# CYCLICITY OF LUSZTIG'S STRATIFICATION OF GRASSMANNIANS AND POISSON GEOMETRY

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Abstract. We prove that the standard Poisson structure on the Grassmannian $\mathrm{Gr}(k,n)$ is invariant under the action of the Coxeter element $c = (12\dots n)$. In particular, its symplectic foliation is invariant under $c$. As a corollary, we obtain a second, Poisson geometric proof of the result of Knutson, Lam, and Speyer that the Coxeter element interchanges the Lusztig strata of $\mathrm{Gr}(k,n)$. We also relate the main result to known anti-invariance properties of the standard Poisson structures on cominuscule flag varieties.

## 1. Introduction

For the purpose of the study of canonical bases, Lusztig defined [4] the totally nonnegative part $(G/P)_{\geq 0}$ of an arbitrary complex flag variety $G/P$. He also constructed an algebro-geometric stratification of $G/P$ and conjectured that intersecting this stratification with $(G/P)_{\geq 0}$ is producing a cell decomposition of $(G/P)_{\geq 0}$. This was latter proved by Rietsch in [5]. Both the non-negative part $(G/P)_{\geq 0}$ and the Lusztig stratification of a flag variety were studied in recent years from many different combinatorial and Lie theoretic points of view.

In a recent work Knutson, Lam, and Speyer proved that the Lusztig stratification of the Grassmannian $\mathrm{Gr}(k,n)$ has a remarkable cyclicity property. If $c$ denotes the Coxeter element $(12\dots n)$ of $S_n$ and the permutation matrix in $\mathrm{GL}_n(\mathbb{C})$ which represents it, then $c$ permutes the strata of the Lusztig stratification of $\mathrm{Gr}(k,n)$.

In this note we give a Poisson geometric proof of this fact. We also prove a stronger invariance property of a Poisson structure on $\mathrm{Gr}(k,n)$. In [2], jointly with Goodearl, we found a Poisson geometric interpretation of the Lusztig stratification of any flag variety $G/P$. For a choice of opposite Borel subgroups $B$ and $B^-$ of $G$ such that $B \subset P$ one defines the standard Poisson structure $\pi_{G/P}$ on $G/P$ which is invariant under the action of the maximal torus $T = B \cap B^-$, see [2] for details. According to [2, Theorem 0.4] the $T$-orbits of symplectic leaves of $\pi_{G/P}$ are exactly the Lusztig strata.

In the case of the complex Grassmannian $\mathrm{Gr}(k,n)$ the standard Poisson structure is given by

$$\pi_{k,n} = - \sum_{1 \leq i < j \leq n} \chi(E_{ij}) \wedge \chi(E_{ji})$$

where $\chi\colon \mathfrak{gl}_n(\mathbb{C}) \to \mathrm{Vect}(\mathrm{Gr}(k,n))$ denotes the induced infinitesimal action from the left action of $\mathrm{GL}_n(\mathbb{C})$ on $\mathrm{Gr}(k,n)$ and $E_{ij}$ denote the elementary matrices. This Poisson structure is invariant under the action of the maximal torus $T_n$ of diagonal matrices in $\mathrm{GL}_n(\mathbb{C})$. For each $w \in S_n$ denote by the same letter the corresponding permutation matrix in $\mathrm{GL}_n(\mathbb{C})$. As before $c$ denotes the permutation matrix corresponding to the Coxeter element $(12\dots n)$. The main result of this paper is:

**Theorem 1.1.** *Multiplication by $c$ is a Poisson automorphism of $(\mathrm{Gr}(k,n), \pi_{k,n})$.*

It is well known that the action of the permutation matrix $w_\circ$ corresponding to the longest element of $S_n$ is an anti-Poisson automorphism, see Section 2 for details. Thus Theorem 1.1 implies:

**Corollary 1.2.** *The actions of $w_\circ$, $c$, and $T_n$ generate an action of $I_2(n) \ltimes T_n$ by Poisson and anti-Poisson automorphisms of $(\mathrm{Gr}(k,n), \pi_{k,n})$ where $I_2(n)$ denotes the dihedral group of order $2n$.*

The Lusztig stratification of the Grassmannian $\mathrm{Gr}(k,n)$ is defined as follows, see [4] for details. Let $B$ and $B_-$ be the standard Borel subgroups of $\mathrm{GL}_n(\mathbb{C})$ consisting of upper and lower triangular matrices. Denote the maximal parabolic subgroup

$$P_{k,n} = \{ \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \in \mathrm{GL}_n(\mathbb{C}) \mid a \in M_{k,k}, b \in M_{k,n-k}, c \in M_{n-k,n-k} \}$$

of $\mathrm{GL}_n(\mathbb{C})$ and the induced map

$$q \colon \mathrm{GL}_n(\mathbb{C})/B \to \mathrm{GL}_n(\mathbb{C})/P_{k,n} \cong \mathrm{Gr}(k,n).$$

The strata in the Lusztig stratification of $\mathrm{Gr}(k,n)$ are given by

$$R_{v,w} = q(B_- \cdot vB \cap B \cdot wB), \quad v \in (S_n)^{S_k \times S_{n-k}}_{\max}, w \in S_n, v \leq w.$$

Here $\leq$ refers to the Bruhat order. We denote by $S_k \times S_{n-k}$ the subgroup of $S_n$ consisting of of those $u \in S_n$ such that $u(i) \leq k$ for $i \leq k$ and $u(i) \geq k+1$ for $i \geq k+1$. Finally, $(S_n)^{S_k \times S_{n-k}}_{\max}$ denotes the set of maximal length representatives for the cosets $S_n/(S_k \times S_{n-k})$

The symplectic foliation of a Poisson structure is uniquely determined by it. Thus the $T_n$-orbits of leaves of $\pi_{k,n}$ (which are exactly the Lusztig strata) are an invariant of the pair ($\pi_{k,n}$, $T_n$-action). Therefore Theorem 1.1 gives a second proof of the result of Knutson, Lam, and Speyer that the action of the Coxeter element $c$ on $\mathrm{Gr}(k,n)$ interchanges the Lusztig strata. In fact Theorem 1.1 is equivalent to the stronger statement:

*The action of $c$ on $\mathrm{Gr}(k,n)$ restricts to Poisson isomorphisms between various Lusztig strata $(R_{v,w}, \pi_{k,n}|_{R_{v,w}})$ considered as regular Poisson varieties.*

Finally we trace the roots of this phenomenon from a Poisson geometric point of view. It is well known that on any flag variety $G/P$ the standard Poisson structure $\pi_{G/P}$ is anti-invariant under the action of any representative $\dot{w}_\circ$ of the longest element of the Weyl group $W$ of $G$. If, in addition, $P$ is cominuscule, then [2, Proposition 4.2] implies that the standard Poisson structure on $G/P$ is anti-invariant under the action of any representative $\dot{w}^P_\circ$ of the longest element of the corresponding parabolic subgroup of $W$. In the special case of the Grassmannian the specific Coxeter element $c$ happens to be a $k$-th root of $\dot{w}^P_\circ \dot{w}_\circ$. Thus Theorem 1.1 claiming that the standard Poisson structure on $\mathrm{Gr}(k,n)$ is invariant under $c$ is a strengthening of [2, Proposition 4.2]. See Section 2 for more details. We do not know of good Poisson properties of roots of $\dot{w}^P_\circ \dot{w}_\circ$ for any other cominuscule flag varieties.

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## 2. Proof of Theorem 1.1

*Proof of Theorem 1.1.* The statement is equivalent to showing that

$$\sum_{1\le i<j\le n} \chi(\mathrm{Ad}_c(E_{ij})) \wedge \chi(\mathrm{Ad}_c(E_{ji})) - \sum_{1\le i<j\le n} \chi(E_{ij}) \wedge \chi(E_{ji}) = 0;$$

that is

$$V := \sum_{i=2}^{n} \chi(E_{1i}) \wedge \chi(E_{i1}) = 0. \tag{2.1}$$

We will check this on the open Schubert cell $B_- \cdot P_{k,n} \subset Gr(k,n)$. Since $V$ is an algebraic bivector field, this will establish (2.1). Identify

$$M_{n-k,k} \cong B^- \cdot P_{k,n} \subset \mathrm{Gr}(k,n), \quad X \mapsto \begin{pmatrix} I_k & 0 \\ X & I_{n-k} \end{pmatrix} \cdot P_{k,n} \tag{2.2}$$

where $M_{n-k,k}$ denotes the space of $(n-k)\times k$ complex matrices. Applying [1, eq. (3.17)] we get that under (2.2)

$$\chi(E_{1,i+k}) \mapsto -\sum_{p=1}^{n-k}\sum_{q=1}^{k} x_{p1}x_{iq}\frac{\partial}{\partial x_{pq}}, \quad \text{for } i=1,\dots,n-k. \tag{2.3}$$

It is obvious that

$$\chi(E_{i+k,1}) \mapsto \frac{\partial}{\partial x_{i1}}, \quad \text{for } i=1,\dots,n-k. \tag{2.4}$$

Let $1 \le i,j \le k$. Then

$$\mathrm{Ad}_{\exp(sE_{ij})} \begin{pmatrix} I_k & 0 \\ X & I_{n-k} \end{pmatrix} = \begin{pmatrix} I_k & 0 \\ X - s\sum_{p=1}^{n-k} X_{pi}E_{pj} & I_{n-k} \end{pmatrix},$$

which implies that under (2.2)

$$\chi(E_{i,j}) \mapsto -\sum_{p=1}^{n-k} x_{pi}\frac{\partial}{\partial x_{pj}}. \tag{2.5}$$

The summation in (2.1) can be taken from $i = 1$ since $\chi(E_{1,1}) \wedge \chi(E_{1,1}) = 0$. Applying (2.3), (2.4), and (2.5) we obtain that under the identification (2.2)

$$V|_{B_- \cdot P_{k,n}} \mapsto \sum_{i=1}^{k} \sum_{p=1}^{n-k} \sum_{q=1}^{n-k} x_{p1} x_{qi} \frac{\partial}{\partial x_{pi}} \wedge \frac{\partial}{\partial x_{q1}} - \sum_{i=1}^{n-k} \sum_{p=1}^{n-k} \sum_{q=1}^{k} x_{p1} x_{iq} \frac{\partial}{\partial x_{pq}} \wedge \frac{\partial}{\partial x_{i1}} = 0. \tag{2.6}$$

This implies (2.1) and the statement of the Theorem. □

For an arbitrary complex simple group $G$ and a maximal parabolic subgroup $P$, one defines the standard Poisson structure

$$\pi_{G/P} = -\chi(r_G) \tag{2.7}$$

on the flag variety $G/P$ induced from a compatible triangular decomposition of $G$ (a pair of Borel subgroups $B$ and $B_-$, such that $B \cap B_- = T$ is a maximal torus of $G$ and $B \subset P$), see e.g. [2]. Here $\chi \colon \wedge^2 \mathrm{Lie}\,(G) \to \Gamma(TG/P, G/P)$ denotes the induced action from the infinitesimal action of $\mathrm{Lie}\,(G)$. The standard $r$-matrix $r_G \in \wedge^2 \mathrm{Lie}\,(G)$ obtained from the triangular decomposition of $G$ is given by:

$$r_G = \sum_{\alpha \in \Delta_+} e_\alpha \wedge f_\alpha$$

where $e_\alpha$ and $f_\alpha$ are appropriately normalized root vectors of $\mathrm{Lie}\,(G)$ and $\Delta_+$ is the set of positive roots of $G$, cf. [2]. It is obvious that the action of any representative $\dot{w}_\circ$ of the longest element of the Weyl group $W$ of $G$ on $(G/P, \pi_{G/P})$ is anti-Poisson, since $\mathrm{Ad}_{\dot{w}_\circ}$ interchanges $e_\alpha$ and $f_\alpha$.

Denote the Levi factor of $P$ containing $T$ by $L$, and the longest element of the subgroup of $W$ corresponding to $L$ by $w_\circ^P$. Let $\dot{w}_\circ^P$ be any representative of $w_\circ^P$ in the normalizer of $T$.

Recall that among several equivalent definitions/characterizations of cominuscule parabolic subgroups: a parabolic subgroup $P$ of $G$ is cominuscule if and only if its unipotent radical is abelian. According to [2, Proposition 4.2], if $P$ cominuscule, then $\pi_{G/P}$ is also given by

$$\pi_{G/P} = -\chi(r_L).$$

where $r_L \in \wedge^2 \mathrm{Lie}\,(L)$ is the standard $r$-matrix of $L$. Thus the action of $\dot{w}_\circ^P$ on $(G/P, \pi_{G/P})$ is anti-Poisson as well. So:

**Proposition 2.1.** *For any cominuscule parabolic subgroup $P$ of a complex simple Lie group $G$, the action of $\dot{w}_\circ^P \dot{w}_\circ$ on $(G/P, \pi_{G/P})$ is Poisson.*

In the special case of the Grassmannian $\mathrm{Gr}(k, n)$

$$w_\circ^P w_\circ = c^k$$

for the particular Coxeter element $c$. Taking powers of this product, we see that the action of $c^{gcd(k,n)}$ on $\mathrm{Gr}(k, n)$ is Poisson. In the case when $k$ and $n$ are relatively prime this gives yet another proof of Theorem 1.1. One could argue that Theorem 1.1 holds because it is true for relatively prime $k$ and $n$, and its statement (cf. also its proof) is independent of the numerical properties of $k$ and $n$.

Conceptually the invariance of $\pi_{k,n}$ under $c$ is the result of a two step process:

1. From Proposition 2.1 one has the invariance of $\pi_{G/P}$ under the product $\dot{w}_\circ^P \dot{w}_\circ$ of the longest elements of the Weyl groups of $G$ and the Levi subgroup $L$, for arbitrary cominuscule flag variety $G/P$.

2. In the case of the Grassmannian the Coxeter element $c$ which is a $k$-th root of $\dot{w}_\circ^P \dot{w}_\circ$ acts by Poisson automorphisms of $(\mathrm{Gr}(k,n), \pi_{k,n})$ as well.

The special property of the Coxeter element $c = (12 \ldots n)$ is that the other Coxeter elements of $S_n$ are not roots of $\dot{w}_\circ^P \dot{w}_\circ$.

## 3. Corollaries

The symplectic foliation of a Poisson manifold $(M, \pi)$ is an invariant of it. Similarly if a group $H$ act on $(M, \pi)$ by Poisson automorphisms the partition of $M$ into $H$-orbits of symplectic leaves is an invariant of $(M, \pi)$ considered as a Poisson $H$-space. Since the partition of $\mathrm{Gr}(k,n)$ into $T_n$-orbits of symplectic leaves of $(\mathrm{Gr}(k,n), \pi_{k,n})$ is exactly the Lusztig stratification of $\mathrm{Gr}(k,n)$ due to [2, Theorem 0.4] and $c$ normalizes $T_n$, Theorem 1.1 implies the following Theorem of Knutson, Lam, and Speyer:

**Theorem 3.1.** *The action of the permutation matrix corresponding to the Coxeter element $c = (12 \ldots n)$ on $\mathrm{Gr}(k,n)$ permutes the strata $R_{v,w} = q(B_- \cdot vB \cap B \cdot wB)$ of the Lusztig stratification.*

As we pointed out, in addition, when $c$ maps one Lusztig stratum $R_{v_1,w_1}$ to another $R_{v_2,w_2}$, it matches the regular Poisson structures $\pi_{k,n}|_{R_{v_1,w_1}}$ and $\pi_{k,n}|_{R_{v_2,w_2}}$.

Finally let us point out that all constructions and invariance properties are valid over the reals since all constructions are derived from the real split form $\mathrm{GL}_n(\mathbb{R})$.

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