mathcal{H}}^{int, \widehat{\chi}_\lambda}_\kappa$$

where $\widehat{\mathcal{H}}^{int,\widehat{\chi}_\lambda}$ is the subcategory of $\widehat{\mathcal{H}}^{int}_\kappa$ consisting of modules possessing a finite filtration with subquotients isomorphic to quotients of induced modules $\mathrm{Ind}(M)_\kappa$ *for $\chi(M) = \chi_\mu$, $\mu \in W\lambda + pQ^\vee$.*

Remark 4.10. Corollary 4.8 and Corollary 4.9 are easily extended to the infinitely generated version (the category $\widehat{\mathcal{H}}^{int}_{\kappa,inf}$) of the category $\widehat{\mathcal{H}}^{int}_\kappa$. The category $\widehat{\mathcal{H}}^{int}_{\kappa,inf}$ is the full subcategory of $\widehat{\mathcal{H}}_{\kappa,inf}$, consisting of $\tilde{\mathfrak{g}}$-modules V for which all $\mathfrak{g}$-submodules V^ξ have composition series with subquotients with infinitesimal characters in P/W. We state them only in the case of $\widehat{\mathcal{H}}^{int}_\kappa$ to avoid extra technical details.

Now we return to the proof of Theorem 4.7.

Proposition 4.11. *If $\widehat{\chi}_1, \widehat{\chi}_2 \in (\mathfrak{h}^*/\sim)$ and $\widehat{\chi}_1 \neq \widehat{\chi}_2$, then*

$$\mathrm{Ext}^1_{\tilde{\mathfrak{g}},\mathfrak{k},\kappa}(V_1, V_2) = 0, \textit{ for all } V_1 \in \widehat{\mathcal{H}}^{\widehat{\chi}_1}_\kappa,\ V_2 \in \widehat{\mathcal{H}}^{\widehat{\chi}_2}_\kappa.$$

For the proof of Proposition 4.11 we will need the following lemma.

Lemma 4.12. *Let $V \in \widehat{\mathcal{H}}_{\kappa,inf}$. For any real number ζ the module V has a finite filtration by $\tilde{\mathfrak{g}}$-submodules*

$$0 = W_0 \subset W_1 \subset \ldots \subset W_N = V$$

such that there exists a subset $J \subset \{1, \ldots N\}$ with the following properties.

(i) If $j \in J$, then

$$V_j/V_{j-1} \cong \mathrm{Irr}(M_j)_\kappa, \quad \textit{for some irreducible } \mathfrak{g}\textit{-modules } M \in \mathcal{H}.$$

(ii) If $\notin J$, then $(V_j/V_{j-1})^\xi = 0$ for $\mathrm{Re}\xi \leq \zeta$.

For a $\tilde{\mathfrak{g}}$-module $V \in \widehat{\mathcal{H}}_{\kappa,inf}$ and a real number ζ define the $\mathfrak{g}$-module

$$V^{\langle\zeta\rangle} = \bigoplus_{\xi \in \mathbb{C} \colon \mathrm{Re}\xi \leq \zeta} V^\xi \in \mathcal{H}. \tag{4.19}$$

Note that the above sum is finite.

We prove *Lemma 4.12* by induction on the length of the $\mathfrak{g}$-module $V^{\langle\zeta\rangle}$.

If $V^{\langle\zeta\rangle}$ is trivial, then the statement of the lemma is obvious. Otherwise choose $\xi_0 \in \mathbb{C}$ such that $\mathrm{Re}\xi_0 \leq \zeta$,

$$V^{\xi_0} \neq 0 \quad \text{and} \quad V^{\xi_0 - n} = 0 \text{ if } n \in \mathbb{Z}_{>0}. \tag{4.20}$$

Let M be an irreducible $\mathfrak{g}$-submodule of V^{ξ_0}. Then M is annihilated by $\tilde{\mathfrak{n}}_+$ and we get a canonical homomorphism $f\colon \mathrm{Ind}(M)_\kappa \to V$. Denote the image of f by U_2 and the image of the maximal, nontrivial submodule of $\mathrm{Ind}(M)_\kappa$ under f by U_1. Then

$$U_2/U_1 \cong \mathrm{Irr}(M)_\kappa$$

and the lengths of the $\mathfrak{g}$-modules $U_1^{\langle\zeta\rangle}$ and $(V/U_2)^{\langle\zeta\rangle}$ are strictly less than the length $V^{\langle\zeta\rangle}$. This completes the induction argument. □

We need an auxiliary lemma for the proof of Proposition 4.11.

Lemma 4.13. *Let $M \in \mathcal{H}$ be an irreducible $\mathfrak{g}$-module with infinitesimal character χ_λ, $\lambda \in \mathfrak{h}^*$.*

(i) If $V_2 \in \widehat{\mathcal{H}}_\kappa$ is such that

$$V_2^{\langle \mathrm{Re}\phi(\lambda) \rangle} = 0$$

then $\mathrm{Ext}_{\tilde{\mathfrak{g}},\mathfrak{k},\kappa}(\mathrm{Irr}(M)_\kappa, V_2) = 0$, *recall* (3.8).

(ii) If $V_2 \in \widehat{\mathcal{H}}_\kappa^{\widehat{\chi}_2}$, and $\widehat{\chi}_\lambda \neq \widehat{\chi}_2$ then

$$\mathrm{Ext}^1_{\tilde{\mathfrak{g}},\mathfrak{k},\kappa}(\mathrm{Ind}(M)_\kappa, V_2) = 0.$$

Proof. Part (i): We need to show that any extension in $\mathcal{C}_{(\tilde{\mathfrak{g}},\mathfrak{k},\kappa)}$ of $\mathrm{Ind}(M)_\kappa$ by V_2 splits. Consider one such extension

$$0 \to V_2 \to W \to \mathrm{Ind}(M)_\kappa \to 0. \tag{4.21}$$

The functor $V \mapsto V^\xi$ is an exact functor from $\mathcal{C}_{(\tilde{\mathfrak{g}},\mathfrak{k},\kappa)}$ to $\mathcal{H}$ for any complex number ξ. This, combined with the hypothesis, implies that

$$W^{\phi(\lambda)} \cong \mathrm{Ind}(M)_\kappa^{\phi(\lambda)} \cong \mathrm{Ind}(M)_\kappa(0) \cong M.$$

Moreover $W^{\phi(\lambda)-n} = 0$ for $n \in \mathbb{Z}_{>0}$ because of the assumption and (3.7). Thus $W^{\phi(\lambda)} \subset W(0)$ and there exists a canonical homomorphism from $\mathrm{Ind}(M)_\kappa$ to W which defines a splitting of (4.21).

Using the long exact sequence for the Ext groups and Lemma 4.12, one sees that to prove *part (ii)* it suffices to to show that

$$\mathrm{Ext}_{\tilde{\mathfrak{g}},\mathfrak{k},\kappa}(\mathrm{Ind}(M)_\kappa, \mathrm{Irr}(M_2)_\kappa) = 0$$

for any irreducible $\mathfrak{g}$-module $M_2 \in \mathcal{H}$ with infinitesimal character $\chi(M_2) = \chi_{\lambda_2}$ such that

$$\lambda_2 \notin W\lambda + pQ.$$

This is done analogously to Lemma 4.6. Recall from (4.15) the exact sequence

$$0 \to \mathrm{Irr}(M_2)_\kappa \to D(\mathrm{Ind}(M_2^d)_\kappa) \to D(X(M_2^d)_\kappa) \to 0.$$

It gives rise to the exact sequence

$$\begin{aligned}\mathrm{Hom}(\mathrm{Ind}(M)_\kappa, D(X(M_2^d)_\kappa)) \to \mathrm{Ext}^1_{\tilde{\mathfrak{g}},\mathfrak{k},\kappa}(\mathrm{Ind}(M)_\kappa, \mathrm{Irr}(M_2)_\kappa) \to \\ \to \mathrm{Ext}^1_{\tilde{\mathfrak{g}},\mathfrak{k},\kappa}(\mathrm{Ind}(M)_\kappa, D(\mathrm{Ind}(M_2^d)_\kappa)).\end{aligned}$$

The first term vanishes because $D(X(M_2^d)_\kappa) \in \widehat{\mathcal{H}}_{\kappa,inf}^{\widehat{\chi}_{\lambda_2}}$. The third term vanishes as a consequence of Proposition 4.2. This completes the proof of part (ii). □

Proof of Proposition 4.11. We will show that

$$\mathrm{Ext}^1_{\tilde{\mathfrak{g}},\mathfrak{k},\kappa}(U, V_2) = 0 \tag{4.22}$$

for any $\tilde{\mathfrak{g}}$-modules that is a quotient of an induced module $\mathrm{Ind}(M)_\kappa$ from an irreducible $\mathfrak{g}$-module $M \in \mathcal{H}^{\chi_\lambda}$ such that λ is in the class of $\widehat{\chi}_1$. This implies the general case using the long exact sequence for the Ext groups and the fact that any module $V_1 \in \widehat{\mathcal{H}}_\kappa^{\widehat{\chi}_1}$ has a finite filtration by such $\tilde{\mathfrak{g}}$-modules U.

Let

$$0 \to X \to \mathrm{Ind}(M)_\kappa \to U \to 0$$

be the long exact sequence defining U. From it we deduce the exact sequence

$$\begin{aligned}\mathrm{Hom}(X, V_2) \to \mathrm{Ext}^1_{\tilde{\mathfrak{g}},\mathfrak{k},\kappa}(U, V_2) \to \\ \to \mathrm{Ext}^1_{\tilde{\mathfrak{g}},\mathfrak{k},\kappa}(\mathrm{Ind}(M)_\kappa, V_2).\end{aligned}$$

Since $X \in \widehat{\mathcal{H}}^{\widehat{\chi}_1}$ and $\widehat{\chi}_1 \neq \widehat{\chi}_2$ the Hom group vanishes. The last Ext group vanishes due to Lemma 4.13. This implies that the middle term vanishes which proves (4.22). □

5. Compatibility of the affine Jacquet functor with the Kazhdan–Lusztig tensor product

In this section we prove a compatibility relation between the affine Jacquet functor and the Kazhdan–Lusztig tensor product which can be considered as an affine (fusion) analog of Proposition 2.3.

Throughout the section we will assume

$$\kappa \notin \mathbb{Q}_{\geq 0}$$

and will use the notation from the appendix.

5.1. **Main result.**

Theorem 5.1. *For any two $\tilde{\mathfrak{g}}$-modules $U \in \mathcal{KL}_\kappa$ and $V \in \widehat{\mathcal{H}}_\kappa$ there exists a canonical isomorphism*

$$\hat{j}(U \dot{\otimes} V) \cong D\left[U \dot{\otimes} D\hat{j}(V)\right]. \tag{5.1}$$

Theorem 5.1 follows from Proposition 5.4 and Proposition 5.5 below.

We will need some notation. For $M \in \mathcal{H}$ set

$$\varphi(M) = \left(M^d\right)^*. \tag{5.2}$$

It defines an exact functor from the category of Harish-Chandra modules to the category of $\mathfrak{g}$-modules. For $V \in \widehat{\mathcal{H}}_\kappa$ set

$$\Phi(V) = [D(V)]^{*\sharp}(\infty). \tag{5.3}$$

If V has the decomposition (2.8) with respect to the action of L_0, then the underlining $\mathfrak{g}$-module of $\Phi(V)$ is canonically identified with

$$\bigoplus_{\xi\colon\, \xi-\xi_i \in \mathbb{Z}_{\geq 0}} \varphi(V^\xi).$$

Clearly Φ is an exact functor from $\widehat{\mathcal{H}}_\kappa$ to the category of strictly smooth $\tilde{\mathfrak{g}}$-modules of central charge $\kappa - h^\vee$.

Given $V \in \mathcal{H}$, the embedding of $\mathfrak{g}$-modules $V(N) \hookrightarrow V$ induces the embedding

$$\varphi(V(N)) \hookrightarrow \Phi(V). \tag{5.4}$$

It is easy to see:

Lemma 5.2. *For any $V \in \widehat{\mathcal{H}}_\kappa$ the embedding* (5.4) *is an isomorphism of $\mathfrak{g}$-modules.*

For a $\mathfrak{g}$-module M define also its $\mathfrak{g}$-submodule

$$j^0(M) = \cup_{N \in \mathbb{Z}_{\geq 0}} \mathrm{Ann}_{\mathfrak{n}_0^N} M.$$

The Jacquet module of $M \in \mathcal{H}$ is given by $j(M) \cong j^0(M^*)$. If V is a $\tilde{\mathfrak{g}}$-module then $j^0(V)$ is a $\tilde{\mathfrak{g}}$-module as well.

Fix two $\tilde{\mathfrak{g}}$-modules U and V as in Theorem 5.1. Set for shortness

$$W = U \otimes V. \tag{5.5}$$

The natural embedding $\mathfrak{g} \hookrightarrow \Gamma$ by constant functions induces structures of $\mathfrak{g}$-modules on the spaces W and W/G_kW for all $k \in \mathbb{Z}_{>0}$, as well as on their full and restricted duals. Note the canonical isomorphism of $\mathfrak{g}$-modules

$$\mathrm{Ann}_{G_k} W^d = (W/G_kW)^d, \quad \mathrm{Ann}_{G_k} W^* = (W/G_kW)^*. \tag{5.6}$$

Lemma 6.1 and the fact that $\mathcal{H}$ is stable under tensoring with finite dimensional $\mathfrak{g}$-modules imply

$$W/G_kW \in \mathcal{H},\ k \in \mathbb{Z}_{>0}. \tag{5.7}$$

Lemma 5.3. *Under the above assumptions on the $\tilde{\mathfrak{g}}$-modules U, V, and W there exists a canonical isomorphism of $\mathfrak{g}$-modules*

$$\left[\left(\mathrm{Ann}_{G_k} W^d\right)^d\right]^* \cong \mathrm{Ann}_{G_k} W^*,$$

for all $k \in \mathbb{Z}_{>0}$.

Proof. Because of (5.7) we have the canonical isomorphism of $\mathfrak{g}$-modules

$$\left((W/G_kW)^d\right)^d \cong W/G_kW.$$

Combining this with (5.6) gives the need canonical isomorphism of $\mathfrak{g}$-modules

$$\left[\left(\mathrm{Ann}_{G_k} W^d\right)^d\right]^* \cong \left[\left((W/G_kW)^d\right)^d\right]^* \cong (W/G_kW)^* \cong \mathrm{Ann}_{G_k} W^*.$$

□

One can define a structure of a strictly smooth $\tilde{\mathfrak{g}}$-module on the space

$$\cup_{k\in\mathbb{Z}_{>0}} \mathrm{Ann}_{G_k} W^*$$

in exactly the same way as the $\mathfrak{g}$-action on

$$T'(U,V) = \cup_{k\in\mathbb{Z}_{>0}} \mathrm{Ann}_{G_k} W^d$$

from the appendix.

Proposition 5.4. *In the above notation for U, V, and W there exists a canonical isomorphism of $\tilde{\mathfrak{g}}$-modules*

$$\Phi\left[\cup_{k\in\mathbb{Z}_{>0}} \mathrm{Ann}_{G_k} W^d\right] \cong \cup_{k\in\mathbb{Z}_{>0}} \mathrm{Ann}_{G_k} W^*.$$

Under it the $\tilde{\mathfrak{g}}$-submodule $\hat{j}(U\dot{\otimes}V)$ of the first module is sent to the $\tilde{\mathfrak{g}}$-submodule $j_0(\cup_{k\in\mathbb{Z}_{>0}} \mathrm{Ann}_{G_k} W^*)$ of the second one.

Proof. Lemma 5.2, Lemma 5.3, and (6.6) give rise to the canonical isomorphisms of $\mathfrak{g}$-modules

$$\Phi\left[\cup_{k\in\mathbb{Z}_{>0}} \mathrm{Ann}_{G_k} W^d\right](N) \cong \varphi(\mathrm{Ann}_{G_N} W^d) \cong \mathrm{Ann}_{G_N} W^*. \tag{5.8}$$

One checks directly that they are consistent for different $N \in \mathbb{Z}_{>0}$ and when put together, define an isomorphism of $\tilde{\mathfrak{g}}$-modules as stated in (5.8).

The second part is straightforward. □

Similarly to the above discussion, there is a canonical embedding of $\mathfrak{g}$-modules

$$\left(D\hat{j}(V)\right)^d \hookrightarrow j^0V^*.$$

It gives rise to the embedding of Γ-modules

$$\left(U \otimes D\hat{j}(V)\right)^d \hookrightarrow U^* \otimes \left(D\hat{j}(V)\right)^d \hookrightarrow U^* \otimes j^0V^*.$$

One further has the embedding of Γ-modules

$$U^* \otimes j^0 V^* \hookrightarrow (U \otimes j^0 V)^*.$$

Proposition 5.5. *The above embeddings of* Γ*-modules*

$$\Big(U \otimes D\hat{j}(V)\Big)^d \hookrightarrow U^* \otimes j^0 V^* \hookrightarrow (U \otimes j^0 V)^*$$

gives rise to the isomorphisms of Γ*-modules*

$$\cup_{k \in \mathbb{Z}>0} \mathrm{Ann}_{G_k} \Big(U \otimes (D\hat{j}(V))\Big)^d \cong \cup_{k \in \mathbb{Z}>0} \mathrm{Ann}_{G_k} \big(U^* \otimes j^0(V^*))\big) \cong \cong \cup_{k \in \mathbb{Z}>0} j_0 \mathrm{Ann}_{G_k} (U \otimes V)^*. \tag{5.9}$$

Each of the three spaces in (5.9) *is equipped with a structure of* $\mathfrak{g}$*-module analogously to* $T'(U,V)$.

5.2. **Finiteness of the fusion tensoring of $\widehat{\mathcal{H}}_\kappa$ by $\mathcal{KL}_\kappa$.** Here we modify Theorem 5.1 to give another proof of the case of Theorem 1.6 from our work [17] when the reductive subalgebra $\mathfrak{f}$ of $\mathfrak{g}$ is equal to $\mathfrak{k}$ (see Theorem 6.2 below).

For any positive integer N denote by $\mathcal{O}'^N$ the truncation of the category $\mathcal{O}'$ for $\mathfrak{g}$, consisting of finitely generated $\mathfrak{g}$-modules M with (generalized) weight space decompositions

$$M = \bigoplus_{\lambda \in \mathfrak{h}^*} M^\lambda$$

such that

$$\prod_{i=1}^N (h_i - \lambda(h_i))^N m = 0, \text{ for all } m \in M, \ h_i \in \mathfrak{h}.$$

Similarly the categories $\widehat{\mathcal{O}}'_\kappa$ posses the truncations $\widehat{\mathcal{O}}'^N_\kappa$ consisting of finitely generated, strictly smooth $\tilde{\mathfrak{g}}$-modules V for which

$$V(N) \in \mathcal{O}'^N, \ \forall N \in \mathbb{Z}_{>0}.$$

The union of the categories $\widehat{\mathcal{O}}'^N_\kappa$ is the category $\widehat{\mathcal{O}}'_\kappa$.

The categories $\widehat{\mathcal{O}}'^N_\kappa$ can be also described in terms of the Borel and Cartan subalgebras of the extended affine Kac–Moody algebra $\widehat{\mathfrak{g}}$, see section 2.4. Consider the categories of finitely generated modules V of central charge $\kappa - h^\vee$ for $\widehat{\mathfrak{g}}$ with the properties:

(i) The subalgebra $\mathfrak{n}_+ \oplus t\mathfrak{g}[t]$ of $\widehat{\mathfrak{g}}$ acts locally nilpotently on V.

(ii) The module V has a generalized weight space decomposition with respect to $\widehat{\mathfrak{h}}$

$$V = \bigoplus_{\widehat{\lambda} \in \widehat{\mathfrak{h}}^*} V^{\widehat{\lambda}}, \text{ such that for all } v \in V^{\widehat{\lambda}} \ \prod_{i=1}^N (h_i - \widehat{\lambda}(h_i))^N v = 0 \ \forall h_i \in \widehat{\mathfrak{h}}.$$

The categories $\widehat{\mathcal{O}}'^N_\kappa$ are essentially the above truncations of the generalized BGG category for $\widehat{\mathfrak{g}}$: for each module from $\widehat{\mathcal{O}}'^N$, consider infinitely many copies of it on which the operator d acts by $const - L_0$ for an arbitrary choice of the constant.

Lemma 5.6. *Any strictly smooth* $\tilde{\mathfrak{g}}$*-module* V *of central charge* N *such that*

$$V(k) \in \mathcal{O}'^N, \ \forall\, k \in \mathbb{Z}_{>0} \tag{5.10}$$

has a finite filtration by $\tilde{\mathfrak{g}}$ *submodules*

$$0 = W_0 \subset W_1 \subset \ldots \subset W_L = V$$

with strictly smooth $\tilde{\mathfrak{g}}$*-submodules* W_i *such that*

$$(W_{i+1}/W_i)(k) \in \mathcal{O},\ \forall\, k \in \mathbb{Z}_{>0}. \tag{5.11}$$

Proof. For $j \in \mathbb{Z}_{>0}$ define the subspace W_j of V by

$$V_j = \operatorname{span}\{v \in V \mid \text{ there exists } \lambda \in \mathfrak{h}^* \text{ such that } \prod_{i=1}^{j} (h_i - \lambda(h_i))^i v = 0,\ \forall h_i \in \mathfrak{h}\}.$$

It is a $\tilde{\mathfrak{g}}$-submodule of V because the adjoint action of $\mathfrak{h}$ on $\tilde{\mathfrak{g}}$ is semisimple. For some sufficiently large integer $L \leq N. \dim \mathfrak{h}$ we have $W_L = V$.

Define the generalized weight spaces of V with respect to $\mathfrak{h}$ by

$$V^\lambda = \{v \in V \mid (h - \lambda(h))^N v = 0 \text{ for all } h \in \mathfrak{h}\},$$

for $\lambda \in \mathfrak{h}^*$. The assumption (5.10) implies that $V \bigoplus_{\lambda \in \mathfrak{h}^*} V^\lambda$.

Observe that for each $h \in \mathfrak{h}$ the linear transformation

$$\psi_h(v) = (h - \lambda(h))v, \text{ if } v \in V^\lambda$$

is a $\tilde{\mathfrak{g}}$-endomorphism of V.

We will show that the $\tilde{\mathfrak{g}}$-module V/W_1 satisfies

$$(V/W_1)(k) \in \mathcal{O}^{'(L-1)},\ \forall k \in \mathbb{Z}_{>0}.$$

From this (5.11) follows by induction. Moreover it is obvious that $(V/W_1)(k)$ is spanned by vectors v such that

$$\prod_{i=1}^{L-1} (h_i - \lambda(h))v = 0,\ \forall h_i \in \mathfrak{h}$$

and $\mathfrak{n}_+$ acts locally nilpotently on (V/W_1). So we only need to show that $(V/W_1)(k)$ are finitely generated $\mathfrak{g}$-modules.

Denote the projection $p\colon V \to V/W_1$. Then

$$(V/W_1)(k) \cong p^{-1}\left[(V/W_1)(k)\right] / \left(p^{-1}\left[(V/W_1)(k)\right] \cap W_1\right). \tag{5.12}$$

Besides this

$$\begin{aligned} p^{-1}\left[(V/W_1)(k)\right] &= \{v \in V \mid \psi_h(uv) = 0,\ \forall u \in U(\tilde{\mathfrak{n}}_+)^{-k},\ h \in \mathfrak{h}\} \\ &= \cap_{h\in\mathfrak{h}} \psi_h^{-1}\left[V(N)\right]. \end{aligned} \tag{5.13}$$

Fix a basis $h_1, \ldots, h_r$, of $\mathfrak{h}$. Then (5.12) and (5.13) imply that $(V/W_1)(k)$ is isomorphic to the image of the $\mathfrak{g}$-homomorphism

$$(\psi_{h_1}, \ldots, \psi_{h_r})\colon p^{-1}\left[(V/W_1)(k)\right] \to V(k) \oplus \ldots \oplus V(k),\ (r \text{ times}).$$

Since $V(k)$ is finitely generated this gives that $(V/W_1)(k)$ is a finitely generated $\mathfrak{g}$-module. □

The main ingredient of Kazhdan–Lusztig's finiteness result [11] that $\mathcal{KL}_\kappa$ is closed under the fusion tensor product is a hard characterization of the category $\widehat{\mathcal{O}}_\kappa$ only in terms of the modules $V(k)$. It is based on the BGG reciprocity for Kac–Moody algebras of Rocha-Caridi and Wallach [15] or equivalently Soergel's generalized BGG reciprocity [2]. Later this characterization was extended by Finkelberg to the category $\widehat{\mathcal{O}}_\kappa$, [6].

Theorem 5.7. *(Kazhdan–Lusztig [11], Finkelberg [6]) If V is a strictly smooth $\tilde{\mathfrak{g}}$-module of central charge $\kappa - h^\vee$ such that*

$$V(k) \in \mathcal{O},\ \forall k \in \mathbb{Z}_{>0}$$

then V has finite length and thus $V \in \widehat{\mathcal{O}}_\kappa$.

Theorem 5.7, combined with Lemma 5.6, implies:

Proposition 5.8. *If V is a strictly smooth $\tilde{\mathfrak{g}}$-module of central charge $\kappa - h^\vee$ such that*

$$V(k) \in \mathcal{O}'^{N},\ \forall k \in \mathbb{Z}_{>0}$$

for some fixed integer N, then V has finite length and $V \in \widehat{\mathcal{O}}'^{N}_\kappa$.

Finally using affine Jacquet functors and Proposition 5.8 we obtain a second proof of the following special case of our finiteness result [17, Theorem 1.6] for the Kazhdan–Lusztig tensor product.

Theorem 5.9. *Assume that $\kappa \notin \mathbb{Q}_{\geq 0}$. Then the categories $\widehat{\mathcal{O}}'^{N}_\kappa$, $\widehat{\mathcal{O}}'_\kappa$, and $\widehat{\mathcal{H}}_\kappa$ are stable under Kazhdan–Lusztig tensoring with $\mathcal{KL}_\kappa$.*

We will need the following lemma.

Lemma 5.10. *Every $\tilde{\mathfrak{g}}$-modules of central charge $\kappa - h^\vee$ such that*

$$V(k) \in \mathcal{H},\ \text{respectively } \mathcal{O}' \tag{5.14}$$

belongs to $\widehat{\mathcal{H}}_{\kappa,inf}$, respectively $\widehat{\mathcal{O}}'_{\kappa,inf}$.

Proof. Fix a $\tilde{\mathfrak{g}}$-module with the property (5.14). The action of L_0 on V preserves $V(k)$. On each $V(k)/V(k-1)$, L_0 acts by $1/\kappa\Omega$ where Ω is the Casimir operator of $\mathfrak{g}$. This implies that L_0 acts locally semisimply on V. Denote the generalized eigenvalues of L_0 on $V(1)$ by $\xi_1, \ldots, \xi_n$. Since $\mathfrak{g}t$ maps V^ξ to $V^{\xi-1}$ and $V(k)\backslash V(1)$ isomorphically to $V(k-1)$ we have

$$V^\xi = 0 \text{ unless } \xi - \xi_i \in \mathbb{Z}_{>0} \text{ for some } i = 1, \ldots, n.$$

Then for any $\xi \in \mathbb{C}$, $V^\xi \subset V(k)$ for some sufficiently large k and therefore $V^\xi \in \mathcal{H}$ or $V^\xi \mathcal{O}'$ for the two cases in (5.14). □

Proof of Theorem 5.9. Fix two $\tilde{\mathfrak{g}}$-modules $U \in \mathcal{KL}_\kappa$ and $V \in \widehat{\mathcal{H}}_\kappa$. Due to Lemma 6.1

$$T'(U,V)(k) \in \mathcal{H},\ \forall k \in \mathbb{Z}_{>0}.$$

Lemma 5.10 implies

$$T'(U,V) \text{ and } U \dot{\otimes} V \in \widehat{\mathcal{H}}_{\kappa,inf}.$$

We need to show that $U \dot{\otimes} V$ has finite length.

The proof of Theorem 5.1 can be easily extended to show that (5.1) holds in this situation, where in the lhs of (5.1) $\hat{j}$ denotes the affine Jacquet functor from $\widehat{\mathcal{H}}_{\kappa,inf}$ to $\widehat{\mathcal{O}}'_{\kappa,inf}$.

Since $V \in \widehat{\mathcal{H}}_\kappa$ from Proposition 3.2 we obtain $\hat{j}(V) \in \widehat{\mathcal{O}}'^{N}_\kappa$ for some sufficiently large integer N. Then Lemma 6.1 implies that

$$T'(U, D(\hat{j}(V)))(k) \in \mathcal{O}'^{N}, \forall k \in \mathbb{Z}_{>0}.$$

Applying the characterization of the categories $\widehat{\mathcal{O}}'^{N}_\kappa$ from Proposition 5.8, we get that $T'(U, D(\hat{j}(V)))$ belongs to $\widehat{\mathcal{O}}'^{N}_\kappa$. Then (5.1) gives that $\hat{j}(U \dot{\otimes} V)$ belongs to $\widehat{\mathcal{O}}'_\kappa$, as well, and in particular $\hat{j}(U\dot{\otimes}V)$ has finite length. Since the affine Jacquet functor is faithful we finally obtain that $U\dot{\otimes}V$ has finite length. $\square$

6. Appendix: The Kazhdan–Lusztig fusion tensor product

In this appendix we review the definition of the Kazhdan–Lusztig fusion tensor product [11]. Consider the Riemann sphere $\mathbb{CP}^1$ with three fixed distinct points p_i, $i = 0, 1, 2$ on it. Choose local coordinates (charts) at each of them, i.e. isomorphisms $\gamma_i : \mathbb{CP}^1 \to \mathbb{CP}^1$ such that $\gamma_i(p_i) = 0$ where the second copy of $\mathbb{CP}^1$ is equipped with a fixed coordinate function t vanishing at 0.

Set $R = \mathbb{C}[\mathbb{CP}^1\backslash\{p_0, p_1, p_2\}]$ and denote by Γ the central extension of the Lie algebra $\mathfrak{g} \otimes R$ by

$$[f_1x_1, f_2x_2] := f_1f_2[x_1, x_2] + \mathrm{Res}_{p_0}(f_2df_1)(x_1, x_2)K, \tag{6.1}$$

for $f_i \in R$ and $x_i \in \mathfrak{g}$. Here $(.,.)$ is invariant bilinear form on $\mathfrak{g}$ from section 2.3. There is a canonical homomorphism

$$\Gamma \to \widehat{\mathfrak{g} \oplus \mathfrak{g}}, \quad xf \mapsto (x\mathrm{Ex}(\gamma_1^*)^{-1}(f), x\mathrm{Ex}(\gamma_2^*)^{-1}(f)),\ K \mapsto -K \tag{6.2}$$

where $\mathrm{Ex}(.)$ denotes the power series expansion of a rational function on $\mathbb{CP}^1$ at 0 in terms of the coordinate function t.

Define

$$G_N = \mathrm{span}\{(f_1x_1)\dots(f_Nx_N) \mid f_i \text{ vanish at } p_0,\ x_i \in \mathfrak{g}\} \subset U(\Gamma).$$

Let f_0 be a rational function on $\mathbb{CP}^1$ (unique up to a multiplication by a nonzero complex number) having only one (simple) zero at p_0 and only one (simple) pole at p_1. For instance when $\gamma_0(p_1)$ is finite $f_0(t) = a\gamma_0^*(t)/(\gamma_0^*(t) - \gamma_0(p_1))$, $a \neq 0$. Set

$$X_N = \mathrm{span}\{(f_0x_1)\dots(f_0x_N) \mid x_i \in \mathfrak{g}\} \subset U(\Gamma). \tag{6.3}$$

Clearly $X_N \subset G_N$.

Let $\mathfrak{f}$ be a subalgebra of $\mathfrak{g}$ which is reductive in $\mathfrak{g}$. Denote by $\mathcal{H}_{\mathfrak{g},\mathfrak{f}}$ the category of finite length $\mathfrak{g}$-modules which are $\mathfrak{f}$-locally finite, semisimple, and admissible. The last condition on a $\mathfrak{g}$-module M means that every finite dimensional $\mathfrak{f}$-module appears with finite multiplicity in $M|_{\mathfrak{f}}$. By $V \mapsto V^d$ we will denote the duality functor in $\mathcal{H}_{\mathfrak{g},\mathfrak{f}}$ given by $M^d = (M^*)^{U(\mathfrak{f})-fin}$. By $D(.)$ we will denote the duality functor in $\mathcal{AFF}(\mathcal{H}_{\mathfrak{g},\mathfrak{f}})_\kappa$ given by $D(V) = (V^d)^\sharp(\infty)$, see [17, Section 4] for details.

Fix two smooth $\tilde{\mathfrak{g}}$ modules U and V of central charge $\kappa - h^\vee$ such that

$$U \in \mathcal{KL}_\kappa \text{ and } V \in \mathcal{AFF}(\mathcal{H}_{\mathfrak{g},\mathfrak{k}})_\kappa.$$

Using the homomorphism (6.2) $U \otimes_{\mathbb{C}} V$ becomes a Γ module of central charge $-\kappa + h^\vee$. Note that the restricted dual $(U \otimes V)^d$ is naturally a Γ submodule of $(U \otimes V)$, having central charge $\kappa - h^\vee$. (The restricted dual is taken with respect to the embedding $\mathfrak{k} \hookrightarrow \mathfrak{g} \hookrightarrow \Gamma$ using constant functions on $\mathbb{CP}^1$.) Following [11] define the following Γ submodule of $(U \otimes V)^d$

$$T'(U, V) := \cup_{N\geq 1} T(U, V)\{N\} \tag{6.4}$$

where

$$T(U, V)\{N\} := \mathrm{Ann}_{G_N}(U \otimes V)^d.$$

Eq. (6.4) indeed defines a Γ submodule of $(U \otimes V)^d$ since for any $y \in \Gamma$ and any integer N there exists an integer i such that $G_{N+i} y \in \Gamma G_N$. In other words $T'(U,V)\{N\}$ is defined as

$$T'(U,V)\{N\} = \{\eta \in (U \otimes V)^* \mid \eta(G_N W) = 0, \ \dim U(\mathfrak{f})\eta < \infty\}.$$

The spaces $T'(U,V)\{N\}$ inherit a natural $\mathfrak{g}$-action, using the embedding of $\mathfrak{g}$ in Γ by constant functions. Kostant's theorem [12] that $\mathcal{H}_{\mathfrak{g},\mathfrak{k}}$ is stable under tensoring with finite dimensional $\mathfrak{g}$-modules and the following lemma of Kazhdan and Lusztig imply that the spaces $T'(U,V)\{N\}$, equipped with this $\mathfrak{g}$-action belong to $\mathcal{H}_{\mathfrak{g},\mathfrak{k}}$.

Lemma 6.1. [11, Proposition 7.4] *Assume that V_i are two strictly smooth $\tilde{\mathfrak{g}}$-modules of central charge $\kappa - h^\vee$, generated by $V_i(N_i)$, respectively. Then*

$$U \otimes V = \sum_{k=0}^{N-1} X_k(U(N_1) \otimes V(N_2)) + G_N(U \otimes V)$$

for all $N \in \mathbb{Z}_{>0}$.

There is a canonical action of $\tilde{\mathfrak{g}}$ ("the copy attached to p_0") on the space $T'(U,V)$, defined as follows. Let $\eta \in T(U,V)\{N\}$. Fix $\omega \in \mathbb{C}[t,t^{-1}]$, $x \in \mathfrak{g}$ and choose $f \in R$ such that $f - \gamma_0^*(\omega)$ has a zero of order at least N at p_0. Then

$$(\omega x)\eta := (f x)\eta \tag{6.5}$$

correctly defines a structure of smooth $\tilde{\mathfrak{g}}$-module on $T'(U,V)$ of central charge $\kappa - h^\vee$. Moreover

$$T'(U,V)(N) = T'(U,V)\{N\}. \tag{6.6}$$

The Kazhdan–Lusztig fusion tensor product of the strictly smooth $\tilde{\mathfrak{g}}$-modules U and V is defined by

$$U \dot{\otimes} V = DT'(U,V) \tag{6.7}$$

In [17] we proved the following affine version of Kostant's theorem that the categories $\mathcal{H}_{\mathfrak{g},\mathfrak{k}}$ are stable under tensoring with finite dimensional $\mathfrak{g}$-modules.

Theorem 6.2. *Assume that $\mathfrak{f}$ is a subalgebra of $\mathfrak{g}$ which is reductive in $\mathfrak{g}$ and $\kappa \notin \mathbb{R}_{\geq 0}$. Then the categories $\mathcal{AFF}(\mathcal{H}_{\mathfrak{g},\mathfrak{f}})_\kappa$ are stable under Kazhdan–Lusztig tensoring with modules from the Kazhdan–Lusztig's category $\mathcal{KL}_\kappa$.*

The hardest step in Theorem 6.2 is to show that $U \dot{\otimes} V$ is finitely generated (equivalently has finite length) for any $U \in \mathcal{KL}_\kappa$ and $V \in \mathcal{AFF}(\mathcal{H}_{\mathfrak{g},\mathfrak{f}})_\kappa$.

The case $\mathfrak{f} = \mathfrak{g}$ recovers the finiteness result of Kazhdan and Lusztig [11] that $\mathcal{KL}_\kappa$ is a monoidal category.

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