THE PROLATE SPHEROIDAL PHENOMENON AND BISPECTRAL ALGEBRAS OF ORDINARY DIFFERENTIAL OPERATORS

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Abstract. Landau, Pollak, Slepian, and Tracy, Widom discovered that certain integral operators with so called Bessel and Airy kernels possess commuting differential operators and found important applications of this phenomenon in time-band limiting and random matrix theory. In this paper we announce that very large classes of integral operators derived from bispectral algebras of rank 1 and 2 (parametrized by lagrangian grassmannians of infinitely large size) have this property. The above examples come from special points in these grassmannians.

1. Introduction

It was discovered by Landau, Pollak, Slepian [23, 18, 19, 22] and Tracy, Widom [25, 26] that certain integral operators associated to the Airy and Bessel special functions posses commuting differential operators. They found important applications of this to time-band limiting, and to the study of asymptotics of Fredholm determinants, relevant to scaling limits of random matrix models. We call this phenomenon the prolate spheroidal phenomenon.

On the other hand, the problem of bispectrality was posed [8] about 20 years ago by one of us (A. G.) and J. J. Duistermaat as a tool to understand this prolate spheroidal property of integral operators. The aim was to extend it to larger classes and search for possible applications. Despite the dramatic recent developments in the areas of random matrices and bispectrality the two problems remained isolated except for several common examples, see [10, 11]. In addition only few integral operators possessing the prolate spheroidal phenomenon were found.

Here we prove that any selfadjoint bispectral algebra of ordinary differential operators (see Definition 3.3 and Definition 4.4 and Section 2 for general definitions) of rank 1 and 2 induces an integral operator possessing the prolate spheroidal property. The kernel of such an operator is of the form

$$K(x,z) = \int_{\Gamma_2} \Psi(x,z)\Psi(y,z)dz$$

where $\Psi(x,z)$ is the corresponding bispectral (eigen)function and Γ_2 is a contour in the complex plane with 1 or 2 end points. It acts on the space $L^2(\Gamma_1)$ again for a contour with the same property. The main results are stated in Theorem 3.10 and Theorem 4.11.

The integral operators of Landau, Pollak, Slepian, and Tracy, Widom correspond to the cases $\Psi(x,z) = \sqrt{xz}J_{\nu+1/2}(ixz)$ and $\Psi(x,z) = A(x+z)$ which are known to be "basic" bispectral functions, in the sense that the other rank 1 and 2 bispectral functions are obtained from them by certain type of Darboux transformations, see

[8, 29, 4, 3, 17]. (Here $J_\nu(.)$ denote the Bessel functions of first kind and $A(.)$ denotes the Airy function.) Although in these cases [23, 18, 19, 22, 25, 26] the commuting differential operator is of order 2, in general it is of arbitrarily large order.

In the rest of the introduction we describe our strategy for proving Theorem 3.10 and Theorem 4.11 which relies on a very interesting property of the "size" of bispectral algebras of rank 1 and 2.

Consider, more generally, a holomorphic function $\Psi(x,z)$ in some domain of $\mathbb{C} \times \mathbb{C}$ which is not an eigenfunction of any differential operator in x or z. Denote by $\mathcal{B}_\Psi$ the algebra of differential operators $R(x, \partial_x)$ with rational coefficients for which there exists an operator $S(z, \partial_z)$ with rational coefficients such that

$$R(x, \partial_x)\Psi(x,z) = S(z, \partial_z)\Psi(x,z). \tag{1.1}$$

The algebra of all differential operators $S(z, \partial_z)$ obtained in this way will be denoted by $\mathcal{C}_\Psi$. The equality

$$b_\Psi(R(x,\partial_x)) := S(z, \partial_z)$$

correctly defines an antiisomorphism from $\mathcal{B}_\Psi$ to C_Ψ. Recall that such a function $\Psi(x,z)$ is called bispectral if both algebras $\mathcal{B}_\Psi$ and $\mathcal{C}_\Psi$ contain rational functions. The subalgebra of $\mathcal{B}_\Psi$ and $\mathcal{C}_\Psi$, consisting of differential operators in x and z for which $\Psi(x,z)$ is an eigenfunction, are commutative. Algebras obtained in this way are called bispectral algebras. Their rank (which is equal and is also called rank of the bispectral function $\Psi(x,z)$) is the greatest common divisor of the orders of the operators of these algebras.

We derive our main result from the following remarkable property:

Consider the $\mathbb{Z}_+ \times \mathbb{Z}_+$ filtration of the algebra $\mathcal{B}_\Psi$ given by

$$\mathcal{B}_\Psi^{l_1,l_2} = \{R(x,\partial_x) \in \mathcal{B}_\Psi \mid \operatorname{ord} R(x,\partial_x) \leq 2l_1,\ \operatorname{ord}(b_\Psi R)(x,\partial_x) \leq 2l_2\}.$$

If $\Psi(x,z)$ is a bispectral function of rank $r = 1$ or 2 then the the size of the spaces $\mathcal{B}_\Psi^{l_1,l_2}$ in this filtration is large in the sense

$$\dim \mathcal{B}_\Psi^{l_1,l_2} \geq \frac{2}{r}(2l_1l_2 + l_1 + l_2) - const$$

where the constant is independent of l_1 and l_2.

For the basic bispectral functions e^{xz} of rank 1, and $A(x+z)$, $\sqrt{xz}J_{\nu+1/2}(ixz)$, $\nu \in \mathbb{C}\backslash\mathbb{Z}$ of rank 2, the dimension of these spaces is exactly equal to the right hand side with $const = -1$. This is remarkable since it shows that for all bispectral functions $\Psi(x,z)$ of rank 1 and 2 the spaces of the above natural filtration of the algebra $\mathcal{B}_\Psi$ are approximately of the same dimension as the basic (Bessel and Airy) ones.

It is very interesting to understand the relation between our results and the approach of Adler, Shiota, and van Moerbeke [1, 2] to the Tracy–Widom system of differential equations via Virasoro constraints. One possible way would be to incorporate the representation theoretic meaning of the Bessel tau functions from [5] in terms of representations of the $W_{1+\infty}$ algebra, see [9].

Another important problem is to understand the relation of this work to the isomonodromic deformations approach to random matrices from the works of Palmer [21], Harnad–Tracy–Widom [14, 27], Its–Harnad [13], and Borodin–Deift [7].

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2. Bispectrality and commutativity of Integral and Differential Operators

We will denote the algebra of differential operators in one variable with rational coefficients by $\mathcal{W}_{rat}$ and the Weyl algebra of differential operators in one variable with polynomial coefficients by $\mathcal{W}_{poly}$.

The formal adjoint of an operator

$$D(x, \partial_x) = \sum_{k=0}^{n} b_k(x)\partial_x^n \in \mathcal{W}_{rat} \tag{2.1}$$

is given by

$$aD(x, \partial_x) = \sum_{k=0}^{n} (-\partial_x)^n b_k(x). \tag{2.2}$$

To a differential operator $D \in \mathcal{W}_{rat}$ as in (2.1) we associate the bidifferential operator

$$D \mapsto \phi_x(D) = \sum_{k=1}^{n} \sum_{j=0}^{k-1} (-1)^j \partial_x^{k-i-1} \otimes \partial_x^i b_k(x) \tag{2.3}$$

acting on two functions $f(x)$ and $g(x)$ by

$$\phi_x(D)\,(f, g) = \sum_{k=1}^{n} \sum_{j=0}^{k-1} (-1)^j f^{(k-i-1)}(x)(b_k g)^{(i)}(x). \tag{2.4}$$

(We keep the complex variable x in the notation for $\phi(D)$ because later we will often need to evaluate $\phi(D)$.)

The right hand side of (2.4) is known as the *bilinear concomitant* associated to the operator D and is denoted by

$$\langle f, g\rangle_D(x). \tag{2.5}$$

We refer the reader to [28, Section 3] for a comprehensive exposition of the properties of the bilinear concomitant and to the classical text [15, Chapter 5]. We will primarily need it in Section. The main property of (2.5) is

$$(D(x, \partial_x)f(x))g(x) - f(x)(aD(x, \partial_x)g(x)) = \partial_x\langle f, g\rangle_D. \tag{2.6}$$

Given an oriented contour Γ in $\mathbb{C}$, denote its endpoints by $e(\Gamma)$. For any $\xi \in e(\Gamma)$ set $\pi(\xi) = 1$ or 0 depending on whether ξ is a left or right endpoint of Γ. Assume that the differential operator $D(x, \partial_x)$, given by (2.1), is regular along Γ.

The standard integration by parts formula states that for any two functions $f(x)$ and $g(x)$ on Γ that are smooth and decrease rapidly when $x \to \infty$ (i.e.

$\lim_{x\to\infty} p(x)f^{(k)}(x) = 0$ for any polynomial $p(x)$ and any integer k, similarly for $g(x)$)

(2.7)
$$\int_\Gamma (D(x,\partial_x)f(x))\, g(x)\mathrm{d}x = \int_\Gamma f(x)\,(aD(x,\partial_x)g(x))\,\mathrm{d}x + \sum_{\xi\in e(\Gamma)} (-1)^{\pi(\xi)}\langle f,g\rangle_D(\xi)$$

Let us fix a holomorphic function $\Psi(x,z)$ on some domain in $\mathbb{C}\times\mathbb{C}$ which is not an eigenfunction of a differential operator in x or z. In the introduction we defined two algebras of differential operators $\mathcal{B}_\Psi$ and $\mathcal{C}_\Psi$, associated to such a function, see (1.1). Let $\widetilde{\mathcal{B}}_\Psi$ and $\widetilde{\mathcal{C}}_\Psi$ be two subalgebras of $\mathcal{B}_\Psi$ and $\mathcal{C}_\Psi$, respectively, which are stable under the formal adjoint map (2.2) and such that

$$b_\Psi(\widetilde{\mathcal{B}}_\Psi) = \widetilde{\mathcal{C}}_\Psi.$$

Fix a second function $\Phi(x,z)$ which is holomorphic on a possibly another domain but satisfies

$$P(x,\partial_x)\Phi(x,z) = (b_\Psi P)(z,\partial_z)\Phi(x,z),\ \forall P(x,\partial_x)\in\mathcal{B}_\Psi$$

as $\Psi(x,z)$, i.e

$$\widetilde{\mathcal{B}}_\Psi \subset \mathcal{B}_\Phi \text{ and } b_\Psi|_{\widetilde{\mathcal{B}}_\Psi} = b_\Phi|_{\widetilde{\mathcal{B}}_\Psi}.$$

Proposition 2.1. *Assume that Γ_1 and Γ_2 are two contours in $\mathbb{C}$ such that $\Gamma_1\times\Gamma_2$ is in the domain of $\Psi(x,z)$ and $\Phi(x,z)$. Assume also that both $\Psi(x,z)$ and $\Phi(x,z)$ decrease rapidly when x or z go to ∞ along Γ_1 and Γ_2 (if these contours extend to infinity).*

If $D(x,\partial_x)\in\widetilde{\mathcal{B}}_\Psi$ is such that

$$ab_\Psi(D(x,\partial_x)) = b_\Psi a(D(x,\partial_x)) \tag{2.8}$$

and

$$\phi_\xi(D(x,\partial_x)) = 0,\ \forall\xi\in e(\Gamma_1);\quad \phi_\xi((bD)(z,\partial_z)) = 0,\ \forall\xi\in e(\Gamma_2) \tag{2.9}$$

then the integral operator with kernel

$$K(x,y) = \int_{\Gamma_2}\Psi(x,z)\Phi(y,z)\mathrm{d}z \tag{2.10}$$

on $L^2(\Gamma_1)$ commutes with the differential operator $D(x,\partial_x)$ with domain all smooth functions on Γ_1 that decrease rapidly as $x\to\infty$.

Proof. The integral operator with kernel (2.10) will be also denoted by K. We need to show that

$$KD(f) = DK(f)$$

for any smooth rapidly decreasing function f on Γ_1. Compute

$$\begin{aligned}(DKf)(x) &= \int_{\Gamma_1}\int_{\Gamma_2}(D(x,\partial_x)\Psi(x,z))\Phi(y,z)\mathrm{d}zf(y)\mathrm{d}y\\ &= \int_{\Gamma_1}\int_{\Gamma_2}((bD)(z,\partial_z)\Psi(x,z))\Phi(y,z)\mathrm{d}zf(y)\mathrm{d}y\\ &= \int_{\Gamma_1}\int_{\Gamma_2}\Psi(x,z)(abD)(z,\partial_z)\Phi(y,z)\mathrm{d}zf(y)\mathrm{d}y.\end{aligned}$$

The third equality follows from the second assumption in (2.9) and (2.7). Similarly

$$\begin{aligned}(KDf)(x) &= \int_{\Gamma_1}\int_{\Gamma_2} \Psi(x,z)\Phi(y,z)\mathrm{d}z D(y,\partial_y)f(y)\mathrm{d}y \\ &= \int_{\Gamma_1}\int_{\Gamma_2} \Psi(x,z)(aD)(y,\partial_y)\Phi(y,z)\mathrm{d}z f(y)\mathrm{d}y \\ &= \int_{\Gamma_1}\int_{\Gamma_2} (D(x,\partial_x)\Psi(x,z))(baD)(z,\partial_z)\Phi(y,z)\mathrm{d}z f(y)\mathrm{d}y\end{aligned}$$

and in deducing the second equality we used the first assumption in (2.9) and (2.7). The Proposition now follows from these two sequences of equalities. □

Definition 2.2. We will call a differential operator $D(x,\partial_x) \in \mathcal{W}_{rat}$ *formally symmetric* if

$$(aD)(x,\partial_x) = D(x,\partial_x)$$

and *formally skewsymmetric* if

$$(aD)(x,\partial_x) = -D(x,\partial_x).$$

Lemma 2.3. *A differential operator $D(x,\partial_x) \in \mathcal{W}_{rat}$ is formally symmetric if and only if it has the form*

$$D(x,\partial_x) = \sum_{i=0}^{n} \partial_x^n c_i(x)\partial_x^n \tag{2.11}$$

for some integer n and some rational functions $c_i(x)$.

Proof. Obviously the operator in the RHS of (2.11) is formally symmetric. In the other direction if $D(x,\partial_x) \in \mathcal{W}_{rat}$ is formally symmetric then it is of even order, say $2n$. If its leading coefficient is $c_n(x)$ then $D(x,\partial_x) - \partial_x^n c_n(x)\partial_x^n$ is a formally symmetric operator of order at most $2n-2$ and one can continue by induction. □

The following Lemma is straightforward.

Lemma 2.4. *Fix $\xi \in \mathbb{C}$. The operator $D(x,\partial_x)$ given by (2.11) satisfies*

$$\phi_\xi(D(x,\partial_x)) = 0, \ i.e\, \langle f,g\rangle_D(\xi) = 0$$

for all smooth functions in a neighborhood of ξ if and only if

$$(\partial_x^i c_k)(\xi) = 0, \, for\, k = 1,\dots,n, \ i = 0,\dots,k-1.$$

For a given function $\Psi(x,z)$ as above denote by $\mathcal{B}_{\Psi,sym}$ the subalgebra of $\mathcal{B}_\Psi$ that consists of differential operators $R(x,\partial_x)$ for which both $R(x,\partial_x)$ and $(bR)(z,\partial_z)$ are formally symmetric. Set also $\mathcal{C}_{\Psi,sym} := b\mathcal{B}_{\Psi,sym}$. Consider the linear spaces

$$\mathcal{B}^{l_1,l_2}_{\Psi,sym} = \{R(x,\partial_x) \in \mathcal{B}_{\Psi,sym} \mid \operatorname{ord} R(x,\partial_x) \le 2l_1 \text{ and } \operatorname{ord}(bR)(z,\partial_z) \le 2l_2\}, \tag{2.12}$$

$$\mathcal{C}^{l_1,l_2}_{\Psi,sym} = \{S(z,\partial_z) \in \mathcal{C}_{\Psi,sym} \mid \operatorname{ord}(b^{-1}S)(x,\partial_x) \le 2l_1 \text{ and } \operatorname{ord} S(z,\partial_z) \le 2l_2\}. \tag{2.13}$$

Clearly $\mathcal{C}^{l_1,l_2}_{\Psi,sym} = b(\mathcal{B}^{l_1,l_2}_{\Psi,sym})$.

From Proposition 2.1, Lemma 2.3, and Lemma 2.4 we obtain:

Theorem 2.5. *Assume the conditions from Proposition 2.1 for the functions* $\Psi(x,z)$, $\Phi(x,z)$ *and the contours* Γ_1, Γ_2. *If either of the following two conditions is satisfied then the integral operator with kernel* (2.10) *possesses a formally commuting symmetric differential operator of order less than or equal to* $2l_1$ *and domain – the space of rapidly decreasing smooth functions on* Γ_1.

Condition (i):

$$\mathcal{B}^{l_1,l_2}_{\Psi,sym} > l_1(l_1+1)e(\Gamma_1)/2 + l_2(l_2+1)e(\Gamma_2)/2,$$

Condition (ii): $-e(\Gamma_1) = e(\Gamma_1)$, $-e(\Gamma_2) = e(\Gamma_2)$, *all operators in* $\mathcal{B}_{\Psi,sym}$ *are invariant under the transformation* $x \mapsto -x$, *and*

$$\mathcal{B}^{l_1,l_2}_{\Psi,sym} > l_1(l_1+1)e(\Gamma_1)/4 + l_2(l_2+1)e(\Gamma_2)/4$$

3. Integral operators associated to selfadjoint Darboux transformations of Airy functions

3.1. **The Airy bispectral function.**

Denote by $A(x)$ the classical Airy function, see ??. Recall that it decreases rapidly when $x \to \infty$ in the sector $-\pi/3 < \arg x < \pi/3$. By abuse of notation the function

$$\Psi_A(x,z) = A(x+z) \tag{3.1}$$

will be also called Airy function.

If $L_A(x,\partial_x)$ denotes the Airy differential operator

$$L_A(x,\partial_x) = \partial_x^2 - x$$

then $\Psi_A(x,z)$ satisfies

$$L_A(x,\partial_x)\Psi_A(x,z) = z\Psi_A(x,z), \tag{3.2}$$

$$\partial_x\Psi_A(x,z) = \partial_z\Psi_A(x,z), \tag{3.3}$$

$$x\Psi_A(x,z) = L_A(z,\partial_z)\Psi_A(x,z). \tag{3.4}$$

The space of holomorphic functions $\Psi(x,z)$ in x and z that satisfy (3.2)–(3.4) is two dimensional. They are obtained from functions $f(x)$ in the kernel of $L_A(x,\partial_x)$ by putting $\Psi(x,z) = f(x+z)$.

For shortness we will denote the algebras $\mathcal{B}_{\Psi_A}$ and $\mathcal{C}_{\Psi_A}$ of differential operators with rational coefficients associated to the Airy function $\Psi_A(x,z)$, see (1.1), by $\mathcal{B}_A$ and $\mathcal{C}_A$. It is easy to describe explicitly the algebras $\mathcal{B}_A$ and $\mathcal{C}_A$.

Proposition 3.1. *The algebras* $\mathcal{B}_A$ *and* $\mathcal{C}_A$ *coincide with the Weyl algebra* $\mathcal{W}_{poly}$ *of differential operators in one variable with polynomial coefficients. Moreover the antiisomorphism* $b_A\colon \mathcal{B}_A \to \mathcal{C}_A$ *associated to the Airy function* $\Psi_A(x,z)$, *recall* (3.1), *is uniquely defined by the relations*

$$b_A(x) = (L_A(z,\partial_z)), \quad b_A(\partial_x) = \partial_z, \quad b_A(L_A(x,\partial_x)) = z. \tag{3.5}$$

Proof. The relations (3.2)–(3.3) imply that $\mathcal{B}_A$ and $\mathcal{C}_A$ contain $\mathcal{W}_{poly}$ and the validity of (3.5).

Fix $D(x,\partial_x) \in \mathcal{B}_A$ and assume that the leading term of $D(x,\partial_x)$ is $p(x)\partial_x^n$ for some rational function $p(x)$. We need to show that $D(x,\partial_x) \in \mathcal{W}_{poly}$. Since $b_A\colon \mathcal{B}_A\mathcal{C}_A$ is an antiisomorphism

$$\operatorname{ad}^k_{L_A(x,\partial_x)}(D(x,\partial_x)) = 0 \tag{3.6}$$

for some sufficiently large k. We will more generally show that the algebra of differential operators $\mathcal{D}(x, \partial_x)$ with the property (3.6) coincides with $\mathcal{W}_{poly}$.

The leading term of the lhs of (3.6) is

$$2^k p^{(k)}(x) \partial_x^{n+k}.$$

Because it vanishes $p^{(k)}(x) = 0$ and thus $p(x)$ is a polynomial. Then

$$p(x)\partial_x^n \in \mathcal{W}_{poly}, \ D(x, \partial_x) - p(x)\partial_x^n \in \mathcal{B}_A, \text{ and } \operatorname{ord}(D(x, \partial_x) - p(x)\partial_x^n) < n.$$

The proof is now easily completed by induction on the order of $D(x, \partial_x)$. □

3.2. **Selfadjoint Darboux transformations from the Airy function.** Proposition 3.1 implies

$$\mathcal{B}_A \cap \mathbb{C}(x) = \mathbb{C}[x]$$

and

$$b_A^{-1}(\mathcal{C}_A \cap \mathbb{C}(z)) = \mathbb{C}[L_A(x, \partial_x)].$$

The paper [4] defines a set $\mathcal{D}_A$ of *rational Darboux transformations from the Airy function* $\Psi_A(x, z)$. It consists of all functions $\Psi(x, z)$ for which there exist differential operators

$$P(x, \partial_x), Q(x, \partial_x) \in (\mathcal{B}_A)_{(\mathbb{C}[x]\backslash 0)} = \mathcal{W}_{rat} \tag{3.7}$$

such that

$$f(L_A(x, \partial_x)) = Q(x, \partial_x) P(x, \partial_x), \tag{3.8}$$

$$\Psi(x, z) = \frac{1}{p(z)} P(x, \partial_x) \Psi_A(x, z), \tag{3.9}$$

for some monic polynomials $f(t)$ and $p(z)$. After an appropriate division of $\Psi(x, z)$ by a polynomial in x, the differential operators $P(x, \partial_x)$ and $Q(x, \partial_x)$ can be normalized to have leading coefficient 1. (The polynomial $p(z)$ appears only for normalization purposes. The reader can consult Section 5 for details.) The quotient ring of $\mathcal{B}_A$ by $\mathbb{C}[x]\backslash\{0\}$ in (3.7) is well defined since the subset $\mathbb{C}[x]\backslash\{0\}$ of $\mathcal{B}_A = \mathcal{W}_{poly}$ satisfies the Ore condition, see [20].

The main property of the rational Darboux transformations from the Airy function is that they are bispectral functions. This was shown in in [4, 17] and more conceptually proved in [3].

Theorem 3.2. *All rational Darboux transformations from the Airy function* $\Psi(x, z)$ *are bispectral functions of rank* 2.

Definition 3.3. Define the set $\mathcal{SD}_A$ of selfadjoint Darboux transformations from the Airy function $\Psi(x, z)$ to consist of those functions $\Psi(x, z)$ for which there exists a differential operator $P(x, \partial_x) \in \mathcal{W}_{rat}$ with leading coefficient 1 such that

$$g(L_A(x, \partial_x))^2 = (aP)(x, \partial_x) P(x, \partial_x), \tag{3.10}$$

$$\Psi(x, z) = \frac{1}{g(z)} P(x, \partial_x) \Psi_A(x, z), \tag{3.11}$$

for some monic polynomial $g(t)$.

In Section 5 we will provide an explicit classification of all selfadjoint Darboux transformations from the Airy function. We will further show $\mathcal{SD}_A$ that consists exactly of those $\Psi(x, z) \in \mathcal{D}_A$ for which $Q(x, \partial_x) = (aP)(x, \partial_x)$ in (3.8)–(3.9) with an appropriate normalization of the polynomial $p(z)$. (As a consequence of this it is obtained that $f(t)$ is the square of some polynomial $g(t)$, compare to (3.10)–(3.11).)

3.3. **Size of the algebra $\mathcal{B}_A$ relative to the antiisomorphism b_A.** As in (2.12) the algebra $\mathcal{B}_A(=\mathcal{W}_{poly})$ has a natural $\mathbb{Z}_+\times\mathbb{Z}_+$ filtration induced by the antiisomorphism $b_A\colon \mathcal{B}_A\to\mathcal{C}_A$ and the natural filtration on $\mathcal{W}_{poly}$ by the order of differential operators:

$$\mathcal{B}_A^{l_1,l_2}=\{R(x,\partial_x)\in\mathcal{B}_A \mid \operatorname{ord}R(x,\partial_x)\le 2l_1, \operatorname{ord}(b_AR)(z,\partial_z)\le 2l_2\}. \tag{3.12}$$

The following Lemma describes $\mathcal{B}_A^{l_1,l_2}$ explicitly.

Lemma 3.4. *The vector space $\mathcal{B}_A^{l_1,l_2}$ has a basis that consists of the differential operators*

$$\{x^m(L_A(x,\partial_x))^n \mid n\le l_1, m\le l_2\}\cup\{x^m\partial_x(L_A(x,\partial_x))^n \mid n< l_1, m< l_2\}.$$

Lemma 3.4 easily follows from the facts that the set of operators

$$\{x^m(L_A(x,\partial_x))^n, x^m\partial_x(L_A(x,\partial_x))^n\}_{n,m=0}^{\infty}$$

is a basis for $\mathcal{B}_A=\mathcal{W}_{poly}$ and the antiisomorphism $b_A\colon\mathcal{B}_A\to\mathcal{C}_A$ is given by

$$\begin{aligned} b_A(x^m(L_A(x,\partial_x))^n) &= z^n(L_A(x,\partial_x))^m,\\ b_A(x^m\partial_x(L_A(x,\partial_x))^n) &= z^n\partial_z(L_A(x,\partial_x))^m. \end{aligned}$$

Note that the formal adjoint antiinvolution a of $\mathcal{W}_{rat}$ preserves the spaces $\mathcal{B}_A^{l_1,l_2}$. Because $a^2=\mathrm{id}$ the space $\mathcal{B}_A^{l_1,l_2}$ is the direct sum of the eigenspaces of a with eigenvalues ±1. Let us denote the eigenvalue 1 subspace of $\mathcal{B}_A^{l_1,l_2}$ by $\mathcal{B}_{A,sym}^{l_1,l_2}$. In addition we have $ab_A=b_Aa$ which is easily checked on the generators x and ∂_x of $\mathcal{W}_{poly}$. Therefore

$$\begin{aligned}\mathcal{B}_{A,sym}^{l_1,l_2}=\{R(x,\partial_x)\in\mathcal{B}_A \mid &\operatorname{ord}R(x,\partial_x)\le 2l_1,\ \operatorname{ord}(b_AR)(z,\partial_z)\le 2l_2,\\ &aR(x,\partial_x)=R(x,\partial_x),\ a(b_AR(z,\partial_z))=b_AR(z,\partial_z)\}.\end{aligned}$$

The following Lemma contains the main result of this subsection.

Lemma 3.5. *The set of operators*

$$\{x^m(L_A(x,\partial_x))^n+(L_A(x,\partial_x))^nx^m \mid n\le l_1, m\le l_2\} \tag{3.13}$$

is a basis for the space $\mathcal{B}_{A,sym}^{l_1,l_2}$. In particular

$$\dim\mathcal{B}_{A,sym}^{l_1,l_2}=(l_1+1)(l_2+1).$$

Proof. Because

$$a(L_A(x,\partial_x))=L_A(x,\partial_x),\quad a(\partial_x)=-\partial_x$$

the operators in (3.13) and

$$\{x^m\partial_x(L_A(x,\partial_x))^n+(L_A(x,\partial_x))^n\partial_xx^m \mid n< l_1, m< l_2\} \tag{3.14}$$

are eigenvectors of a with eigenvalues 1 and -1, respectively. Obviously (3.13) and (3.14) are subsets of $\mathcal{B}_A^{l_1,l_2}$.

Using the commutation relations

$$[L_A(x,\partial_x),x]=2\partial_x,\quad [\partial_x,L_A(x,\partial_x)]=-1,\quad [\partial_x,x]=1 \tag{3.15}$$

one deduces that

$$L_A(x,\partial_x)^n x^m = x^m L_A(x,\partial_x)^n + \sum_{\substack{0\le i<m\\0\le j<n}} c_{i,j} x^i L_A(x,\partial_x)^j + \sum_{\substack{0\le i<m\\0\le j<n}} d_{i,j} x^i \partial_x L_A(x,\partial_x)^j \tag{3.16}$$

and

$$L_A(x,\partial_x)^n \partial_x x^m = x^m \partial_x L_A(x,\partial_x) + \sum_{\substack{0\le i\le m\\0\le j\le n\\i+j\le m+n-1}} c'_{i,j} x^i L_A(x,\partial_x)^j + \sum_{\substack{0\le i<m\\0\le j<n}} d'_{i,j} x^i \partial_x L_A(x,\partial_x)^j \tag{3.17}$$

for some integers $c_{i,j}$, $d_{i,j}$, $c'_{i,j}$, $d'_{i,j}$. Now (3.16) and (3.17) imply that the operators in (3.13) and (3.14) are linearly independent elements of $\mathcal{B}^{l_1,l_2}_{A,sym}$ and $\mathcal{B}^{l_1,l_2}_{A,-}$, respectively. Thus the dimension of the direct sum of the spaces spanned by the operators in (3.13) and (3.14) is

$$(l_1+1)(l_2+1)+l_1 l_2$$

which is exactly the dimension of

$$\mathcal{B}^{l_1,l_2}_{A} = \mathcal{B}^{l_1,l_2}_{A,sym} \oplus \mathcal{B}^{l_1,l_2}_{A,-}.$$

This completes the proof of the Lemma. □

Example 3.6. Consider the integral operator with kernel

$$K_A(x,y) = \int_s^\infty \Psi_A(x,z)\Psi_A(y,z)\mathrm{d}z = \int_s^\infty A(x+z)A(y+z)\mathrm{d}z \tag{3.18}$$

acting on $L^2([t,\infty))$, $s,t\in\mathbb{R}$.

Theorem 2.5 and Lemma 3.4 imply that there exists a second order differential operator which commutes with K_A. Here we construct such a differential operator by a straightforward application of Proposition 2.1 and Lemma 3.4.

According to Lemma 3.4

$$\mathcal{B}^{1,1}_{A,sym} = \mathrm{Span}\{1, x, L_A(x,\partial_x), (xL_A(x,\partial_x)+L_A(x,\partial_x)x)/2 = \partial_x x\partial_x - x^2\}. \tag{3.19}$$

Proposition 2.1 implies that the operator

$$D(x,\partial_x) = ux + vL_A(x,\partial_x) + w\left(xL_A(x,\partial_x)+L_A(x,\partial_x)x\right)/2$$

commutes with K_A if

$$\phi_t(D(x,\partial_x)) = 0 \text{ and } \phi_s((b_A D)(z,\partial_z)) = 0. \tag{3.20}$$

(The constant 1 in (3.19) is disregarded in the construction of $D(x,\partial_x)$ since it commutes with K_A.) But

$$D(x,\partial_x) = \partial_x(wx+v)\partial_x - (wx^2+(v-u)x)$$

and

$$\begin{aligned} D(x,\partial_x) &= uL_A(z,\partial_z) + vz + w(zL_A(z,\partial_z)+L_A(z,\partial_z)z)/2 \\ &= \partial_z(wz+u)\partial_z - (wz^2+(u-v)z). \end{aligned}$$

Finally because of Lemma 2.4, (3.20) is equivalent to

$$wt + v = 0,\ ws + u = 0$$

and we can choose $w = 1$, $v = -t$, $u = -s$.

The differential operator which commutes with K_A, obtained in this way, is

$$D(x, \partial_x) = -sx - tL_A(x, \partial_x) + (xL_A(x, \partial_x) + L_A(x, \partial_x)x)/2 \tag{3.21}$$

$$= \partial_x(x - t)\partial_x - (x^2 + (s - t)x). \tag{3.22}$$

This is the operator found by Tracy and Widom in [26]. Note that although (3.22) is simpler than (3.21), it is (3.21) that reveals the natural symmetry between s and t, recall (3.5).

Let us also note that K_A is an integrable integral operator in the sense of Its–Izergin–Korepin–Slavnov [16]. The kernel (3.18) is also given by the formula

$$K_A(x, y) = \frac{A(x)A'(y + s) - A(y)A'(x + s)}{x - y} \tag{3.23}$$

which can be seen from

$$\begin{aligned}
&(x - y)K_A(x, y) \\
&= \int_s^\infty (L_A(z, \partial_z)\Psi_A(x, z))\, \Psi_A(y, z)\mathrm{d}z - \int_s^\infty \Psi_A(x, z)\, (L_A(z, \partial_z)\Psi_A(y, z))\, \mathrm{d}z \\
&= A(x)A'(y + s) - A(y)A'(x + s).
\end{aligned}$$

All integral operators, considered subsequently, will be integrable integral but will be much more complicated than (3.23). □

3.4. **Size of the algebra $\mathcal{B}_{\Psi,sym}$ for selfadjoint Darboux transformations from the Airy function, relative to the involution b_Ψ.** Let us fix an arbitrary selfadjoint Darboux transformation $\Psi(x, z) \in \mathcal{SD}_A$ from the Airy function given by (3.10)–(3.11) for some $P(x, \partial_x) \in \mathcal{W}_{rat}$ and $g(t) \in \mathbb{C}[t]$. The goal of this subsection is to show that the spaces $\mathcal{B}_\Psi^{l_1,l_2}$ are sufficiently large, more precisely that

$$\dim \mathcal{B}_A^{l_1,l_2} - \dim \mathcal{B}_\Psi^{l_1,l_2}$$

is bounded from below as a function of l_1 and $l_2 \in \mathbb{Z}_{\geq 0}$.

Let

$$P(x, \partial_x) = \frac{1}{v(x)} R(x, \partial_x) \tag{3.24}$$

for some $R(x, \partial_x) \in \mathcal{B}_A (= \mathcal{W}_{poly})$ and $v(x) \in \mathbb{C}[x]$. Set

$$\operatorname{ord} R(x, \partial_x) = \rho_1 \quad \text{and} \quad \operatorname{ord}(b_A R)(x, \partial_x) = \rho_2. \tag{3.25}$$

Define

(3.26)

$$\mathcal{S}_{\Psi,1} = \mathrm{Span}\{\frac{1}{v(x)} R(x, \partial_x) M(x, \partial_x)(aR)(x, \partial_x) \frac{1}{v(x)} \mid M(x, \partial_x) \in \mathcal{B}_{A,sym}^{l_1-\rho_1,l_2}\},$$

(3.27)

$$\mathcal{S}_{\Psi,2} = \mathrm{Span}\{v(x)M(x, \partial_x)v(x) \mid M(x, \partial_x) \in \mathcal{B}_{A,sym}^{l_1,l_2-\rho_2}\}.$$

Proposition 3.7. *The spaces of differential operators $\mathcal{S}_{\Psi,1}$ and $\mathcal{S}_{\Psi,2}$ are subspaces of $\mathcal{B}^{l_1,l_2}_{\Psi,sym}$ and*

$$b_\Psi\left(\frac{1}{v(x)}R(x,\partial_x)M(x,\partial_x)(aR)(x,\partial_x)\frac{1}{v(x)}\right) = g(z)(b_A M)(z,\partial_z)g(z), \tag{3.28}$$

$$b_\Psi\left(v(x)M(x,\partial_x)v(x)\right) = \frac{1}{g(z)}(b_A R)(z,\partial_z)(b_A M)(z,\partial_z)(ab_A R)(z,\partial_z)\frac{1}{g(z)} \tag{3.29}$$

for the operators in $\mathcal{S}_{\Psi,1}$ and $\mathcal{S}_{\Psi,2}$

Proof. Let

$$M(x,\partial_x)\in\mathcal{B}^{l_1-\rho_1,l_2}_{A,sym}.$$

Start with the equality

$$M(x,\partial_x)g(L_A(x,\partial_x))^2\Psi_A(x,z) = g(z)^2(b_A M)(z,\partial_z)\Psi_A(x,z).$$

Using the definition (3.10)–(3.11) of $\Psi(x,z)$ one gets

$$M(x,\partial_x)(aR)(x,\partial_x)\frac{1}{v(x)}g(z)\Psi(x,z) = g^2(z)(b_A M)(z,\partial_z)\Psi_A(x,z).$$

Acting on both sides by $\frac{1}{v(x)g(z)}R(x,\partial_x)$ and taking into account (3.11) gives

$$\frac{1}{v(x)}R(x,\partial_x)M(x,\partial_x)(aR)(x,\partial_x)\frac{1}{v(x)}\Psi(x,z) = g(z)(b_A M)(z,\partial_z)g(z)\Psi(x,z).$$

Note that

$$(a-\mathrm{id})\left(\frac{1}{v(x)}R(x,\partial_x)M(x,\partial_x)(aR)(x,\partial_x)\frac{1}{v(x)}\right) = 0$$

since $(aM)(x,\partial_x) = M(x,\partial_x)$. Analogously

$$(a-\mathrm{id})\left(g(z)(b_A M)(z,\partial_z)g(z)\right) = 0$$

because of $(ab_A M)(z,\partial_z) = (b_A a M)(z,\partial_z) = (b_A M)(z,\partial_z)$. This finally implies

$$\frac{1}{v(x)}R(x,\partial_x)M(x,\partial_x)(aR)(x,\partial_x)\frac{1}{v(x)}\in\mathcal{B}^{l_1,l_2}_{\Psi,sym} \tag{3.30}$$

(that is $\mathcal{S}_{\Psi,1}\subset\mathcal{B}^{l_1,l_2}_{\Psi,sym}$) and the validity of (3.28).

For the second part we will need Theorem 4.2 of [3]. In our setting it states that (3.10)–(3.11) are equivalent to

$$v(L_A(z,\partial_z))^2 = (ab_A R)(z,\partial_z)\frac{1}{g(z)^2}(b_A R)(z,\partial_z), \tag{3.31}$$

$$\Psi(x,z) = \frac{1}{v(x)g(z)}(b_A R)(z,\partial_z)\Psi_A(x,z). \tag{3.32}$$

(The hard step is to show the first equality.) Now the fact that $\mathcal{S}_{\Psi,2}$ is a subspace of $\mathcal{B}^{l_1,l_2}_{\Psi,sym}$ and (3.29) follow from the above relations and (3.30), (3.28) by exchanging the roles of x and z. □

Proposition 3.8. *The dimension of the intersection $\mathcal{S}_{\Psi,1}\cap\mathcal{S}_{\Psi,2}$ is less than or equal to*

$$(l_1-\rho_1+1)(l_2-\rho_2+1). \tag{3.33}$$

Proof. Comparing the orders of the operators in (3.10) and using (3.24) gives

$$\rho_1 (= \operatorname{ord} R) = 2 \deg g. \tag{3.34}$$

In particular ρ_1 is even. Similarly from (3.31) we deduce

$$\rho_2 (= \operatorname{ord} b_A R) = 2 \deg v. \tag{3.35}$$

Comparing the leading terms in (3.10) (and again using (3.24)) we get that the leading term of $R(x, \partial_x)$ is $v(x)\partial_x^{\rho_1}$. Combining the above facts implies that

$$R(x, \partial_x) = v(x) L_A(x, \partial_x)^r + \sum_{j=0}^{r-1} (h_j(x) + g_j(x)\partial_x) L_A(x, \partial_x)^j \tag{3.36}$$

for some polynomials $h_j(x)$, $g_j(x)$.

We need to show that the dimension of the space of differential operators

$$M(x, \partial_x) \in \mathcal{B}_{A,sym}^{l_1 - \rho_1, l_2} \text{ and } N(x, \partial_x) \in \mathcal{B}_{A,sym}^{l_1, l_2 - \rho_2}$$

such that

$$R(x, \partial_x) M(x, \partial_x) (aR)(x, \partial_x) = v(x)^2 N(x, \partial_x) v(x)^2 \tag{3.37}$$

is less than or equal to $(l_1 - \rho_1 + 1)(l_2 - \rho_2 + 1)$. Write two such operators $M(x, \partial_x)$ and $N(x, \partial_x)$ in the form

$$M(x, \partial_x) = \sum_{j=0}^{l_1 - \rho_1} \left(b_j(x) L_A(x, \partial_x)^j + L_A(x, \partial_x)^j b_j(x) \right), \tag{3.38}$$

$$N(x, \partial_x) = \sum_{j=0}^{l_1} \left(c_j(x) L_A(x, \partial_x)^j + L_A(x, \partial_x)^j c_j(x) \right) \tag{3.39}$$

where $b_j(x)$ and $c_j(x)$ are polynomials and

$$\deg b_j \leq l_2, \ \deg c_j \leq l_2 - \rho_2. \tag{3.40}$$

In terms of the those we need to show that the dimension of the space of of polynomials $\{b_j(x)\}_{j=0}^{l_1-\rho_1}$ and $\{c_j(x)\}_{j=0}^{l_2}$, satisfying (3.37), is less than or equal to $(l_1 - \rho_1 + 1)(l_2 - \rho_2 + 1)$. (The polynomials $v(x)$, $h_j(x)$, and $g_j(x)$ are fixed. They determine the function $\Psi(x, z)$.)

Using the commutation relations (3.15) we get

$$M(x, \partial_x) = 2 \sum_{j=0}^{l_1 - \rho_1} b_j(x) L_A(x, \partial_x)^j + \sum_{i=0}^{l_1 - \rho_1 - 1} \left(\eta_i(x) + \theta_i(x) \partial_x \right) L_A(x, \partial_x)^i$$

where $\eta_i(x)$ and $\theta_i(x)$ are polynomials determined from $b_k(x)$ for $k \in [i+1, l_1 - \rho_1]$.

Again applying the commutation relations (3.15) we obtain

$$\begin{aligned} R(x, \partial_x) M(x, \partial_x) (aR)(x, \partial_x) = & \sum_{j=0}^{l_1 - \rho_1} 2 v(x)^2 b_j(x) L_A(x, \partial_x)^{j + \rho_1} \\ & + \sum_{i=0}^{l_1 - 1} \left(\alpha_i(x) + \beta_i(x) \partial_x \right) L_A(x, \partial_x)^i \end{aligned} \tag{3.41}$$

for some polynomials $\alpha_i(x)$ and $\beta_i(x)$ depending only on the polynomials $b_k(x)$ for $k \in [i - \rho_1 + 1, l_1 - \rho_1]$.

Analogously

$$v(x)^2 N(x,\partial_x) v(x)^2 = 2v(x)^4 \sum_{i=0}^{l_1} c_i(x) L_A(x,\partial_x)^i + v(x)^2 \sum_{i=0}^{l_1-1} \left(\gamma_i(x) + \delta_i(x)\partial_x\right) L_A(x,\partial_x)^i \tag{3.42}$$

for some polynomials $\gamma_i(x)$ and $\delta_i(x)$ depending only on the polynomials $c_k(x)$ for $k \in [i+1, l_1]$.

Now we substitute (3.41) and (3.42) in (3.37) and use the fact that

$$\{x^m L_A(x,\partial_x)^n, x^m \partial_x L_A(x,\partial_x)^n\}_{n,m=0}^{\infty}$$

are linearly independent:

1. From the coefficient of $L_A(x,\partial_x)^{l_1}$ we obtain

$$b_{l_1-\rho_1}(x) = v(x)^2 c_{l_1}(x).$$

Since $\deg b_{l_1-\rho_1}(x) \leq l_2$ and $2\deg v(x) = \rho_2$, see (3.35) the polynomial $b_{l_1-\rho_1}(x)$ depends (linearly) on at most $l_2 - \rho_2 + 1$ parameters. Fixing these parameters determines $c_{l_1}(x)$ completely.

2. Arguing by induction, assume that $b_j(x), \ldots, b_{l_1-\rho_1}(x)$ and $c_{j+l_1}(x), \ldots, c_{l_1}(x)$ and are fixed for j running from $l_1 - \rho_1$ down to 1. From the coefficient of $L_A(x,\partial_x)^{j-1}$ we obtain

$$b_{j-1}(x) = v(x)^2 c_{j+\rho_1-1}(x) + \left(\alpha_{j+\rho_1-1}(x) - \eta_{j+\rho_1-1}(x)\right)/2v(x)^2.$$

Thus the reminder of $b_{j-1}(x)$ modulo $v(x)^2$ is fixed. Since $\deg b_{j-1}(x) \leq l_2$ and $\deg v(x)^2 = \rho_2$ the choice of $b_{j-1}(x)$ depends (linearly) on at most $l_2 - \rho_2 + 1$ parameters. Fixing $b_{j-1}(x)$ completely determines $c_{j+l_1-1}(x)$.

Adding up these “maximal degrees of freedom” for the polynomials $b_0(x), \ldots, b_{l_1-\rho_1}(x)$ completes the proof. □

Because of Lemma 3.5

$$\dim \mathcal{S}_{\Psi,1} = (l_1 - \rho_1 + 1)(l_2 + 1) \quad \text{and} \quad \dim \mathcal{S}_{\Psi,2} = (l_1 + 1)(l_2 - \rho_2 + 1).$$

Taking into account Proposition 3.8 we obtain the following Theorem.

Theorem 3.9. *For any selfadjoint Darboux transformation from the Airy function $\Psi(x,z) \in \mathcal{SD}_A$ the dimension of the space of differential operators $\mathcal{S}_{\Psi,1} + \mathcal{S}_{\Psi,2}$ is greater than or equal to $(l_1+1)(l_2+1) - \rho_1\rho_2$. In particular*

$$\dim \mathcal{B}_{\Psi,sym}^{l_1,l_2} \geq (l_1+1)(l_2+1) - \rho_1\rho_2. \tag{3.43}$$

Theorem 3.9 and Theorem 2.5 imply our final result for integral operators derived from Darboux transformations from the Airy function.

Theorem 3.10. *Let $\Psi(x,z) \in \mathcal{SD}_A$ be a selfadjoint Darboux transformation from the Airy function, given by* (3.10), (3.11), (3.24). *Let Γ_1, Γ_2 be two connected contours in $\mathbb{C}$ that do not contain the roots of the polynomials $v(t)$ and $g(t)$ respectively and that begin at some finite points and go to infinity in the sector $-\pi/3 < \arg x < \pi/3$. Then the integral operator*

$$K(x,y) = \int_{\Gamma_2} \Psi(x,z)\Psi(y,z)\mathrm{d}z$$

on $L^2(\Gamma_1)$ commutes with a formally symmetric differential operator with rational coefficients of order less than or equal to $2\rho_1\rho_2$ and domain all smooth functions on Γ_1 that decrease rapidly as $x \to \infty$.

Proposition 3.11. *Let Γ_1 and Γ_2 be two contours as in Theorem 3.10 with end points t and s, respectively. For any selfadjoint Darboux transformation $\Psi(x,z) \in \mathcal{SD}_A$ from the Airy function, given by* (3.10), (3.11), (3.24) *the integral operator with kernel*

$$K_\Psi(x,y) = \int_{\Gamma_2} \Psi(x,z)\Psi(y,z)\mathrm{d}z$$

acting on $L^2(\Gamma_1)$ is integrable in the sense of Its–Izergin–Korepin–Slavnov [16]. *Moreover there exist functions $a_i(x)$, $b_i(y)$, $c_i(x)$, and $d_i(y)$ that are smooth on Γ_1 such that*

$$K(x,y) = \frac{v(x)}{v(y)} K_A(x,y) + \sum_i^N a_i(x) b_i(y) \tag{3.44}$$

$$= \frac{v(y)}{v(x)} K_A(x,y) + \sum_i^N c_i(x) d_i(y) \tag{3.45}$$

where $K_A(x,y)$ is the Airy kernel given by (3.23)

$$K_A(x,y) = \int_{\Gamma_2} \Psi_A(x,z)\Psi_A(y,z)\mathrm{d}z = \frac{A(x)A'(y+s) - A(y)A'(x+s)}{x-y}.$$

Proof. According to (3.32)

$$K_\Psi(x,y) = \frac{1}{v(x)v(y)} \int_{\Gamma_2} \frac{1}{g(z)} (b_A R(z,\partial_z)\Psi_A(x,z)) \frac{1}{g(z)} (b_A R(z,\partial_z)\Psi_A(y,z))\mathrm{d}z. \tag{3.46}$$

Integrating by parts (2.7) the second operator $(1/g(z))b_A R(z,\partial_z)$ we get

$$K_\Psi(x,y) = \frac{1}{v(x)v(y)} \int_{\Gamma_2} \left(a b_A R(z,\partial_z) \frac{1}{g(z)^2} b_A R(z,\partial_z) \Psi_A(x,z) \right) \Psi_A(y,z)\mathrm{d}z$$
$$- \left\langle \frac{1}{g(z)} (b_A R(z,\partial_z)\Psi_A(x,z)), \Psi_A(x,z) \right\rangle_{(1/g(z)) b_A R(z,\partial_z)} (s).$$

The second term above gives rise to the second term in (3.44). Eq. (3.45) is proved by intergrating out the second operator $(1/g(z))b_A R(z,\partial_z)$ in (3.46). □

4. Integral operators associated to selfadjoint Darboux transformations of Bessel functions

4.1. **The Bessel bispectral function.** Recall that the *Bessel functions of third kind* are given by

$$I_\nu(t) = (t/2)^\nu \sum_{m=0}^{\infty} \frac{(t/2)^{2m}}{m!\Gamma(m+\mu+1)}, \ t \in \mathbb{C}\backslash\{0\} \tag{4.1}$$

for $\nu \in \mathbb{C}\backslash\mathbb{Z}$, see e.g. [6, Chapter 7.2.2]. For those values of ν one defines the functions

$$K_\nu(t) = \frac{I_{-\nu}(t) - I_\nu(t)}{2\pi \sin(\nu\pi)} \tag{4.2}$$

which are called *modified Bessel functions of third kind.* Both $I_\nu(t)$ and $K_\nu(t)$ are multiple valued holomorphic functions of $t \in \mathbb{C}\{0\}$. By continuity one defines $I_\nu(t)$ and $K_\nu(t)$ for $\nu \in \mathbb{Z}$, see [6, Sections 7.2.4-7.2.5] for details.

By abuse of notation the functions

$$\Psi_\nu(x,z) = (xz)^{1/2} K_{\nu+1/2}(xz) \tag{4.3}$$

will be also called Bessel functions. Consider the Euler operator

$$D_x = x\partial_x$$

and the operators

$$L_\nu(x,\partial_x) = \partial_x^2 - \frac{\nu(\nu+1)}{x^2} = \frac{1}{x^2}(D_x+\nu)(D_x-\nu-1) \tag{4.4}$$

to be called Bessel operators. The Bessel functions satisfy the equations

$$L_\nu(x,\partial_x)\Psi_\nu(x,z) = z^2\Psi_\nu(x,z), \tag{4.5}$$

$$D_x\Psi_\nu(x,z) = D_z\Psi_\nu(x,z), \tag{4.6}$$

$$x^2\Psi_\nu(x,z) = L_\nu(z,\partial_z)\Psi_\nu(x,z). \tag{4.7}$$

For shortness the algebras $\mathcal{B}_{\Psi_\nu}$ and $\mathcal{C}_{\Psi_\nu}$ associated to the Bessel function $\Psi_\nu(x,z)$, recall (1.1), will be denoted by $\mathcal{B}_\nu$ and $\mathcal{C}_\nu$.

Because of (4.5)–(4.7) the algebras of differential operators $\mathcal{B}_\nu$ and $\mathcal{C}_\nu$ contain the operators $L_\nu(x,\partial_x)$, D_x, and x^2. *We will denote their subalgebras generated by those operators by* $\widetilde{\mathcal{B}}_\nu$ *and* $\widetilde{\mathcal{C}}_\nu$, respectively. Clearly (4.5)–(4.7) imply

$$b_\nu(\widetilde{\mathcal{B}}_\nu) = \widetilde{\mathcal{C}}_\nu.$$

The algebras $\mathcal{B}_\nu$ and $\mathcal{C}_\nu$ are described in the following Proposition.

Proposition 4.1. *(i) For* $\nu \in \mathbb{C}\backslash\mathbb{Z}$ *the algebras* $\mathcal{B}_\nu$ *and* $\mathcal{C}_\nu$ *are generated by the operators* $L_\nu(x,\partial_x)$, D_x, *and* x^2, *i.e.*

$$\mathcal{B}_\nu = \widetilde{\mathcal{B}}_\nu, \quad \mathcal{C}_\nu = \widetilde{\mathcal{C}}_\nu.$$

(ii) If $\nu \in \mathbb{Z}$ *the algebras* $\mathcal{B}_\nu$ *and* $\mathcal{C}_\nu$ *are larger than* $\widetilde{\mathcal{B}}_\nu$ *and* $\widetilde{\mathcal{C}}_\nu$, *e.g.* $\mathcal{B}_0 = \mathcal{C}_0 = \mathcal{W}_{poly}$.

For those values of ν *the subalgebras* $\widetilde{\mathcal{B}}_\nu$ *and* $\widetilde{\mathcal{C}}_\nu$ *of* $\mathcal{B}_\nu$ *and* $\mathcal{C}_\nu$ *consist of exactly those differential operators in* $\mathcal{B}_\nu$ *and* $\mathcal{C}_\nu$ *that are invariant under the transformation* $x \mapsto -x$.

Remark 4.2. In the definition of $\Psi_\nu(x,z)$ instead of $K_\nu(\cdot)$ we could use the Bessel functions of first kind $J_\nu(i\cdot)$ or the modified Bessel functions of first kind $I_\nu(\cdot)$. In that case the corresponding algebras $\mathcal{B}_\nu$ and $\mathcal{C}_\nu$ would be smaller, in fact equal to $\widetilde{\mathcal{B}}_\nu$ and $\widetilde{\mathcal{C}}_\nu$ above. *This will not reflect the fact that the operators* $L_\nu(x,\partial_x)$ *generate rank 1 maximal bispectral algebras for* $\nu \in \mathbb{Z}$, *see* [4, 8].

Proof of Proposition 4.1. Part(i): We will need Proposition 2.4 from [4] which states that $\mathbb{C}[L_\nu(x,\partial_x)]$ is a maximal bispectral algebra of rank 2 for $\nu \in \mathbb{C}\backslash\mathbb{Z}$. In particular this means that

$$[L_\nu(x,\partial_x), C(x,d_x)] = 0 \text{ implies } C(x,\partial_x) \in \mathbb{C}[L_\nu(x,d_x)]. \tag{4.8}$$

Since $b_\nu\colon \mathcal{B}_\nu \to \mathcal{C}_\nu$ is an antiisomorphism, as in Proposition 3.1, we have that for any $D(x,\partial_x) \in \mathcal{B}_\nu$ there exists an integer k such that

$$\mathrm{ad}^k_{L_\nu(x,\partial_x)}(D(x,\partial_x)) = 0. \tag{4.9}$$

The minimal such k for a given $D(x, \partial_x)$ will be denoted by $k(D)$.

First we will prove by induction on $k(D)$ that $D(x, \partial_x) \in \mathcal{B}_\nu$ implies that $D(x, \partial_x)$ is invariant under the transformation $x \mapsto -x$. (In the case $k(D) = 1$ this follows from (4.8).)

Note that $D(-x, -\partial_x) \in \mathcal{B}_\nu$ and

$$b_\nu(D(-x, -\partial_x)) = (b_\nu D)(-z, -\partial_z)$$

because $\Psi_\nu(x, z)$ depends only on xz. Now (4.9) implies that

$$\mathrm{ad}^{k(D)-1}_{L_\nu(x,\partial_x)}(D(x, \partial_x)) = q(L_\nu(x, \partial_x))$$

for some polynomial $q(t)$. Since $L_\nu(x, \partial_x)$ is invariant under the transformation $x \mapsto -x$ we get

$$\mathrm{ad}^{k(D)-1}_{L_\nu(x,\partial_x)}(D(x, \partial_x)) = \mathrm{ad}^{k(D)-1}_{L_\nu(x,\partial_x)}(D(-x, -\partial_x)).$$

That is

$$k(D(x, \partial_x) - D(-x, -\partial_x)) \leq k(D) - 1,$$

and due to the inductive assumption the operator

$$D(x, \partial_x) - D(-x, -\partial_x) \tag{4.10}$$

is invariant under the transformation $x \mapsto -x$. At the same time the operator (4.10) is clearly skewinvariant under $x \mapsto -x$. This can only happen if $D(x, \partial_x)$ is by itself invariant under $x \mapsto -x$.

Now fix $D(x, \partial_x) \in \mathcal{B}_\nu$. We will show that $D(x, \partial_x) \in \widetilde{\mathcal{B}}_\nu$. If the leading term of $D(x, \partial_x)$ is $p(x)\partial_x^n$, we have $p(x) = p(-x)$ because $D(x, \partial_x)$ is invariant under $x \mapsto -x$.

Analogously to Proposition 3.1, using (4.9) one shows that $p(x)$ is a polynomial.

If n is even then $p(x) = h(x^2)$ for some polynomial $h(t)$ and

$$D(x, \partial_x) - h(x^2) L_\nu^{n/2}(x, \partial_x) \in \mathcal{B}_\nu$$

and has order less than $\mathrm{ord}\, D$.

If n is odd then $p(x) = xh(x^2)$ for some polynomial $h(t)$ and

$$D(x, \partial_x) - h(x^2) D_x L_\nu^{n/2}(x, \partial_x) \in \mathcal{B}_\nu$$

and has order less than $\mathrm{ord}\, D$.

In either cases the proof of part (i) is completed by induction on $\mathrm{ord}\, D$.

Part (ii): Recall that

$$K_{1/2}(t) = (\pi/2t)^{1/2} e^{-t}$$

and thus

$$\Psi_0(x, z) = (\pi/2)^{1/2} e^{-xz}.$$

Analogously to Proposition 3.1 one proves that $\mathcal{B}_0 = \mathcal{C}_0 = \mathcal{W}_{poly}$.

For $\nu \in \mathbb{Z}$ the algebras $b_\nu^{-1}(\mathcal{C}_\nu \cap \mathbb{C}[z])$ are maximal bispectral algebras of rank 1, [4, 8, 29]. This means that that they contain differential operators of odd order with leading coefficient 1, and hence they contain differential operators that are not invariant under $x \mapsto -x$. That is for $\nu \in \mathbb{Z}$ the algebras $\mathcal{B}_\nu$ and $\mathcal{C}_\nu$ are larger than $\widetilde{\mathcal{B}}_\nu$ and $\widetilde{\mathcal{C}}_\nu$.

The last statement in part (ii) is proved analogously to part (i). $\square$

The Bessel functions corresponding to $\nu_1, \nu_2 \in \mathbb{C}$ that differ by an integer can be obtained by a Darboux transformation from each other:

$$\Psi_{\nu+1}(x,z) = \frac{1}{xz}(D_x - \nu - 1)\Psi_\nu(x,z), \tag{4.11}$$

$$\Psi_\nu(x,z) = \frac{1}{xz}(D_x + \nu + 1)\Psi_{\nu+1}(x,z), \tag{4.12}$$

which corresponds to the factorizations

$$L_\nu = x^{-1}(D_x + \nu + 1)x^{-1}(D_x - \nu - 1), \quad L_{\nu+1} = x(D_x - \nu - 1)x^{-1}(D_x + \nu + 1).$$

The above Darboux transformations are the reason for the fact that for $\nu \in \mathbb{Z}$ the Bessel operators $L_\nu(x, \partial_x)$ belong to maximal bispectral algebras of rank 2.

4.2. **Selfadjoint Darboux transformations from the Bessel functions.** The set of *rational Darboux transformations* $\mathcal{D}_\nu$ *from the Bessel function* $\Psi_\nu(x,z)$, defined in [4], consists of those functions $\Psi(x,z)$ for which there exist differential operators

$$P(x,\partial_x), Q(x,\partial_x) \in (\mathcal{B}_\nu)_{(\mathcal{B}_\nu \cap \mathbb{C}(x)\backslash\{0\})} \tag{4.13}$$

such that

$$f(L_\nu(x,\partial_x)) = Q(x,\partial_x)P(x,\partial_x), \tag{4.14}$$

$$\Psi(x,z) = \frac{1}{p(z)}P(x,\partial_x)\Psi_\nu(x,z), \tag{4.15}$$

for some polynomials $f(t)$ and $p(z)$. It is **not hard** to show that the subset $\mathcal{B}_\nu \cap \mathbb{C}(x)\backslash\{0\}$ of $\mathcal{B}_\nu$ satisfies the Ore condition and the quotient ring in (4.13) makes sense. The function $\Psi(x,z)$ is further normalized by dividing it by the leading coefficient of $P(x,\partial_x)$, see [4].

Let us note that for $\nu \in \mathbb{Z}$ the intersection $\mathcal{B}_\nu \cap \mathbb{C}(x)$ is larger than $\mathbb{C}[x^2]$. For the present paper we will need the smaller algebra $\widetilde{\mathcal{B}}_\nu \cap \mathbb{C}(x)$, for which part (ii) of Proposition 4.1 implies

$$\mathbb{C}[x^2] = \widetilde{\mathcal{B}}_\nu \cap \mathbb{C}(x),$$
$$\mathbb{C}[L_\nu(x,\partial_x)] = b_\nu^{-1}(\widetilde{\mathcal{C}}_\nu \cap \mathbb{C}(z)).$$

(Note that for $\nu \in \mathbb{Z}$ the intersection $\mathcal{B}_\nu \cap \mathbb{C}(x)$ is larger than $\mathbb{C}[x^2]$, see part (ii) of Proposition 4.1.)

The following theorem was proved in [29] for $\nu = 1$ and in [4, 3] in the general case.

Theorem 4.3. *All rational Darboux transformations from the Bessel functions are bispectral of rank* 2 *if* $\nu \in \mathbb{C}\backslash\mathbb{Z}$ *and of rank 1 if* $\nu \in \mathbb{Z}$.

Definition 4.4. Define the set of selfadjoint ("even, selfadjoint" in the case $\nu \in \mathbb{Z}$) Darboux transformations $\mathcal{SD}_\nu$ from the Bessel functions $\Psi_\nu(x,z)$ to consist of all functions $\Psi(x,z)$ for which there exists a monic polynomial $f(t)$ of the form

$$f(t) = t^m g(t)^2, \ m = 0 \text{ or } 1 \tag{4.16}$$

and a differential operator

$$P(x,\partial_x) \in (\widetilde{\mathcal{B}}_\nu)(\mathbb{C}[x^2]\backslash\{0\})^{-1}$$

with leading term x^m such that

$$f(L_\nu(x,\partial_x)) = (-1)^m (aP)(x,\partial_x) x^{-2m} P(x,\partial_x) \tag{4.17}$$

$$\Psi(x,z) = \frac{1}{z^m g(z^2) x^m} P(x,\partial_x)\Psi_\nu(x,z). \tag{4.18}$$

In general the polynomial $g(t)$ can vanish at 0.

Equations (4.11)–(4.12) show that $\Psi_{\nu+n}(x,z)$ are (even) selfadjoint Darboux transformations from $\Psi_\nu(x,z)$.

Below we explain the reason for the terminology *even.* In Section 5 we will show that the set $\mathcal{SD}_\nu$ consists of those rational Darboux transformations $\Psi(x,z)$ from the Bessel function $\Psi_\nu(x,z)$ for which in the notation (4.14)–(4.15)

$$\widetilde{Q}(x,\partial_x) = (-1)^{\operatorname{ord} P} (a\widetilde{P})(x,\partial_x) \tag{4.19}$$

with the additional property in the case $\nu \in \mathbb{Z}$

$$P(-x,-\partial_x) = P(x,\partial_x). \tag{4.20}$$

In (4.19), $\widetilde{Q}(x,\partial_x)$ and $\widetilde{P}(x,\partial_x)$ denote the differenial operators obtained from $Q(x,\partial_x)$ and $P(x,\partial_x)$ by dividing them by their leading coefficients on the right and on the left, respectively. As a consequence it is obtained that the polynomial $f(t)$ in (4.14) has the form (4.16). (An appropriate normalization of $p(z)$ is made as well.)

According to part (i) of Proposition 4.1 in the rank 2 case $\nu \in \mathbb{C}\backslash\mathbb{Z}$ the condition (4.20) is a consequence of (4.13) because in that case

$$\mathcal{B}_\nu = \widetilde{\mathcal{B}}_\nu.$$

In the rank 1 case $\nu \in \mathbb{Z}$ the term *even* reflects this extra condition, recall that due to part (ii) of Proposition 4.1 the algebra of operators in $\mathcal{B}_\nu$ invariant under $x \mapsto -x$ is exactly $\widetilde{\mathcal{B}}_\nu$. The extra condition (4.20) is needed since in the case of Darboux transformations from the Bessel function the prolate spheroidal property will be deduced from the second condition in Theorem 2.5.

4.3. **Size of the algebras $\widetilde{\mathcal{B}}_\nu$ relative to the antiisomorphisms b_ν.** Similarly to (2.12) the algebra $\widetilde{\mathcal{B}}_\nu$ has a $\mathbb{Z}_+ \times \mathbb{Z}_+$ filtration induced by the antiisomorphism $b_\nu \colon \widetilde{\mathcal{B}}_\nu \to \widetilde{\mathcal{C}}_\nu$ and the standard $\mathbb{Z}_+$ filtration on $\mathcal{W}_{rat}$ by the order of the operator:

$$\widetilde{\mathcal{B}}_\nu^{l_1,l_2} = \{R(x,\partial_x) \in \widetilde{\mathcal{B}}_\nu \mid \operatorname{ord} R(x,\partial_x) \le 2l_1, \operatorname{ord}(b_\nu R)(z,\partial_z) \le 2l_2\} \tag{4.21}$$

Lemma 4.5. *The set of operators*

$$\{x^{2m}(L_\nu(x,\partial_x))^n \mid n \le l_1, m \le l_2\} \cup \{x^{2m} D_x (L_\nu(x,\partial_x))^n \mid n < l_1, m < l_2\}.$$

is a basis of the space $\mathcal{B}_\nu^{l_1,l_2}$.

Proof. The operators $2L_\nu(x,\partial_x)$, $2x^2$, and $-D_x - 1/2$ satisfy the commutation relations for the Cartan generators of $sl_2(\mathbb{C})$:

(4.22)

$$[D_x, L_\nu(x,\partial_x)] = -2L_\nu(x,\partial_x),\ [D_x, x^2] = -2x^2,\ [x^2, L_\nu(x,\partial_x)] = -4D_x - 2.$$

Using in addition the fact the operators $L_\nu(x,\partial_x)$, D_x, and x^2 generate $\widetilde{\mathcal{B}}_\nu$ one easily obtains that the following set is a basis of $\widetilde{\mathcal{B}}_\nu$

$$\{x^{2m}\,(L_\nu(x,\partial_x))^n\,, x^{2m} D_x\,(L_\nu(x,\partial_x))^n\}_{m,n=0}^\infty.$$

In terms of this basis the antiisomorphism $b_\nu \colon \widetilde{\mathcal{B}}_\nu \to \widetilde{\mathcal{C}}_\nu$ is given by

$$b_\nu \left(x^{2m} (L_\nu(x, \partial_x))^n \right) = (L_\nu(z, \partial_z))^n z^{2m},$$
$$b_\nu \left(x^{2m} D_x (L_\nu(x, \partial_x))^n \right) = (L_\nu(z, \partial_z))^n D_z z^{2m}.$$

Now the statement of the Lemma is obvious. □

Note that the formal adjoint involution a of $\mathcal{W}_{rat}$ preserves the spaces $\mathcal{B}_\nu^{l_1,l_2}$. Since $a^2 = \mathrm{id}$ the space $\widetilde{\mathcal{B}}_\nu^{l_1,l_2}$ is the direct sum of the eigenspaces of a with eigenvalues ± 1. The eigenvalue 1 subspace of $\widetilde{\mathcal{B}}_\nu^{l_1,l_2}$ will be denoted by $\widetilde{\mathcal{B}}_{\nu,sym}^{l_1,l_2}$.

Let us also note that bispectral involution b_ν and the formal adjoint involution a commute on $\widetilde{\mathcal{B}}_\nu$:

$$ab_\nu(P(x, \partial_x)) = b_\nu a(P(x, \partial_x)), \ \forall P(x, \partial_x) \in \widetilde{\mathcal{B}}_\nu. \tag{4.23}$$

(This is proved by a direct check on the generators $L_\nu(x, \partial_x)$, D_x, and x^2 of $\widetilde{\mathcal{B}}_\nu$.)

From the commutativity (4.23) we obtain

$$\widetilde{\mathcal{B}}_{\nu,sym}^{l_1,l_2} = \{ R(x, \partial_x) \in \widetilde{\mathcal{B}}_\nu \mid \operatorname{ord} R(x, \partial_x) \le 2l_1, \ \operatorname{ord}(b_\nu R)(z, \partial_z) \le 2l_2,$$
$$aR(x, \partial_x) = R(x, \partial_x), \ a(b_\nu R(x, \partial_x)) = b_\nu R(x, \partial_x) \}.$$

Lemma 4.6. *The set of operators*

$$\{ x^{2m} (L_\nu(x, \partial_x))^n + (L_\nu(x, \partial_x))^n x^{2m} \mid n \le l_1, m \le l_2 \} \tag{4.24}$$

is a basis for the space $\widetilde{\mathcal{B}}_{\nu,sym}^{l_1,l_2}$. *In particular*

$$\dim \widetilde{\mathcal{B}}_{\nu,sym}^{l_1,l_2} = (l_1 + 1)(l_2 + 1).$$

The *Proof of Lemma 4.6* is similar to the one of Lemma 3.5. We start with

$$a(L_\nu(x, \partial_x)) = L_\nu(x, \partial_x), \quad a(D_x + 1/2) = -(D_x + 1/2).$$

Thus the operators in (4.24) and

$$\{ x^m \partial_x (L_A(x, \partial_x))^n + (L_A(x, \partial_x))^n \partial_x x^m \mid n < l_1, m < l_2 \} \tag{4.25}$$

are eigenvectors of a with eigenvalues 1 and -1, respectively. Obviously these are subsets of $\mathcal{B}_A^{l_1,l_2}$.

From the commutation relations (4.22) we get

$$L_\nu(x, \partial_x)^n x^{2m} = x^{2m} L_\nu(x, \partial_x)^n \qquad \sum_{\substack{0 \le i < m \\ 0 \le j < n}} c_{i,j} x^{2i} L_\nu(x, \partial_x)^j$$
$$+ \sum_{\substack{0 \le i < m \\ 0 \le j < n}} d_{i,j} x^i (D_x + 1/2) L_\nu(x, \partial_x)^j$$

$$L_\nu(x, \partial_x)^n (D_x + 1/2) x^m = x^m (D_x + 1/2) L_\nu(x, \partial_x) + \sum_{\substack{0 \le i \le m \\ 0 \le j \le n \\ i + j \le m + n - 1}} c'_{i,j} x^i L_\nu(x, \partial_x)^j$$
$$+ \sum_{\substack{0 \le i < m \\ 0 \le j < n}} d'_{i,j} x^i (D_x + 1/2) L_\nu(x, \partial_x)^j$$

for some integers $c_{i,j}$, $d_{i,j}$, $c'_{i,j}$, $d'_{i,j}$. Applying Lemma 4.5 we get that the operators in (4.24) and (4.25) are linearly independent elements of $\mathcal{B}_\nu^{l_1,l_2}$. Thus the dimension of the direct sum of the spaces spanned by (4.24) and (4.25) is

$$(l_1+1)(l_2+1)+l_1l_2$$

which is exactly the dimension of $\mathcal{B}_\nu^{l_1,l_2}$. This is only possible if the set (4.24) is basis of $\mathcal{B}_{\nu,sym}^{l_1,l_2}$. □

Example 4.7. Consider the integral operator with kernel

$$K_{\sin}(x,y) = \frac{\sin s(x-y)}{(x-y)}, \tag{4.26}$$

acting on $L^2[-t,t]$, $s,t\in\mathbb{R}$. The fact that it commutes with the differential operator

$$D_0(x,\partial_x) = \partial_x(x^2-t^2)\partial_x + s^2x^2 \tag{4.27}$$

is an essential property of (4.26), having important application to time and band limiting, and the bulk scaling limit of the gaussian unitary ensemble, as shown by Landau, Pollak, Slepian [23, 18] and Tracy, Widom [24]. (Typically one uses $s=\pi$ and divides the above kernel $K_{\sin}$ by π. We introduce the extra parameter s since there is a natural duality between s and t to be described below.)

Here we show how $D_0(x,\partial_x)$ is easily constructed from Theorem 2.5 and Lemma 4.5.

First of all observe that the sine kernel (4.26) is also given by

$$K_{\sin}(x,y) = \frac{e^{is(x-y)}-e^{-is(x-y)}}{2i(x-y)} = \frac{1}{2i}\int_{-is}^{is} e^{xz}e^{-yz}\mathrm{d}z \tag{4.28}$$

Note that e^{xz} and e^{-xz} span the space of functions satisfying (4.5)–(4.7) for $\nu=0$ and according to Lemma 4.5

$$\widetilde{B}_{0,sym}^{1,1} = \mathrm{Span}\{1, x^2, \partial_x^2, x^2\partial_x^2+\partial_x^2x^2\}. \tag{4.29}$$

Now Theorem 2.5 immediately implies that the sine kernel possesses a commuting second order differential operator but such is also easy to construct on the basis of Proposition 2.1 and Lemma 2.4 as follows. From Proposition 2.1 we get that

$$D(x,\partial_x) = ux^2 + v\partial_x^2 + w(x^2\partial_x^2+\partial_x^2x^2)/2$$

commutes with $K_{\sin}$ if

$$\phi_t(D(x,\partial_x)) = 0 \text{ and } \phi_{is}((b_0D)(z,\partial_z)) = 0. \tag{4.30}$$

But

$$D(x,\partial_x) = \partial_x(wx^2+v)\partial_x + ux^2 + w$$

and

$$\begin{aligned}(b_0D)(z,\partial_z) &= u\partial_z^2 + vz^2 + w(z^2\partial_z^2+\partial_z^2z^2)/2\\ &= \partial_z(wz^2+u)\partial_z + vz^2 + w\end{aligned}$$

Lemma 2.4 implies that (4.30) is equivalent to

$$wt^2+v=0,\ -ws^2+u=0.$$

The triple $w=1$, $v=-t^2$, $u=s^2$ is a solution of this system producing the commuting differential operator

$$D(x,\partial_x) = s^2x^2 - t^2\partial_x^2 + (x^2\partial_x^2+\partial_x^2x^2)/2 = D_0(x,\partial_x)+1/2,$$

cf. (4.27). Again in the above form the duality between t and is is preserved while in (4.27) it is not transparent. □

Example 4.8. Bessel kernel. □

4.4. **Size of the algebra $\mathcal{B}_{\Psi,sym}$ for an (even) selfadjoint Darboux transformation from a Bessel function.** Fix an arbitrary selfadjoint (and in addition even in the rank 1 case $\nu \in \mathbb{Z}$) Darboux transformation from a Bessel function $\Psi_\nu(x,z)$, $\Psi \in \mathcal{SD}_\nu$ given by (4.17)–(4.18) for some $P(x,\partial_x) \in (\mathbb{C}[x^2]\backslash\{0\})^{-1}\widetilde{\mathcal{B}}_\nu$ and $f(t) = t^m g(t)^2$, $g(t) \in \mathbb{C}[t]$. Let

$$P(x,\partial_x) = \frac{1}{v(x^2)} R(x,\partial_x) \tag{4.31}$$

for some $R(x,\partial_x) \in \widetilde{\mathcal{B}}_\nu$ and $v(x) \in \mathbb{C}[x]$. Set

$$\operatorname{ord} R(x,\partial_x) = \rho_1 \quad \text{and} \quad \operatorname{ord}(b_\nu R)(x,\partial_x) = \rho_2. \tag{4.32}$$

Denote

$$\mathcal{S}_{\Psi,1} = \operatorname{Span}\{\frac{1}{x^m v(x^2)} R(x,\partial_x) M(x,\partial_x)(aR)(x,\partial_x) \frac{1}{x^m v(x^2)} \mid M(x,\partial_x) \in \widetilde{\mathcal{B}}^{l_1-\rho_1,l_2}_{\nu,sym}\},$$

$$\mathcal{S}_{\Psi,2} = \operatorname{Span}\{x^m v(x^2) M(x,\partial_x) x^m v(x^2) \mid M(x,\partial_x) \in \widetilde{\mathcal{B}}^{l_1,l_2-\rho_2}_{\nu,sym}\}.$$

Proposition 4.9. *(i) The spaces of differential operators $\mathcal{S}_{\Psi,1}$ and $\mathcal{S}_{\Psi,2}$ are subspaces of $\mathcal{B}^{l_1,l_2}_{\Psi,sym}$ and are invariant under the transformation $x \mapsto -x$. The bispectral antiinvolution b_Ψ associated to the function $\Psi(x,z)$ acts on these spaces by*

$$b_\Psi\left(\frac{1}{x^m v(x^2)} R(x,\partial_x) M(x,\partial_x)(aR)(x,\partial_x)\frac{1}{x^m v(x^2)}\right) = z^m g(z^2)(b_\nu M)(z,\partial_z) z^m g(z^2), \tag{4.33}$$

$$b_\Psi\left(x^m v(x^2) M(x,\partial_x) x^m v(x^2)\right) = \frac{1}{z^m g(z^2)}(b_\nu R)(z,\partial_z)(b_\nu M)(z,\partial_z)(ab_\nu R)(z,\partial_z)\frac{1}{z^m g(z^2)}. \tag{4.34}$$

(ii) The dimension of the intersection $\mathcal{S}_{\Psi,1} \cap \mathcal{S}_{\Psi,2}$ is less than or equal to

$$(l_1 - \rho_1 + 1)(l_2 - \rho_2 + 1).$$

Proof. Applying Theorem 4.2 from [3] we obtain that (4.17)–(4.18), (4.31) are equivalent to

$$L_\nu(z,\partial_z)^m v(L_\nu(z,\partial_z))^2 = (ab_\nu R)(z,\partial_z)\frac{1}{z^{2m} g(z^2)^2}(b_\nu R)(z,\partial_z) \tag{4.35}$$

$$\Psi(x,z) = \frac{1}{(xz)^m v(x^2) g(z^2)}(b_\nu R)(z,\partial_z)\Psi_\nu(x,z), \tag{4.36}$$

cf. the dual expressions for Darboux transformations in the Airy case (3.31)–(3.32).

Part (i) is proved analogously to Proposition 3.7 using (4.35)–(4.36).

Part (ii) is similar to Proposition 3.8 but there are several differences:

The orders ρ_1 and ρ_2 of the operator $R(x,\partial_x)$ and $(b_\nu R)(z,\partial_z)$ are not necessarily even. First

$$\rho_1 (= \operatorname{ord} R) = m + 2\deg g, \text{ that is } \rho_1 \equiv m (\mod 2). \tag{4.37}$$

This is obtained by comparison of the orders of the operators in (4.17), using (4.16) and (4.31). Second from (4.35) one gets

$$\rho_2(= \operatorname{ord} b_\nu R) = m + 2\deg v, \ \textit{i.e.}\ \rho_1 \equiv m(\mod 2). \tag{4.38}$$

Since $P(x,\partial_x) = \frac{1}{v(x^2)}R(x,\partial_x)$ has leading coefficient x^m the differential operator $R(x,\partial_x)$ must have leading coefficient $x^m v(x^2)$.

Thus if m is even $R(x,\partial_x)$ has the form

$$R(x,\partial_x) = v(x^2)L_\nu(x,\partial_x)^{\deg g} + \sum_{j=0}^{\deg g-1}(q_j(x^2) + r_j(x^2)D_x)L_\nu(x,\partial_x)^j \tag{4.39}$$

and if m is odd $R(x,\partial_x)$ has the form

$$R(x,\partial_x) = (v(x^2)D_x + q_{\deg g}(x^2))L_\nu(x,\partial_x)^{\deg g} + \sum_{j=0}^{\deg g-1}(q_j(x^2) + r_j(x^2)D_x)L_\nu(x,\partial_x)^j \tag{4.40}$$

for some polynomials $q_j(t)$, $r_j(t)$.

Similarly to the Airy case we need to show that the dimension of the space of differential operators

$$M(x,\partial_x) \in \widetilde{\mathcal{B}}^{l_1-\rho_1,l_2}_{\nu,sym} \text{ and } N(x,\partial_x) \in \widetilde{\mathcal{B}}^{l_1,l_2-\rho_2}_{\nu,sym}$$

such that

$$R(x,\partial_x)M(x,\partial_x)(aR)(x,\partial_x) = x^{2m}v(x^2)^2 N(x,\partial_x)x^{2m}v(x^2)^2 \tag{4.41}$$

is less than or equal to $(l_1-\rho_1+1)(l_2-\rho_2+1)$. Write two such operators $M(x,\partial_x)$ and $N(x,\partial_x)$ in the form

$$M(x,\partial_x) = \sum_{j=0}^{l_1-\rho_1}\left(b_j(x^2)L_\nu(x,\partial_x)^j + L_\nu(x,\partial_x)^j b_j(x^2)\right), \tag{4.42}$$

$$N(x,\partial_x) = \sum_{j=0}^{l_1}\left(c_j(x^2)L_\nu(x,\partial_x)^j + L_\nu(x,\partial_x)^j c_j(x^2)\right) \tag{4.43}$$

where $b_j(t)$ and $c_j(t)$ are polynomials and

$$\deg b_j \le l_2, \ \deg c_j \le l_2 - \rho_2, \tag{4.44}$$

recall Lemma 4.5. As in the Airy case in terms of those we need to show that the dimension of the space of of polynomials $\{b_j(t)\}_{j=0}^{l_1-\rho_1}$ and $\{c_j(t)\}_{j=0}^{l_2}$, satisfying (4.41), is less than or equal to $(l_1-\rho_1+1)(l_2-\rho_2+1)$. (The polynomials $v(t)$, $q_j(t)$, and $r_j(t)$ are fixed. They determine the function $\Psi(x,z)$.)

Using the commutation relations (4.22) from (4.39)-(4.40) and (4.42) we deduce

$$R(x,\partial_x)M(x,\partial_x)(aR)(x,\partial_x) = \sum_{j=0}^{l_1-\rho_1} 2x^{2m}v(x^2)^2 b_j(x^2)L_\nu(x,\partial_x)^{j+\rho_1} + \sum_{i=0}^{l_1-1}\left(\alpha_i(x^2) + \beta_i(x^2)D_x\right)L_\nu(x,\partial_x)^i \tag{4.45}$$

for some polynomials $\alpha_i(t)$ and $\beta_i(t)$ depending only on the polynomials $b_k(t)$ for $k \in [i-\rho_1+1, l_1-\rho_1]$. Similarly the right hand side of (4.41) is given by

$$x^{2m}v(x^2)^2 N(x,\partial_x) x^{2m} v(x^2)^2 = 2x^{2m}v(x^2)^4 \sum_{i=0}^{l_1} c_i(x^2) L_\nu(x,\partial_x)^i + v(x^2)^2 \sum_{i=0}^{l_1-1} \left(\gamma_i(x^2) + \delta_i(x^2) D_x\right) L_\nu(x,\partial_x)^i \tag{4.46}$$

for some polynomials $\gamma_i(t)$ and $\delta_i(t)$ depending only on the polynomials $c_k(t)$ for $k \in [i+1, l_1]$.

Finally we substitute (4.45) and (4.46) in (4.41) and compare the two sides, based on the fact that

$$\{x^{2k} L_\nu(x,\partial_x)^n, x^{2k} D_x L_\nu(x,\partial_x)^n\}_{k,n=0}^{\infty}$$

are linearly independent.

1. First from the coefficient of $L_\nu(x,\partial_x)^{l_1}$ we have

$$b_{l_1-\rho_1}(x^2) = x^{2m}v(x^2)^2 c_{l_1}(x^2), \text{ that is } b_{l_1-\rho_1}(t) = t^m v(t)^2 c_{l_1}(t).$$

Because $\deg t^m v(t)^2 = \rho_1$, see (4.38) and $\deg b_{l_1-\rho_1} \leq l_2$, the polynomial $b_{l_1-\rho_1}(t)$ depends (linearly) on at most $l_2 - \rho_2 + 1$ parameters. Fixing those parameters determines the polynomial $c_{l_1}(t)$ as well.

2. By induction, assume that $b_j(t), \ldots, b_{l_1-\rho_1}(t)$ and $c_{j+l_1}(t), \ldots, c_{l_1}(t)$ and are fixed for j running from $l_1 - \rho_1$ down to 1. From the coefficient of $L_\nu(x,\partial_x)^{j-1}$ we obtain

$$b_{j-1}(x^2) = x^{2m}v(x^2)^2 c_{j+\rho_1-1}(x^2) + \left(\alpha_{j+\rho_1-1}(x^2) - \eta_{j+\rho_1-1}(x^2)\right)/2x^{2m}v(x^2)^2.$$

Therefore the remainder of $b_{j-1}(t)$ modulo $t^m v(t)^2$ is fixed from the previous steps of the induction. Because $\deg b_{j-1}(t) \leq l_2$ and $\deg t^m v(t)^2 = \rho_2$ the choice of $b_{j-1}(t)$ depends (linearly) on at most $l_2 - \rho_2 + 1$ parameters. Fixing $b_{j-1}(t)$ completely determines $c_{j+l_1-1}(t)$.

Adding up these "maximal degrees of freedom" for the polynomials $b_0(t), \ldots, b_{l_1-\rho_1}(t)$ completes the proof. □

Theorem 4.10. *For any selfadjoint (and even, selfadjoint in the case $\nu \in \mathbb{Z}$) Darboux transformation from a Bessel function $\Psi_\nu(x,z)$, $\Psi(x,z) \in \mathcal{SD}_\nu$ the space $\mathcal{S}_{\Psi,1} + \mathcal{S}_{\Psi,2}$ consists of differential operators invariant under the transformation $x \mapsto -x$*

$$\dim(\mathcal{S}_{\Psi,1} + \mathcal{S}_{\Psi,2}) \geq (l_1+1)(l_2+1) - \rho_1\rho_2.$$

In particular the dimension of the subspace of $\mathcal{B}^{l_1,l_2}_{\Psi,sym}$ of differential operators invariant under $x \mapsto -x$ is greater than or equal to $(l_1+1)(l_2+1) - \rho_1\rho_2$.

Theorem 4.10 and Theorem 2.5 imply our final result for Darboux transformations from the Bessel functions:

Theorem 4.11. *Let $\Psi(x,z) \in \mathcal{SD}_\nu$ be a selfadjoint (and in addition even if $\nu \in \mathbb{Z}$) Darboux transformation from the Bessel function $\Psi_\nu(x,z)$, given by (4.17), (4.18), (4.16), (4.31). Let Γ_1, Γ_2 be two connected finite contours that do not contain the roots of the polynomials $v(t)$ and $g(t)$, respectively and such that $e(\Gamma_i) = -e(\Gamma_i)$.*

Then the integral operator with kernel

$$K(x,y) = \int_{\Gamma_2} \Psi(x,z)\Psi(y,z)\mathrm{d}z$$

on $L^2(\Gamma_1)$ commutes with a formally symmetric differential operator with rational coefficients of order less than or equal to $2\rho_1\rho_2$ and domain all smooth functions on Γ_1.

Remark 4.12. Let $\Phi_1(x,z)$ and $\Phi_2(x,z)$ be two functions satisfying (3.2)–(3.4) or (4.5)–(4.6). E.g. in the rank 1 case (section 4, $\nu = 0$) one can choose

$$\Phi_1(x,z) = e^{xz}, \quad \Phi_2(x,z) = e^{-xz}$$

and in the rank 2, Bessel case (section 4)

$$\Phi_1(x,z) = \Psi_\nu(x,z), \quad \Phi_2(x,z) = \Psi_\nu(x,-z).$$

Fix a differential operator $P(x,\partial_x)$ and a polynomial $g(z)$ that define an (even) selfadjoint Darboux transformation from the Airy or Bessel functions as in Definition 3.3 and Definition 4.4. Define

$$\Psi_i(x,z) = \frac{1}{g(z)}P(x,\partial_x), \text{ and } \Psi_i(x,z) = \frac{1}{z^m g(z^2)}P(x,\partial_x), \; i = 1,2$$

in the Airy and Bessel cases, respectively. Then Theorem 3.10 and Theorem 4.11 hold for the integral operator with kernel

$$K(x,y) = \int_{\Gamma_2} \Psi_1(x,z)\Psi_2(y,z)\mathrm{d}z$$

and exactly the same commuting differential operator that appears in those Theorems.

5. Classification of selfadjoint Darboux transformations of Bessel and Airy functions

5.1. **Properties of the bilinear concomitant.**

5.2. **Rank 1 case.**

5.3. **Airy case.**

5.4. **Bessel rank 2 case.**

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